GENERALIZED n-LAPLACIAN: QUASILINEAR NONHOMOGENOUS PROBLEM WITH THE CRITICAL GROWTH

ROBERT ČERNÝ

ABSTRACT. Applying the generalized Moser-Trudinger inequality, the Mountain Pass Theorem and the Ekeland variational principle we study the existence of non-trivial weak solutions to the problem

$$u \in W^1L^\Phi(\mathbb{R}^n) \qquad \text{and} \qquad -\operatorname{div}\Big(\Phi'(|\nabla u|)\frac{\nabla u}{|\nabla u|}\Big) + V(x)\Phi'(|u|)\frac{u}{|u|} = f(x,u) + \mu h(x) \quad \text{in } \mathbb{R}^n ,$$

where Φ is a Young function such that the space $W^1L^\Phi(\mathbb{R}^n)$ is embedded into exponential or multiple exponential Orlicz space, the nonlinearity $f(x,t)$ has the corresponding critical growth, $V(x)$ is a continuous potential, $h \in (L^\Phi(\mathbb{R}^n))^*$ is a nontrivial continuous function and $\mu > 0$ is a small parameter.

1. Introduction

It is an often studied problem to find solutions to the Laplace equation

$$u \in W_0^{1,2}(\Omega) \qquad \text{and} \qquad -\Delta u = f(x,u) \quad \text{in } \Omega \subset \mathbb{R}^2 . \tag{1.1}$$

For $n \geq 3$ and f satisfying $\lim_{t\to\infty} \frac{f(x,t)}{t^q} = 0$ uniformly on Ω with $q < \frac{n+2}{n-2}$, there are many results using the compactness of the embedding of the space $W_0^{1,2}(\Omega)$ into $L^r(\Omega)$ with $r \in [1, \frac{2n}{n-2})$ (see a review article by Lions [21] and the references given there). Problem (1.1) under condition $\lim_{t\to\infty} \frac{f(x,t)}{t^{\frac{n+2}{n-2}}} = 0$ becomes much more difficult thanks to the fact that the embedding of $W_0^{1,2}(\Omega)$ into $L^{\frac{2n}{n-2}}(\Omega)$ is no longer compact. This difficulty has been overcame by Brezis and Nirenberg [6]. Their method uses the Mountain Pass Theorem by Ambrosetti and Rabinowitz [3].

When $n = 2$, we do not only have the Sobolev embedding into $L^r(\Omega)$ for any $r \in [0,\infty)$ but there is also the Trudinger embedding [30] into the Orlicz space $\exp L^{\frac{n}{n-1}}(\Omega)$. In particular, there is so called Moser-Trudinger inequality by Moser [23]

$$\sup_{||u||_{W_0^{1,n}(\Omega)} \leq 1} \int_\Omega \exp(K|u|^{\frac{n}{n-1}}) \leq C(n, \mathcal{L}_n(\Omega)) \qquad \text{if and only if} \qquad K \leq n\omega_{n-1}^{n-1} .$$

Therefore, in the literature, there is often used the variational approach by Brezis and Nirenberg [6] together with the Moser-Trudinger inequality to study the n-Laplace equation

$$u \in W_0^{1,n}(\Omega) \qquad \text{and} \qquad -\Delta_n u = f(x,u) \quad \text{in } \Omega , \tag{1.2}$$

where $\Delta_n u := \operatorname{div}(|\nabla u|^{n-2}\nabla u)$ and $f(x,t) \approx \exp(b|t|^{\frac{n}{n-1}})$ for some $b > 0$. See for example Adimurthi [1], de Figueiredo, Miyagaki, Ruf [18] and do Ó [26].

In recent paper [11], above techniques are modified for a differential equation corresponding to the embedding of the Orlicz-Sobolev space $W_0L^n \log^\alpha L(\Omega)$, $n \geq 2$, $\alpha < n-1$, into the Orlicz space $\exp L^{\frac{n}{n-1-\alpha}}(\Omega)$ (this embedding is due to Fusco, Lions, Sbordone [19] and Edmunds, Gurka, Opic [14]). The result is the existence of a non-trivial weak solution to the equation

$$u \in W_0L^\Phi(\Omega) \qquad \text{and} \qquad -\operatorname{div}\Big(\Phi'(|\nabla u|)\frac{\nabla u}{|\nabla u|}\Big) = f(x,u) \quad \text{in } \Omega, \tag{1.3}$$

Date: December 29, 2010.

2000 *Mathematics Subject Classification.* 46E35,46E30,26D910.

Key words and phrases. Orlicz-Sobolev spaces, Mountain Pass Theorem, Palais-Smale sequence, Ekeland variational principle.

with Φ being a Young function that behaves like $t^n\log^\alpha(t)$, $\alpha<n-1$, for large t and with the nonlinearity f having so called critical growth (corresponding to the choice of the Young function Φ).

The results from paper [11] were further generalized in papers [9] and [8] in several ways (generalized n-Laplace equation corresponding to the embedding into multiple exponential spaces, singular nonlinearity and the case of $WL^\Phi(\mathbb{R}^n)$) motivated by some recent results concerning the n-Laplace equation, see for example [2] and [24].

In this paper, similarly as in papers [29] and [25], for a version of (1.3) we employ the Ekeland variational principle to show that besides the Mountain Pass-type weak solution there is also a distinct minimum-type weak solution. When showing that the two solutions are distinct we use a new estimate concerning the Concentration-Compactness alternative for generalized Trudinger inequalities.

On embedding into exponential and multiple exponential spaces. If $\ell\in\mathbb{N}$, $n\geq 2$ and $\alpha<n-1$, we set

$$\gamma=\frac{n}{n-1-\alpha}>0\ ,\qquad B=1-\frac{\alpha}{n-1}=\frac{n}{(n-1)\gamma}>0$$
$$\text{and}\qquad K_{\ell,n,\alpha}=\begin{cases}B^{\frac{1}{B}}n\omega_{n-1}^{\frac{\gamma}{n}} & \text{for } \ell=1\\ B^{\frac{1}{B}}\omega_{n-1}^{\frac{\gamma}{n}} & \text{for } \ell\geq 2\ .\end{cases}\tag{1.4}$$

The space $W_0L^n\log^\alpha L(\Omega)$ of the Sobolev type, modeled on the Zygmund space $L^n\log^\alpha L(\Omega)$, is continuously embedded into the Orlicz space with the Young function that behaves like $\exp(t^\gamma)$ for large t (see [19] and [14]). Moreover it is shown in [14] (see also [15]) that in the limiting case $\alpha=n-1$ we have the embedding into a double exponential space, i.e. the space $W_0L^n\log^{n-1}L\log^\alpha\log L(\Omega)$, $\alpha<n-1$, is continuously embedded into the Orlicz space with the Young function that behaves like $\exp(\exp(t^\gamma))$ for large t. Further in the limiting case $\alpha=n-1$ we have the embedding into triple exponential space and so on. The borderline case is always $\alpha=n-1$ and for $\alpha>n-1$ we have embedding into $L^\infty(\Omega)$. It is well-known that the Zygmund space $L^n\log^\alpha L(\Omega)$ coincides with the Orlicz space $L^\Phi(\Omega)$, where the Young function Φ satisfies

$$\lim_{t\to\infty}\frac{\Phi(t)}{t^n\log^\alpha(t)}=1\ ,$$

the space $L^n\log^{n-1}L\log^\alpha\log L(\Omega)$ coincides with $L^\Phi(\Omega)$ where

$$\lim_{t\to\infty}\frac{\Phi(t)}{t^n\log^{n-1}(t)\log^\alpha(\log(t))}=1\ ,$$

and so on. For other results concerning these spaces we refer the reader to [14], [15] and [16].

