AN EXAMPLE OF A SPACE $L^{p(x)}$ ON WHICH THE HARDY–LITTLEWOOD MAXIMAL OPERATOR IS NOT BOUNDED

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ABSTRACT. We give an example of a function p such that the Hardy–Littlewood maximal operator is not bounded on the generalized Lebesgue space $L^{p(x)}$.

1. INTRODUCTION

Let Ω be a domain on $\mathbb{R}^n$, $n \in \mathbb{N}$. Let p be a measurable function defined on Ω having values in $[1,\infty]$. For such function we define the *generalized Lebesgue space* $L^{p(x)}$ by

$$L^{p(x)} = \big\{ f \text{ measurable on } \Omega;\ \int_\Omega |\lambda f(x)|^{p(x)}\,dx < \infty \text{ for some } \lambda > 0 \big\}.$$

Since the functional $\int_\Omega |f(x)|^{p(x)}\,dx$ is a convex modular (cf. [5] for details), the set $L^{p(x)}$ can be endowed with the Luxemburg norm

$$\|f\|_{L^{p(x)}} = \inf \Big\{ \lambda > 0;\ \int_\Omega \left| \frac{f(x)}{\lambda} \right|^{p(x)} dx \le 1 \Big\},$$

turning it into a Banach space. The spaces $L^{p(x)}$ have been investigated since late 1980's because they constitute a natural functional setting in various applications (cf. [7], [6]). These spaces (together with Sobolev spaces built upon them) were first studied in [4]. However, many properties such as Sobolev-type embeddings or density of smooth functions turned out not to be sharp, unlike their analogues in the setting of classical Sobolev spaces. Some of these properties have recently been improved (cf. [6], [3], [1]). For classical operators of harmonic analysis as Riesz potentials, singular integral operators and, most naturally, the Hardy–Littlewood maximal operator acting on generalized Lebesgue spaces almost nothing is known.

The *centered Hardy–Littlewood maximal operator* is defined at every locally integrable function f on Ω and for every $x \in \Omega$ by

$$(Mf)(x) = \sup_{r>0} \frac{1}{|\Omega \cap B(x,r)|} \int_{\Omega \cap B(x,r)} |f(y)|\,dy,$$

where $B(x,r) = \{y \in \mathbb{R}^n;\ |y-x| < r\}$ is a ball and $|E|$ is the Lebesgue measure of a measurable set $E \subset \mathbb{R}^n$. While it has been well-known for many years that the operator M is bounded on the Lebesgue space $L^p(\Omega)$ if and only if $p > 1$, the problem of its boundedness on a generalized Lebesgue space $L^{p(x)}$ remained open. It is quite natural to expect that not only the values of p but also the quality of gradient of p will play a crucial role. In this note we present an example showing that if the function $p(x)$ has a point of a very rapid increase,

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then the operator M cannot be bounded on the corresponding space $L^{p(x)}$. Since the dimension of the underlying domain plays no role and extensions to higher dimensional cases are quite obvious, we present our example in its simplest form, that is, on an interval in $\mathbb{R}^1$. The method we use was inspired by an example due to B. Kirchheim ([6, Proposition 2.35]) who showed that the $L^{p(x)}$–boundedness cannot be inferred from a uniform boundedness in the classical Lebesgue spaces L^q on appropriate level sets. Recently, we were very pleasantly surprised to learn that L. Diening has proved ([1]) that, in a certain sense, our condition on p is sharp. He showed that when the increase of p is less rapid, then the Hardy-Littlewood maximal operator is already bounded on $L^{p(x)}$.

2. The result

Theorem. *Let $n = 1$, $\Omega = (-1, 1) \subset \mathbb{R}^1$ and $1 < p_0 < \infty$. Let φ be a positive increasing function on $[0, 1)$ such that $\varphi(0^+) = 0$ and*

$$\lim_{x \to 0^+} \varphi\left(\frac{x}{2}\right) \log\left(\frac{1}{x}\right) = \infty. \tag{1}$$

Assume that $p(x) \le p_0$ for all $x \in (-1, 0]$ and $p(x) = p_0 + \varphi(x)$ for all $x \in [0, 1)$. Then the maximal operator is not bounded on $L^{p(x)}(\Omega)$.