The following notation enables us to work with the multiple exponential spaces comfortably. For $k\in\mathbb{N}$, let us write

$$\log_{[k]}(t)=\log(\log_{[k-1]}(t)),\qquad\text{where}\qquad\log_{[1]}(t)=\log(t)$$

and

$$\exp_{[k]}(t)=\exp(\exp_{[k-1]}(t)),\qquad\text{where}\qquad\exp_{[1]}(t)=\exp(t)\ .$$

Let $\ell\in\mathbb{N}$, $n\geq 2$ and $\alpha<n-1$. Then we have above mentioned embedding results for any Young function Φ satisfying

$$\lim_{t\to\infty}\frac{\Phi(t)}{t^n\Big(\prod_{j=1}^{\ell-1}\log_{[j]}^{n-1}(t)\Big)\log_{[\ell]}^\alpha(t)}=1\tag{1.5}$$

(for $\ell=1$ we read (1.5) as $\lim_{t\to\infty}\frac{\Phi(t)}{t^n\log_{[1]}^\alpha(t)}=1$). As Ω is bounded, all Young functions satisfying (1.5) give the same Orlicz-Sobolev space.

Assumptions on Φ, V and f. In this paper, we are interested in C^1-Young functions $\Phi : [0,\infty) \mapsto [0,\infty)$ satisfying (1.5) and in addition we suppose that there is $C > 0$ such that

$$\frac{1}{C} t^n \le \Phi(t) \le C t^n \qquad \text{for } t \in \Big[0, \frac{1}{C}\Big) \ . \tag{1.6}$$

Let us also give two conditions that are often used when discussing the critical case concerning the generalized Moser-Trudinger inequality (Theorem 3.1 bellow)

$$\Phi(t) \ge t^n \Big(\prod_{j=1}^{\ell-1} \log_{[j]}^{n-1}(t)\Big) \log_{[\ell]}^{\alpha}(t) \Big(1 + \log_{[\ell]}^{-\beta}(t)\Big) \qquad \text{for } t \in [t_\Phi, \infty) \tag{1.7}$$

for some $\beta \in (0, \min\{1, \frac{1}{\gamma}\})$ and $t_\Phi \ge 1$,

$$\Phi(t) \le t^n \Big(\prod_{j=1}^{\ell-1} \log_{[j]}^{n-1}(t)\Big) \log_{[\ell]}^{\alpha}(t) \Big(1 - \log_{[\ell]}^{-\beta}(t)\Big) \qquad \text{for } t \in [t_\Phi, \infty) \tag{1.8}$$

for some $\beta \in (0, \min\{1, B\})$ and $t_\Phi \ge 1$. Notice that assumptions (1.5) and (1.6) together with the fact that Φ is a C^1-Young function imply the existence of $c_\Phi > 0$ such that

$$c_\Phi \, \Phi'(t)\, t \le \Phi(t) \ , \quad t > 0 \ . \tag{1.9}$$

We are dealing with the differential equation

$$u \in WL^\Phi(\mathbb{R}^n) \qquad \text{and} \qquad -\operatorname{div}\Big(\Phi'(|\nabla u|)\frac{\nabla u}{|\nabla u|}\Big) + V(x)\Phi'(|u|)\frac{u}{|u|} = f(x,u) + \mu h(x) \quad \text{in } \mathbb{R}^n \ . \tag{1.10}$$

Here $\mu > 0$ is a small parameter, $h \in (L^\Phi(\mathbb{R}^n))^*$ is continuous, the potential $V : \mathbb{R}^n \mapsto \mathbb{R}$ satisfies

$$V \text{ is continuous} \qquad \text{and} \qquad V(x) \ge V_0 > 0 \quad \text{for all } x \in \mathbb{R}^n \ , \tag{1.11}$$

$$V(x) \to \infty \qquad \text{as } |x| \to \infty \ . \tag{1.12}$$

Next, let

$$\exp_{[\ell]}(t) = \sum_{j=0}^{\infty} a_j t^j$$

be the Taylor expansion of the the function $\exp_{[\ell]}$. We set

$$S_{\ell,n,\alpha}(t) = \sum_{0 \le j < \frac{n}{\gamma}} a_j t^j \ .$$

The function $f : \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}$ is supposed to satisfy the following conditions.
There are $M > 0$, $t_M > 0$, $A \in (0, \min(c_\Phi, \frac{1}{n}))$, $\tilde{C}_b > 0$, $C_b > 0$ and $b > 0$ satisfying

$$\begin{aligned} &f \text{ is uniformly continuous on } \mathbb{R}^n \times [-t_0, t_0] \text{ for every } t_0 > 0 \ , \\ &f(x,0) = 0 \quad \text{and} \quad t f(x,t) > 0 \qquad \text{for all } x \in \mathbb{R}^n \text{ and } t \ne 0 \ , \end{aligned} \tag{1.13}$$

$$0 < F(x,t) := \int_0^t f(x,s)\, ds \le M|t|^{1-\frac{1}{M}} |f(x,t)| \quad \text{provided } |t| > t_M \text{ and } x \in \mathbb{R}^n \ , \tag{1.14}$$

$$0 < F(x,t) \le A\, f(x,t)\, t \quad \text{provided } t \ne 0 \text{ and } x \in \mathbb{R}^n \ , \tag{1.15}$$

$$|f(x,t)| \le \tilde{C}_b |t|^{n-1} + C_b \Big(\exp_{[\ell]}(b|t|^\gamma) - S_{\ell,n,\alpha}(b|t|^\gamma)\Big) \quad \text{for every } t \in \mathbb{R} \text{ and } x \in \mathbb{R}^n \ , \tag{1.16}$$

$$\limsup_{t \to 0} \frac{F(x,t)}{C_S \Phi(|t|)} < 1 \quad \text{uniformly on } \mathbb{R}^n \ , \tag{1.17}$$

where $C_S \ge V_0$ is the constant from the Sobolev-type inequality

$$C_S \int_{\mathbb{R}^n} \Phi(|u|) \le \int_{\mathbb{R}^n} \Phi(|\nabla u|) + V(x)\Phi(|u|) \ , \qquad u \in WL^\Phi(\mathbb{R}^n) \ , \tag{1.18}$$

and finally

$$\liminf_{t\to\infty}\frac{tf(x,t)}{\exp_{[\ell]}(b|t|^\gamma)}>0\qquad\text{uniformly on }\mathbb{R}^n\ . \tag{1.19}$$

Variational formulation. Let us define the space

$$X(\mathbb{R}^n)=\Big\{u\in WL^\Phi(\mathbb{R}^n):\int_{\mathbb{R}^n}V(x)\Phi(|u|)\,dx<\infty\Big\}$$

endowed with the norm

$$||u||_{X(\mathbb{R}^n)}=||\nabla u||_{L^\Phi(\mathbb{R}^n)}+||u||_{L^\Phi(\mathbb{R}^n,V(x)dx)}\ .$$

Hence $X(\mathbb{R}^n)$ is a Banach space satisfying

$$X(\mathbb{R}^n)\subset WL^\Phi(\mathbb{R}^n)\ ,\qquad X(\mathbb{R}^n)\subset L^r(\mathbb{R}^n)\ ,\ r\in[n,\infty)\qquad\text{and}\qquad X(\mathbb{R}^n)\subset L^\Phi(\mathbb{R}^n)\ ,$$

where the first embedding is obviously continuous and the last two embeddings are compact by Proposition 2.9 bellow. Moreover $C_0^\infty(\mathbb{R}^n)$-functions are dense in $X(\mathbb{R}^n)$ by [8, Proposition 2.9].

We define

$$J_\mu(u)=\int_{\mathbb{R}^n}\Phi\big(|\nabla u|\big)+V(x)\Phi(|u|)-F(x,u)-\mu h(x)u\,dx\ ,\qquad u\in X(\mathbb{R}^n)\ . \tag{1.20}$$

By Proposition 5.1 bellow, this is a C^1-functional on $X(\mathbb{R}^n)$ and its Fréchet derivative is (1.21)

$$\langle J'_\mu(u),v\rangle=\int_{\mathbb{R}^n}\Phi'\big(|\nabla u|\big)\frac{\nabla u}{|\nabla u|}\cdot\nabla v+V(x)\Phi'\big(|u|\big)\frac{u}{|u|}v-f(x,u)v-\mu h(x)v\,dx\ ,\quad u,v\in X(\mathbb{R}^n)\ ,$$

where the symbol $\langle J'_\mu(u),v\rangle$ denotes the value of the linear functional $J'_\mu(u)$ of v.

We say that $u\in X(\mathbb{R}^n)$ is a weak solution to problem (1.10) if

$$\langle J'_\mu(u),v\rangle=0\qquad\text{for every }v\in X(\mathbb{R}^n)\ . \tag{1.22}$$

Now, we can state our main result.

Theorem 1.1. *Let $\ell\in\mathbb{N}$, $n\geq 2$ and $\alpha<n-1$. Suppose that the Young function $\Phi:[0,\infty)\mapsto[0,\infty)$ satisfies (1.5), (1.6) and (1.8) with $\beta\in(0,\frac{1}{\gamma}-B)$. Let $V:\mathbb{R}^n\mapsto\mathbb{R}$ satisfy (1.11) and (1.12) and let $f:\mathbb{R}^n\times\mathbb{R}\mapsto\mathbb{R}$ be a function satisfying (1.13), (1.14), (1.15), (1.16), (1.17) and (1.19). Let $h\in(L^\Phi(\mathbb{R}^n))^*$ be a nontrivial continuous function. Then then there is $\mu_0>0$ such that problem (1.10) has at least two non-trivial weak solutions in $X(\mathbb{R}^n)$ for every $\mu\in(0,\mu_0)$.*

The paper is organized as follows. After Preliminaries we recall the generalized Moser-Trudinger inequality and its version for unbounded domains. In the fourth section, we give an estimate related to the Concentration-Compactness Alternative. This estimate becomes an important tool in the eighth section. The fifth section is devoted to the proof that J_μ is a C^1-functional. In the sixth section, we show that the functional J_μ satisfies the assumptions of the Mountain Pass Theorem. The properties of Palais-Smale sequences are given in the seventh section. Finally, in the eighth we apply the Mountain Pass Theorem and the Ekeland variational principle to obtain two convergent Palais-Smale sequences. Then we show that the limit functions are distinct.

2. Preliminaries

Throughout the paper ω_{n-1} denotes the surface of the unit sphere. The n-dimensional Lebesgue measure is denoted by $\mathcal{L}_n$. By χ_A we mean the characteristic function of $A\subset\mathbb{R}^n$. By $B(x_0,R)$ we denote an open Euclidean ball in $\mathbb{R}^n$ centered at x_0 with the radius $R>0$. If $x_0=0$ we simply write $B(R)$.

For two functions $g,h:[0,\infty)\mapsto[0,\infty)$ we write $g\lesssim h$, if there is $C>0$ such that $g(t)\leq Ch(t)$ for every $t\in[0,\infty)$. If u is a measurable function on A, then by $u=0$ (or $u\neq 0$) we mean that u is equal (or not equal) to the zero function a.e. on A.

By C we denote a generic positive constant which may depend on ℓ, n, α and Φ. This constant may vary from expression to expression as usual.

By $\mathcal{M}(A)$ we denote the set of all Radon measures on a compact set A. We write that $\nu_k\overset{*}{\rightharpoonup}\nu$ in $\mathcal{M}(A)$ if $\int_A\psi\,d\nu_k\to\int_A\psi\,d\nu$ for every $\psi\in C(A)$.

Properties of $\exp_{[\ell]}$. For given $\ell \in \mathbb{N}$, $n \in \mathbb{N}$, $\alpha < n-1$ and $p \geq 1$, one can easily prove that there is $C \geq 1$ such that

$$\Big(\exp_{[\ell]}(t) - S_{\ell,n,\alpha}(t)\Big)^p \leq C\Big(\exp_{[\ell]}(pt) - S_{\ell,n,\alpha}(pt)\Big) \qquad \text{for all } t \geq 0 \ . \tag{2.1}$$

Young functions and Orlicz spaces. A function $\Phi : [0,\infty) \to [0,\infty)$ is a Young function if Φ is increasing, convex, $\Phi(0) = 0$ and $\lim_{t\to\infty} \frac{\Phi(t)}{t} = \infty$.

Denote by $L^\Phi(A, d\nu)$ the Orlicz space corresponding to a Young function Φ on a set A with a measure ν. If $\nu = \mathcal{L}_n$ we simply write $L^\Phi(A)$. The space $L^\Phi(A, d\nu)$ is equipped with the Luxemburg norm

$$||u||_{L^\Phi(A,d\nu)} = \inf\Big\{\lambda > 0 : \int_A \Phi\Big(\frac{|u(x)|}{\lambda}\Big)\, d\nu(x) \leq 1\Big\} \ . \tag{2.2}$$

Given a differentiable Young function Φ we can define the generalized inverse function to $\phi(y) = \Phi'(y)$ by

$$\psi(s) = \inf\{y : \phi(y) > s\} \qquad \text{for} \quad s > 0$$

and further we define the associated Young function Ψ by

$$\Psi(t) = \int_0^t \psi(s)\, ds \qquad \text{for} \quad t \geq 0 \ .$$

The dual space to $L^\Phi(A, d\nu)$ can be identified as the Orlicz space $L^\Psi(A, d\nu)$. We further have the generalized Hölder's inequality

$$\int_A |u(y)v(y)|\ d\nu(y) \leq 2||u||_{L^\Phi(A,d\nu)}||v||_{L^\Psi(A,d\nu)} \ . \tag{2.3}$$

Δ_2-condition. We say that a function Φ satisfies the Δ_2-condition, if there is $C_\Delta > 1$ such that

$$\Phi(2t) \leq C_\Delta \Phi(t) \qquad \text{whenever } t \geq 0 \ .$$

It is not difficult to check the Δ_2-condition for our Young functions satisfying (1.5) and (1.6). Therefore one easily proves

$$\int_\Omega \Phi\Big(\frac{|u|}{||u||_{L^\Phi(A,d\nu)}}\Big) d\nu(x) = 1 \qquad \text{whenever } ||u||_{L^\Phi(A,d\nu)} > 0 \tag{2.4}$$

and

$$||u_k||_{L^\Phi(A,d\nu)} \overset{k\to\infty}{\to} 0 \quad \iff \quad \int_A \Phi(|u_k|)\, d\nu(x) \overset{k\to\infty}{\to} 0 \ . \tag{2.5}$$

We also need the following lemma.

Lemma 2.1. *If Φ is a C^1-Young function satisfying the Δ_2-condition, then also Φ' satisfies the Δ_2-condition. Further, we have*

$$\Phi(t_2) - \Phi(t_1) \leq C\Big(\Phi'(t_2)t_2 - \Phi'(t_1)t_1\Big) \qquad \textit{whenever } 0 \leq t_1 \leq t_2 \ .$$

Proof. Let us prove the first assertion. Set $P = C_\Delta^2$. If the function Φ' does not satisfy the Δ_2-condition, then we can find $T > 0$ such that

$$P\Phi'(T) \leq \Phi'(2T) \ .$$

Hence using the convexity of Φ, $\Phi(0) = 0$ and the Δ_2-condition for Φ we obtain

$$\begin{aligned} C_\Delta^2 \Phi(T) &\geq \Phi(4T) > \Phi(4T) - \Phi(2T) \geq 2T \min_{\tau\in[2T,4T]} \Phi'(\tau) = 2T\Phi'(2T) \geq 2PT\Phi'(T) \\ &= 2PT \max_{\tau\in[0,T]} \Phi'(\tau) \geq 2P(\Phi(T) - \Phi(0)) = 2P\Phi(T) = 2C_\Delta^2\Phi(T) \end{aligned}$$

and we have a contradiction. Thus, the first assertion follows.