Proof. It follows from (1) that

$$\lim_{x \to 0^+} \left(\frac{1}{p(x/2)} - \frac{1}{p_0}\right) \log\left(\frac{1}{x}\right) = -\infty,$$

whence

$$\lim_{x \to 0^+} \left(\frac{1}{x}\right)^{\frac{1}{p(x/2)} - \frac{1}{p_0}} = 0.$$

Therefore, for $k \in \mathbb{N}$, we can choose $b_k \in (0, \frac{1}{3})$ small enough in order that

$$\left(\frac{1}{b_k}\right)^{\frac{1}{p(b_k/2)} - \frac{1}{p_0}} \le 2^{-\frac{k}{p_0}}, \tag{2}$$

and, at the same time $0 < b_{k+1} < \frac{b_k}{2}$. Having fixed such sequence $\{b_k\}$, we denote $a_k = \frac{b_k}{2}$ and $\lambda_k = \left(\frac{1}{b_k}\right)^{\frac{1}{p(a_k)}}$. Note that $\lambda_k > 1$. Now, we set

$$g = \sum_{k=1}^{\infty} \lambda_k \chi_{(a_k, b_k)} \quad \text{and} \quad h(x) = \begin{cases} g(-x) & \text{for } x \in (-1, 0); \\ 0 & \text{for } x \in [0, 1). \end{cases}$$

Since $p(x) \le p_0$ on $(-1, 0)$ and $\lambda_k > 1$ for every $k \in \mathbb{N}$, we have by (2)

$$\begin{aligned} \int_{-1}^{0} h(x)^{p(x)}\,dx &= \sum_{k=1}^{\infty} \int_{a_k}^{b_k} \lambda_k^{p(-x)}\,dx \le \sum_{k=1}^{\infty} (b_k - a_k) \lambda_k^{p_0} \\ &= \sum_{k=1}^{\infty} \frac{1}{2} b_k^{1 - \frac{p_0}{p(a_k)}} \le \sum_{k=1}^{\infty} 2^{-k} < \infty. \end{aligned}$$

Next, let $x \in (a_k, b_k)$. Then $2x + b_k < 1$, since $b_k < \frac{1}{3}$, and

$$(Mh)(x) \geq \frac{1}{2(x+b_k)} \int_{-b_k}^{2x+b_k} h(y)\,dy \geq \frac{1}{4b_k} \int_{-b_k}^{-a_k} h(y)\,dy$$
$$= \frac{1}{4b_k} \int_{-b_k}^{\frac{-b_k}{2}} \lambda_k\,dy = \frac{\lambda_k}{8}.$$

Thus

$$(Mh)(x) \geq \frac{1}{8} \sum_{k=1}^{\infty} \lambda_k \chi_{(a_k,b_k)}(x), \quad x \in (0,1).$$

Therefore

$$\int_0^1 (Mh)(x)^{p(x)}\,dx \geq \frac{1}{8} \sum_{k=1}^{\infty} \int_{a_k}^{b_k} \lambda_k^{p(x)}\,dx \geq \frac{1}{8} \sum_{k=1}^{\infty} \int_{a_k}^{b_k} \lambda_k^{p(a_k)}\,dx$$
$$= \frac{1}{16} \sum_{k=1}^{\infty} \lambda_k^{p(a_k)} b_k = \frac{1}{16} \sum_{k=1}^{\infty} 1 = \infty.$$

□

Example. The functions

$$\varphi(x) = \frac{1}{\log\log(e + \frac{1}{x})} \qquad \text{or} \qquad \varphi(x) = \frac{1}{(\log \frac{1}{x})^{\alpha}},\ 0 < \alpha < 1,$$

satisfy (1). Of course, such functions approach 0 at 0 very slowly.

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