Let us prove the second assertion. First, let $t \geq 0$ and $s \in (1,2]$. Using the convexity of Φ, Δ_2-condition for Φ' and the Mean Value Theorem we obtain

$$\begin{aligned}\Phi(st)-\Phi(t) &\leq \Phi'(st)(s-1)t \leq \Phi'(2t)(s-1)t \leq C\Phi'(t)(s-1)t = C\Big(\Phi'(t)st-\Phi'(t)t\Big)\\ &\leq C\Big(\Phi'(st)st-\Phi'(t)t\Big)\ .\end{aligned} \tag{2.6}$$

This gives the assertion in the case $t_2 \leq 2t_1$. If $t_2 > 2t_1$, then we find $k \in \mathbb{N}$ and $\tau_1 < \tau_2 < \cdots < \tau_k$ so that $\tau_1 = t_1$, $\tau_k = t_2$ and $\tau_{i+1} \leq 2\tau_i$ for each $i = 1, \dots, k-1$. Next we apply (2.6) to each pair (τ_i, τ_{i+1}), $i = 1, \dots, k-1$ and we sum the estimates up. □

We often use the following estimate from [9, Lemma 2.2] together with generalized Hölder's inequality.

Lemma 2.2. *If a Young function Φ satisfies* (1.5) *and* (1.6)*, then $\Psi(\Phi') \lesssim \Phi$.*

Further, we need the Brezis-Lieb lemma from [5, Theorem 2 and Examples(b)].

Lemma 2.3. *Let $\{f_k\}$ be a sequence of ν-measurable functions on $\Omega \subset \mathbb{R}^n$ such that $f_k \to f$ a.e. in Ω. Let Φ be a Young function. Suppose that $f \in L^\Phi(\Omega, d\nu)$ and $||f_k||_{L^\Phi(\Omega,d\nu)} \leq C$. Then*

$$\int_\Omega \Big|\Phi(|f_k|)-\Phi(|f_k-f|)-\Phi(|f|)\Big|\,d\nu \overset{k\to\infty}{\to} 0\ .$$

Next, we need to be able to estimate the norm by the modular and vice versa. We use the following lemma from [9, Lemma 2.4] (the original statement in [9] and [11] concerns $\nu = \mathcal{L}_n$ only, but the proof is also valid for a general measure ν).

Lemma 2.4. *Let Φ be a Young function satisfying* (1.5) *and* (1.6)*. Then for every $\varepsilon > 0$ there is $\delta > 0$ such that*

$$||u||^{n+\varepsilon}_{L^\Phi(\Omega,d\nu)} \leq \int_\Omega \Phi(|u|)\,d\nu \leq ||u||^{n-\varepsilon}_{L^\Phi(\Omega,d\nu)} \qquad \textit{provided } ||u||_{L^\Phi(\Omega,d\nu)} < \delta\ .$$

Orlicz-Sobolev spaces. Let A be an nonempty open set in $\mathbb{R}^n$ and let Φ be a Young function satisfying (1.5). In this subsection we consider Orlicz spaces only with the Lebesgue measure. We define the Orlicz-Sobolev space $WL^\Phi(A)$ as the set

$$WL^\Phi(A) := \{u\colon u, |\nabla u| \in L^\Phi(A)\}$$

equipped with the norm

$$\|u\|_{WL^\Phi(A)} := \|u\|_{L^\Phi(A)} + \|\nabla u\|_{L^\Phi(A)}\ ,$$

where ∇u is the gradient of u and we use its Euclidean norm in $\mathbb{R}^n$.

We put $W_0L^\Phi(A)$ for the closure of $C_0^\infty(A)$ in $WL^\Phi(A)$.

Non-increasing rearrangement. The non-increasing rearrangement u^* of a measurable function u on Ω is

$$u^*(t) = \inf\Big\{s > 0 : \mathcal{L}_n(\{x \in \Omega : |u(x)| > s\}) \leq t\Big\}\ , \quad t > 0\ .$$

We also define the non-increasing radially symmetric rearrangement $u^\#$ by

$$u^\#(x) = u^*\Big(\frac{\omega_{n-1}}{n}|x|^n\Big) \qquad \text{for } x \in B(R)\ , \quad \mathcal{L}_n(B(R)) = \mathcal{L}_n(\Omega)\ .$$

For an introduction to these rearrangements see e.g. [28]. We need the following result concerning $u^\#$ (see [28, Theorem 1.C]).

Theorem 2.5. *Let Φ be a Young function and let u be a Lipschitz continuous function decaying at infinity ($\mathcal{L}_n(\{x \in \mathbb{R}^n : |u(x)| > t\}) < \infty$ for all $t > 0$). Then*

$$\int_{\mathbb{R}^n} \Phi(|\nabla u(x)|)\,dx \geq \int_{\mathbb{R}^n} \Phi(|\nabla u^\#(x)|)\,dx\ .$$

Tools from the Measure Theory. We make use of the following result from [8, Lemma 2.7].

Lemma 2.6. *Let $\Omega \subset \mathbb{R}^n$ be a bounded set, $\theta \in [0,1)$ and let $\{u_k\}$ be a sequence of functions from $L^1(\Omega)$ converging to $u \in L^1(\Omega)$ a.e. in Ω. Let $f : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ be a continuous function bounded on $\Omega \times [-t_0, t_0]$ for every $t_0 > 0$. Suppose that $f(x, u_k)|u_k|^\theta$ and $f(x,u)|u|^\theta$ belong to $L^1(\Omega)$ and*

$$\int_\Omega |f(x, u_k)\, u_k| \leq C \ .$$

Then $f(x, u_k)|u_k|^\theta \to f(x,u)|u|^\theta$ in $L^1(\Omega)$.

Next, we need a suitable estimate of a radially symmetric function $u \in L^\Phi(\mathbb{R}^n)$ on large spheres given by [8, Lemma 2.10].

Lemma 2.7. *Let Φ be a Young function satisfying* (1.5) *and* (1.6). *Let $u \in L^\Phi(\mathbb{R}^n)$ satisfy $||u||_{L^\Phi(\mathbb{R}^n)} \leq P$, for some $P > 0$. Suppose that u is non-negative, radially symmetric and non-increasing with respect to $|x|$. Then there are $R_s > 0$ and $C_s > 0$ independent of u such that*

$$u(x) \leq C_s P \frac{1}{|x|} \qquad \text{for } |x| > R_s \ .$$

We also need the Generalized Lebesgue Dominated Convergence Theorem (see [27, Exercise 5.4.13]).

Proposition 2.8. *Let $\{u_k\}$, $\{v_k\}$ be sequences of measurable functions on $\Omega \subset \mathbb{R}^n$ such that $|u_k| \leq v_k$ for all $k \in \mathbb{N}$. Let u and v be measurable functions on Ω such that $u_k \to u$ a.e. in Ω and $v_k \to v$ a.e. in Ω. Then*

$$\lim_{k\to\infty} \int_\Omega v_k = \int_\Omega v \qquad \Longrightarrow \qquad \lim_{k\to\infty} \int_\Omega u_k = \int_\Omega u \ .$$

Finally, we recall [8, Proposition 2.11].

Proposition 2.9. *Suppose that the Young function Φ satisfies* (1.5) *and* (1.6). *Let $V : \mathbb{R}^n \mapsto \mathbb{R}$ satisfy* (1.11) *and* (1.12). *Let $\{u_k\} \subset X(\mathbb{R}^n)$ be a bounded sequence. Then, passing to a subsequence we can guarantee that there is $u \in X(\mathbb{R}^n)$ such that*

$$\begin{aligned} u_k &\rightharpoonup u && \text{in } WL^\Phi(\mathbb{R}^n) \\ u_k &\to u && \text{in } L^\Phi(\mathbb{R}^n) \\ u_k &\to u && \text{in } L^r(\mathbb{R}^n) \text{ for every } r \in [n, \infty) \\ u_k &\to u && \text{a.e. in } \mathbb{R}^n \ . \end{aligned}$$

Tools from the Calculus of Variations. Our key instrument is the following version of the Mountain Pass Theorem by Ambrosetti and Rabinowitz [3].

Theorem 2.10. *Let X be a real Banach space and $J \in C^1(X, \mathbb{R})$. Suppose that there exist a neighborhood U of 0 in X and $\xi \in \mathbb{R}$ satisfying the following conditions:*
(i) *$J(0) < \xi$,*
(ii) *$J(u) \geq \xi$ on the boundary of U,*
(iii) *there is $w \notin U$ such that $J(w) < \xi$.*
Set

$$c = \inf_{\gamma\in\Gamma} \max_{u\in\gamma} J(u) \geq \xi \ ,$$

where $\Gamma = \{g \in C([0,1], X) : g(0) = 0, g(1) = w\}$. Then there is a sequence $\{u_k\} \subset X$ such that

$$J(u_k) \to c \quad \text{and} \quad J'(u_k) \to 0 \text{ in } X^* \ . \tag{2.7}$$

The sequence satisfying (2.7) is called the Palais-Smale sequence and the constant c is a Palais-Smale level. Notice that that this version slightly differs from often used version of the Mountain Pass Theorem which requires the Palais-Smale condition (the Palais-Smale sequence has a subsequence convergent in the norm) and asserts that there is a critical point $x_0 \in X$ satisfying $J(x_0) = c$. We use this version of the Mountain Pass Theorem, because we need a bit less from the Palais-Smale sequence than the convergence in the norm. Our approach is taken from [6]. See [6, page 459] for the discussion concerning the proof of Theorem 2.10.

The second weak solution to (1.10) is obtained by the following version of the Ekeland variational principle [17].

Theorem 2.11. *Let Y be a complete metric space and let $\Lambda : Y \mapsto \mathbb{R}$ be a C^1-functional which is bounded from bellow. Then for every $\delta > 0$ there is $u_\delta \in Y$ such that*

$$\Lambda(u_\delta) \leq \inf_{u\in Y} \Lambda(u) + \delta \qquad \textit{and} \qquad ||\Lambda'(u_\delta)||_{C(Y,\mathbb{R})} \leq \delta \ .$$

3. On generalized Moser-Trudinger inequality for unbounded domains

First, let us recall the generalized Moser-Trudinger inequality for embedding into exponential and multiple exponential spaces in the case of a bounded domain $\Omega \subset \mathbb{R}^n$.

Theorem 3.1. *Let $K \geq 0$, $\ell \in \mathbb{N}$, $n \geq 2$ and $\alpha < n-1$. Let Φ be a Young function satisfying* (1.5).
(i) *If $u \in W_0L^\Phi(\Omega)$, then*

$$\int_\Omega \exp_{[\ell]}\big(K|u(x)|^\gamma\big)\, dx < \infty \ .$$

(ii) *If $K < K_{\ell,n,\alpha}$ and $u \in W_0L^\Phi(\Omega)$ with $||\nabla u||_{L^\Phi(\Omega)} \leq 1$, then*

$$\int_\Omega \exp_{[\ell]}\big(K|u(x)|^\gamma\big)\, dx \leq C(\ell, n, \alpha, \Phi, \mathcal{L}_n(\Omega), K) \ .$$

(iii) *If $K > K_{\ell,n,\alpha}$, then*

$$\sup_{u\in W_0L^\Phi(\Omega), ||\nabla u||_{L^\Phi(\Omega)}\leq 1} \int_\Omega \exp_{[\ell]}\big(K|u(x)|^\gamma\big)\, dx = \infty \ .$$

(iv) *If $K = K_{\ell,n,\alpha}$ and Φ satisfies* (1.7) *and $u \in W_0L^\Phi(\Omega)$ with $||\nabla u||_{L^\Phi(\Omega)} \leq 1$, then*

$$\int_\Omega \exp_{[\ell]}\big(K|u(x)|^\gamma\big)\, dx \leq C(\ell, n, \alpha, \Phi, \mathcal{L}_n(\Omega)) \ .$$

(v) *If $K = K_{\ell,n,\alpha}$ and Φ satisfies* (1.6) *and* (1.8), *then*

$$\sup_{u\in W_0L^\Phi(\Omega), ||\nabla u||_{L^\Phi(\Omega)}\leq 1} \int_\Omega \exp_{[\ell]}\big(K|u(x)|^\gamma\big)\, dx = \infty \ .$$

The first assertion follows from [14, Remarks 3.11(iv)]. In the case $k \geq 2$, all four remaining assertions of Theorem 3.1 can be found in [12, Theorem 1.1, Theorem 1.2, Theorem 4.2 and Theorem 4.1]. In case $k = 1$, assertions (ii), (iii), (iv) follow from [20, Theorem 1.1, Theorem 1.2 and Theorem 4.2] while assertion (v) is given in [7, Example 5.1].

Remarks 3.2. (i) Theorem 3.1(iv) and (v) tells us that when $K = K_{\ell,n,\alpha}$, we generally do not know whether the supremum is finite. It depends on the choice of the Young function Φ (compare with the fact that all Young functions satisfying (1.5) give the same Orlicz-Sobolev space).

(ii) There is a large gap between conditions (1.7) and (1.8). In paper [13] it was possible to remove this gap in the case $\ell = 1$ and $\alpha = 0$ showing that the borderline Young function behaves like $t^n \log^{-1}(t)$.

(iii) From the proofs of [20, Theorem 1.1], [20, Theorem 4.2], [12, Theorem 1.1] and [12, Theorem 4.2] one can easily see that for any fixed $\tilde{C} \geq 0$, we have versions of Theorem 3.1(ii) and (iv) with

$$\int_\Omega \exp_{[\ell]}(K(\tilde{C} + |u|)^\gamma|) \leq C(\tilde{C}, \ell, n, \alpha, \mathcal{L}_n(\Omega), K) \ .$$

Moreover, from these proofs we also see that the assumption $||\nabla u||_{L^\Phi(\Omega)} \leq 1$ in Theorem 3.1(ii) and (iv) can be replaced by

$$||\nabla u||_{L^\Phi(\tilde{\Omega})} \leq 1 \ , \qquad \text{where } \tilde{\Omega} = \{x \in \Omega : |\nabla u| > G\} \ ,$$

with $G > 0$ being fixed arbitrarily large number.

A version of Theorem 3.1 for unbounded domains is given in [8].

Theorem 3.3. *Let $\ell \in \mathbb{N}$, $n \geq 2$, $\alpha < n-1$ and let $\Omega \subset \mathbb{R}^n$ be a domain. Suppose that the Young function $\Phi : [0,\infty) \mapsto [0,\infty)$ satisfies (1.5) and (1.6). Let $u \in W_0L^\Phi(\Omega)$.*
(i) *If $K \geq 0$ then*

$$\int_\Omega \exp_{[\ell]}(K|u|^\gamma) - S_{\ell,n,\alpha}(K|u|^\gamma) < \infty \ .$$

(ii) *If $0 \leq K < K_{\ell,n,\alpha}$, $||\nabla u||_{L^\Phi(\Omega)} \leq 1$ and $||u||_{L^\Phi(\Omega)} \leq P$ for some $P \geq 0$, then*

$$\int_\Omega \exp_{[\ell]}(K|u|^\gamma) - S_{\ell,n,\alpha}(K|u|^\gamma) \leq C(\ell, n, \alpha, \Phi, P, K) \ .$$

(iii) *If $K > K_{\ell,n,\alpha}$, then there is a sequence $\{u_k\} \subset W_0L^\Phi(\Omega)$ such that $||\nabla u_k||_{L^\Phi(\Omega)} \leq 1$, $||u_k||_{L^\Phi(\Omega)} \to 0$ and*

$$\int_\Omega \exp_{[\ell]}(K|u_k|^\gamma) - S_{\ell,n,\alpha}(K|u_k|^\gamma) \overset{k\to\infty}{\to} \infty \ .$$

(iv) *If $K = K_{\ell,n,\alpha}$, Φ satisfies (1.7), $||\nabla u||_{L^\Phi(\Omega)} \leq 1$ and $||u||_{L^\Phi(\Omega)} \leq P$ for some $P \geq 0$, then*

$$\int_\Omega \exp_{[\ell]}(K|u|^\gamma) - S_{\ell,n,\alpha}(K|u|^\gamma) \leq C(\ell, n, \alpha, \Phi, P) \ .$$

(v) *If $K = K_{\ell,n,\alpha}$ and Φ satisfies (1.8), then there is a sequence $\{u_k\} \subset W_0L^\Phi(\Omega)$ such that $||\nabla u_k||_{L^\Phi(\Omega)} \leq 1$, $||u_k||_{L^\Phi(\Omega)} \to 0$ and*

$$\int_\Omega \exp_{[\ell]}(K|u_k|^\gamma) - S_{\ell,n,\alpha}(K|u_k|^\gamma) \overset{k\to\infty}{\to} \infty \ .$$

4. An estimate concerning the Concentration-Compactness Principle for generalized Moser-Trudinger inequality

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. The Concentration-Compactness Alternative (for the original version concerning $W_0^{1,n}(\Omega)$ see [22], for the Orlicz-Sobolev case see [10] and [7]) states that each bounded sequence in $W_0L^\Phi(\Omega)$ can be decomposed into subsequences which either concentrate around a point $x_0 \in \bar{\Omega}$ or Theorem 3.1(ii) holds for such a subsequence with the constant K slightly larger than $K_{\ell,n,\alpha}$. The following proposition gives us the estimate of the number K in the second case under the additional assumption that there is $u \in W_0L^\Phi(\Omega)$ such that $\nabla u_k \to \nabla u$ a.e. in Ω.

Proposition 4.1. *Let $\ell \in \mathbb{N}$, $n \geq 2$, $\alpha < n-1$ and let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let Φ be a Young function satisfying (1.5) and (1.6). Let $\{u_k\} \subset W_0L^\Phi(\Omega)$ be a sequence satisfying*

$$\text{there is finite } \lim_{k\to\infty} \int_{\mathbb{R}^n} \Phi(|\nabla u_k|) \qquad \text{and} \qquad \nabla u_k \to \nabla u \quad \text{a.e. in } \Omega$$

for some $u \in W_0L^\Phi(\Omega)$. Then for every

$$p < P := \Big(\frac{1}{\lim_{k\to\infty} \int_{\mathbb{R}^n} \Phi(|\nabla u_k|) - \int_\Omega \Phi(|\nabla u|)} \Big)^{\frac{\gamma}{n}}$$

(where we define $P = \infty$ if the denominator is zero) we have

$$\int_\Omega \exp_{[\ell]}(K_{\ell,n,\alpha} p |u_k|^\gamma) \leq C \qquad \text{where } C \text{ is independent of } k \ .$$

Let us note that an estimate corresponding to the one from Proposition 4.1 is also obtained in [22] (after some transformation, it was obtained that it can be supposed that $\nabla u_k \to \nabla u$ a.e. in Ω). This estimate is not contained in [10] and [7]. Since the gradients of Palais-Smale sequences converge a.e. after passing to a subsequence (see Lemma 7.3 bellow), we do not mind the additional assumption concerning the a.e. convergence of the gradients in Proposition 4.1.

Before we prove Proposition 4.1 we need to do some preliminary work.

Lemma 4.2. *Let $\delta > 0$, $0 < C_1 < C_2$ and let $\tilde{\Omega} \subset \mathbb{R}^n$ be a bounded domain. Then there is $G = G(C_1, C_2, \delta) > 0$ with the following property:*
If $v \in L^\Phi(\tilde{\Omega})$ satisfies $C_1 < ||v||_{L^\Phi(\tilde{\Omega})} < C_2$ and $|v| \geq G$ on $\tilde{\Omega}$, then

$$||v||^n_{L^\Phi(\tilde{\Omega})} \leq (1+\delta)^3 \int_{\tilde{\Omega}} \Phi(|v|) \ .$$

Proof. Let us write $\lambda = ||v||_{L^\Phi(\tilde{\Omega})}$ to simplify our notation. If G is large enough, from (1.5) and (2.4) we obtain

$$1 = \int_\Omega \Phi\Big(\frac{|v|}{\lambda}\Big) \leq (1+\delta) \int_\Omega \Big(\frac{|v|}{\lambda}\Big)^n \Big(\prod_{j=1}^{\ell-1} \log^{n-1}_{[j]}\Big(\frac{|v|}{\lambda}\Big)\Big) \log^\alpha_{[\ell]}\Big(\frac{|v|}{\lambda}\Big) \ .$$

Multiplying both sides by λ^n and using (1.5) again (for G large enough) we arrive to

$$\begin{aligned} \lambda^n &\leq (1+\delta) \int_\Omega |v|^n \Big(\prod_{j=1}^{\ell-1} \log^{n-1}_{[j]}\Big(\frac{|v|}{\lambda}\Big)\Big) \log^\alpha_{[\ell]}\Big(\frac{|v|}{\lambda}\Big) \\ &\leq (1+\delta)^2 \int_\Omega |v|^n \Big(\prod_{j=1}^{\ell-1} \log^{n-1}_{[j]}(|v|)\Big) \log^\alpha_{[\ell]}(|v|) \leq (1+\delta)^3 \int_\Omega \Phi(|v|) \ . \end{aligned}$$

Thus, we are done. □

Proof of Proposition 4.1. Fix $p < P$. Let us find $\delta \in (0,1)$ such that

$$(1+\delta)^{\gamma + \frac{4\gamma}{n}} p < (1-\delta) P \ . \tag{4.1}$$

Next, let us define the set $M = \{x \in \Omega : \delta |u_k - u| \geq |u|\}$. Then we can write

$$|u_k|^\gamma \leq (|u_k - u| + |u|)^\gamma \leq (1+\delta)^\gamma |u_k - u|^\gamma \chi_M + \Big(1 + \frac{1}{\delta}\Big)^\gamma |u|^\gamma \chi_{\Omega \setminus M} \ .$$

Therefore we have

$$\begin{aligned} \int_\Omega \exp_{[\ell]}(K_{\ell,n,\alpha} p |u_k|^\gamma) &\leq \int_\Omega \exp_{[\ell]}(K_{\ell,n,\alpha} p\, (1+\delta)^\gamma |u_k - u|^\gamma) + \int_\Omega \exp_{[\ell]}\Big(K_{\ell,n,\alpha} p \, \Big(1 + \frac{1}{\delta}\Big)^\gamma |u|^\gamma\Big) \\ &= I_k + J \ . \end{aligned}$$

By Theorem 3.1(i) we have $J \leq C$. In the rest of the proof, our aim is to obtain an uniform estimate of I_k. The assumptions of the proposition enable us to use the Brezis-Lieb lemma 2.3 and we obtain

$$\int_\Omega |\Phi(|\nabla(u_k - u)|) - \Phi(|\nabla u_k|) + \Phi(|\nabla u|)| \overset{k\to\infty}{\to} 0 \ . \tag{4.2}$$

Now, we distinguish two cases. First, suppose that $\lim_{k\to\infty} \int_\Omega \Phi(|\nabla u_k|) = \int_\Omega \Phi(|\nabla u|)$. In this case, using (2.5) and (4.2) we arrive to

$$||\nabla(u_k - u)||_{L^\Phi(\Omega)} \to 0 \ .$$

Now the uniform estimate of I_k easily follows. Indeed, for k large enough so that

$$p\,(1+\delta)^\gamma ||\nabla(u_k - u)||^\gamma_{L^\Phi(\Omega)} < 1 - \delta$$

we can apply Theorem 3.1(ii) to obtain the uniform estimate of I_k, while for k small (finite number of indexes) we use Theorem 3.1(i) to show that the integral I_k is finite. Thus, all the integrals can be estimated by the same constant.

In the second case we have $\int_\Omega \Phi(|\nabla u|) < \lim_{k\to\infty} \int_\Omega \Phi(|\nabla u_k|)$ (notice that by Fatou's lemma there is no third case). Since the boundedness of modulars implies the boundedness of norms, we see that there is $C_2 > 0$ such that

$$||\nabla(u_k - u)||_{L^\Phi(\Omega)} \leq ||\nabla u_k||_{L^\Phi(\Omega)} + ||\nabla u||_{L^\Phi(\Omega)} \leq C_2 \qquad \text{for every } k \in \mathbb{N} \ . \tag{4.3}$$

We find the number $C_1 \in (0, C_2)$ such that

$$p\,(1+\delta)^\gamma C_1^\gamma \leq 1 - \delta \ . \tag{4.4}$$

For these C_1, C_2 and δ, there is $G>0$ so that the assertion of Lemma 4.2 is satisfied. Therefore we define for each $k\in\mathbb{N}$

$$\lambda_{G,k}=||\nabla(u_k-u)||_{L^\Phi(\{x\in\Omega:|\nabla(u_k-u)|\geq G\})}$$

and we decompose our indexes $k\in\mathbb{N}$ into two sets

$$\Lambda_1=\{k\in\mathbb{N}:\lambda_{G,k}\leq C_1\}\qquad\text{and}\qquad\Lambda_2=\{k\in\mathbb{N}:\lambda_{G,k}>C_1\}\ .$$

Let us find the uniform bound of I_k for $k\in\Lambda_1$. We set $K=(1-\delta)K_{\ell,n,\alpha}$. Hence by (4.4) for all $k\in\Lambda_1$ we have

$$K_{\ell,n,\alpha}p\,(1+\delta)^\gamma\lambda_{G,k}^\gamma\leq K<K_{\ell,n,\alpha}$$

and thus we can use the version of Theorem 3.1(ii) given by Remark 3.2(iii) to obtain an uniform estimate of I_k for all $k\in\Lambda_1$.

Now, we would like to deal with the set Λ_2. By (4.2) for $k\in\Lambda_2$ large enough (in view of Theorem 3.1(i) it is enough to care about large k only) we have

$$\int_\Omega\Phi(|\nabla(u_k-u)|)\leq(1+\delta)\Big(\lim_{k\to\infty}\int_\Omega\Phi(|\nabla u_k|)-\int_\Omega\Phi(|\nabla u|)\Big)\ . \tag{4.5}$$

Finally, using (4.1), (4.5) and Lemma 4.2 we arrive to

$$\begin{aligned}p(1+\delta)^\gamma\lambda_{G,k}^\gamma&\leq p(1+\delta)^{\gamma+\frac{3\gamma}{n}}\Big(\int_{|\nabla(u_k-u)|\geq G\}}\Phi(|\nabla(u_k-u)|)\Big)^{\frac{\gamma}{n}}\\&\leq p(1+\delta)^{\gamma+\frac{3\gamma}{n}}\Big(\int_\Omega\Phi(|\nabla(u_k-u)|)\Big)^{\frac{\gamma}{n}}\\&\leq p(1+\delta)^{\gamma+\frac{4\gamma}{n}}\Big(\lim_{k\to\infty}\int_\Omega\Phi(|\nabla u_k|)-\int_\Omega\Phi(|\nabla u|)\Big)^{\frac{\gamma}{n}}\\&=p(1+\delta)^{\gamma+\frac{4\gamma}{n}}\frac{1}{P}\leq1-\delta\ .\end{aligned}$$

Therefore we can use the version of Theorem 3.1(ii) given by Remark 3.2(iii) also for $k\in\Lambda_2$ large enough to conclude the proof. ☐

If Ω is not bounded, then there is no Concentration-Compactness Alternative (there can also occur "Concentration up to shifts"). However, there is still a version of Proposition 4.1.

Proposition 4.3. *Let $\ell\in\mathbb{N}$, $n\geq2$, $\alpha<n-1$ and $L>0$. Let Φ be a Young function satisfying* (1.5) *and* (1.6). *Let $\{u_k\}\subset WL^\Phi(\mathbb{R}^n)$ be a sequence satisfying*

$$\textit{there is finite}\ \lim_{k\to\infty}\int_{\mathbb{R}^n}\Phi(|\nabla u_k|)\ ,\qquad ||u_k||_{L^\Phi(\mathbb{R}^n)}\leq L\qquad\textit{and}\qquad\nabla u_k\to\nabla u\quad\textit{a.e. in }\mathbb{R}^n$$

for some $u\in WL^\Phi(\mathbb{R}^n)$. Then for every

$$p<P:=\Big(\frac{1}{\lim_{k\to\infty}\int_{\mathbb{R}^n}\Phi(|\nabla u_k|)-\int_\Omega\Phi(|\nabla u|)}\Big)^{\frac{\gamma}{n}}$$

(where we define $P=\infty$ if the denominator is zero) we have

$$\int_{\mathbb{R}^n}\exp_{[\ell]}(K_{\ell,n,\alpha}p|u_k|^\gamma)-S_{\ell,n,\alpha}(K_{\ell,n,\alpha}p|u_k|^\gamma)\leq C\qquad\textit{where } C \textit{ is independent of } k\ .$$

Sketch of proof. The same way as in the proof of Proposition 4.1 we define $\delta>0$ by (4.1) and then we obtain the estimate

$$\begin{aligned}&\int_{\mathbb{R}^n}\exp_{[\ell]}(K_{\ell,n,\alpha}p|u_k|^\gamma)-S_{\ell,n,\alpha}(K_{\ell,n,\alpha}p|u_k|^\gamma)\\&\qquad\leq\int_{\mathbb{R}^n}\exp_{[\ell]}(K_{\ell,n,\alpha}p\,(1+\delta)^\gamma|u_k-u|^\gamma)-S_{\ell,n,\alpha}(K_{\ell,n,\alpha}p\,(1+\delta)^\gamma|u_k-u|^\gamma)\\&\qquad\quad+\int_{\mathbb{R}^n}\exp_{[\ell]}\Big(K_{\ell,n,\alpha}p\,\Big(1+\frac{1}{\delta}\Big)^\gamma|u|^\gamma\Big)-S_{\ell,n,\alpha}\Big(K_{\ell,n,\alpha}p\,\Big(1+\frac{1}{\delta}\Big)^\gamma|u|^\gamma\Big)\\&\qquad=I_k+J\ .\end{aligned}$$

The integral J is finite by Theorem 3.3(i). It remains to deal with I_k, $k \in \mathbb{N}$. In view of Theorem 2.5 and the density of the $C_0^\infty(\mathbb{R}^n)$-functions in $WL^\Phi(\mathbb{R}^n)$, we can apply the Schwarz symmetrization to the functions $v_k = u_k - u$, $k \in \mathbb{N}$, without disturbing the estimates of the modulars following from the Brezis-Lieb lemma. Hence we can suppose that v_k, $k \in \mathbb{N}$, are radially symmetric, non-negative and non-increasing with respect to $|x|$.

Next we apply some ideas from the proof of Theorem 3.3(ii) ([8, Proof of Theorem 1.3(ii)]) using Lemma 2.7. We find the radius $R > R_s$ (R_s comes from Lemma 2.7) so large that

$$\int_{\mathbb{R}^n \setminus B(R)} \exp_{[\ell]}(K_{\ell,n,\alpha} p\,(1+\delta)^\gamma |w|^\gamma) - S_{\ell,n,\alpha}(K_{\ell,n,\alpha} p\,(1+\delta)^\gamma |w|^\gamma) < \infty \ ,$$

where $w(x) := \frac{C}{|x|} \geq v_k$, $k \in \mathbb{N}$ (for more details see [8, Proof of Theorem 1.3(ii)]).

On $B(R)$ we write $v_k = (v_k - V_k) + V_k$, where the constant V_k is the value of v_k on $S(R)$ (which is estimated by $\frac{C}{R}$). We deal with the functions $v_k - V_k \in W_0L^\Phi(B(R))$ the same way as in the proof of Proposition 4.1, while V_k estimated by the same number are harmless additive constants (see Remarks 3.2(iii)). □

5. Functional J_μ is C^1

Proposition 5.1. *For the functional J_μ defined by* (1.20) *we have $J_\mu \in C^1(X(\mathbb{R}^n), \mathbb{R})$ and its Fréchet derivative is* (1.21).

Sketch of proof. We use the approach by [4, Proof of Theorem A.V]. In particular, we show that the functional J_μ has the Gateaux derivative everywhere in $X(\mathbb{R}^n)$ and then we show that $u \mapsto J'_\mu(u)$ is continuous. First, it is convenient for us to split J into four functionals

$$J_1(u) = \int_{\mathbb{R}^n} \Phi(|\nabla u|) \ , \quad J_2(u) = \int_{\mathbb{R}^n} V(x)\Phi(|u|) \ , \quad J_3(u) = \int_{\mathbb{R}^n} F(x,u) \ , \quad J_4(u) = \mu \int_{\mathbb{R}^n} h(x)u \ .$$

The functionals J_1, J_2 and J_3 are handled in [8, Proof of Proposition 4.1]. It remains to deal with the functional J_4. We need to show that

$$\langle J'_4(u), \varphi \rangle = \mu \int_{\mathbb{R}^n} h(x)\varphi \ , \qquad u, \varphi \in X(\mathbb{R}^n)$$

and $u \mapsto J'_4(u)$ is continuous. But this is trivial. □

6. The geometry of the functional J_μ

In this section we mainly check that our functional J_μ has the Mountain Pass Geometry (i.e. assumptions (i), (ii) and (iii) from Theorem 2.10 are satisfied).

First, we observe that it follows from (1.13) and (1.15) that

$$F(x,t) \geq C|t|^{\frac{1}{A}}, \quad t \in \mathbb{R} \ . \tag{6.1}$$

Now, we can start to check the assertions of the Mountain Pass Theorem.

Lemma 6.1. *If $u \in X(\mathbb{R}^n)$ has a compact support, $u \geq 0$ and $u \neq 0$, then*

$$J_\mu(tu) \stackrel{t\to\infty}{\to} -\infty \ .$$

Moreover, this convergence is uniform with respect to μ taken from a bounded set.

Proof. Since $u \neq 0$ and $u \geq 0$, there is $\tau > 0$ such that

$$\mathcal{L}_n(\{u \geq \tau\}) \geq \tau \ .$$

Fix $q \in (n, \frac{1}{A})$. Using (1.5), (6.1), the compactness of $\operatorname{supp} u$ and the continuity of $V(x)$ we obtain

$$\begin{aligned} J(tu) &= \int_{\operatorname{supp} u} \Phi(t|\nabla u|) + V(x)\Phi(t|u|) - F(x,tu) - \mu t h(x)u\,dx \\ &\leq C + t^q \int_{\mathbb{R}^n} |\nabla u|^q + C|u|^q\,dx + \mu t \int_{\mathbb{R}^n} |h(x)u|\,dx - \int_{\{u \geq \tau\}} C(t\tau)^{\frac{1}{A}})\,dx \\ &\leq C + Ct^q + C\mu t - Ct^{\frac{1}{A}} \stackrel{k\to\infty}{\to} -\infty \ . \end{aligned}$$

□

Lemma 6.2. *There is $\mu_0 > 0$ such that for every $\mu \in (0, \mu_0)$ there are $\varrho_\mu > 0$ and $\xi_\mu > 0$ with the following property: If $u \in X(\mathbb{R}^n)$ with $||u||_{X(\mathbb{R}^n)} = \varrho_\mu$, then $J_\mu(u) \geq \xi_\mu$.*
Moreover, $\varrho_\mu > 0$ can be chosen so that $\varrho_\mu \to 0_+$ as $\mu \to 0_+$ and

$$c_0 = c_0(\mu) := \inf_{||u||_{X(\mathbb{R}^n)} < \varrho_\mu} J_\mu(u) \geq C(\mu, \varrho_\mu) \;, \qquad \text{where } C(\mu,\varrho_\mu) \overset{\mu\to 0_+}{\to} 0_- \;.$$

Proof. Fix $q > n$. By assumptions (1.14), (1.16) and (1.17) we can find $\eta > 0$ so that

$$F(x,t) \leq (1-2\eta) C_S \Phi(|t|) + C\Big(\exp_{[\ell]}(b|t|^\gamma) - S_{\ell,n,\alpha}(b|t|^\gamma)\Big)|t|^q = F_1(t) + F_2(t) \;.$$

By (1.18) we obtain

$$\int_{\mathbb{R}^n} F_1(u) = (1-2\eta)C_S \int_{\mathbb{R}^n} \Phi(|u|) \leq (1-2\eta)\int_{\mathbb{R}^n} \Phi(|\nabla u|) + V(x)\Phi(|u|) \;. \tag{6.2}$$

Next, fix $p > 1$. If ϱ is so small that $bp\varrho^\gamma < K_{\ell,n,\alpha}$, then from Hölder's inequality, Theorem 3.3(ii) (with $P = \varrho$), (2.1) and from the fact that $X(\mathbb{R}^n)$ is continuously embedded into $L^r(\mathbb{R}^n)$, for every $r \in [n, \infty)$, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^n} F_2(u) &= C\int_{\mathbb{R}^n} \Big(\exp_{[\ell]}(b|u|^\gamma) - S_{\ell,n,\alpha}(b|u|^\gamma)\Big)|u|^q \\
&\leq C\Big(\int_{\mathbb{R}^n} \exp_\ell\Big(bp\varrho^\gamma\Big(\frac{|u|}{||u||_{X(\mathbb{R}^n)}}\Big)^\gamma\Big) - S_{\ell,n,\alpha}\Big(bp\varrho^\gamma\Big(\frac{|u|}{||u||_{X(\mathbb{R}^n)}}\Big)^\gamma\Big)\Big)^{\frac{1}{p}}\Big(\int_{\mathbb{R}^n}|u|^{qp'}\Big)^{\frac{1}{p'}} \\
&\leq C||u||^q_{L^{qp'}}(\mathbb{R}^n) \leq C||u||^q_{X(\mathbb{R}^n)} \leq C||\nabla u||^q_{L^\Phi(\mathbb{R}^n)} + C||u||^q_{L^\Phi(\mathbb{R}^n, V(x)dx)} \;.
\end{aligned}$$

Hence for $\varrho > 0$ small enough Lemma 2.4 with $\varepsilon \in (0, q-n)$ gives

$$\begin{aligned}
\int_{\mathbb{R}^n} F_2(u) &\leq C||\nabla u||^{q-n-\varepsilon}_{L^\Phi(\mathbb{R}^n)}||\nabla u||^{n+\varepsilon}_{L^\Phi(\mathbb{R}^n)} + C||u||^{q-n-\varepsilon}_{L^\Phi(\mathbb{R}^n,V(x)dx)}||u||^{n+\varepsilon}_{L^\Phi(\mathbb{R}^n,V(x)dx)} \\
&\leq \eta\int_{\mathbb{R}^n}\Phi(|\nabla u|) + V(x)\Phi(|u|) \;.
\end{aligned} \tag{6.3}$$

Thus, we obtain from (6.2) and (6.3) and generalized Hölder's inequality

$$\begin{aligned}
J_\mu(u) &= \int_{\mathbb{R}^n}\Phi(|\nabla u|) + V(x)\Phi(|u|) - F(x,u) - \mu h(x)u \\
&\geq \eta\int_{\mathbb{R}^n}\Phi(|\nabla u|) + V(x)\Phi(|u|) - 2\mu||h||_{L^\Psi(\mathbb{R}^n)}||u||_{L^\Phi(\mathbb{R}^n)} \;.
\end{aligned}$$

Next $||u||_{X(\mathbb{R}^n)} = \varrho$ implies that $||\nabla u||_{L^\Phi(\mathbb{R}^n)} \geq \frac{\varrho}{2}$ or $||\nabla u||_{L^\Phi(\mathbb{R}^n,V(x)dx)} \geq \frac{\varrho}{2}$. Hence Lemma 2.4 with $\varepsilon = 1$ and $||u||_{L^\Phi(\mathbb{R}^n)} \leq C||u||_{X(\mathbb{R}^n)}$ imply for all $\varrho > 0$ small enough

$$J_\mu(u) \geq \eta\Big(\frac{\varrho}{2}\Big)^{n+1} - C\mu\varrho$$

and the results follow. □

Lemma 6.3. *There is $v \in X(\mathbb{R}^n)$ with $||v||_{X(\mathbb{R}^n)} = 1$ such that for every $\mu > 0$ there is $t_\mu > 0$ with the following property: For every $t \in (0, t_\mu)$ we have $J_\mu(tv) < 0$.*

In particular

$$\inf_{||u||_{X(\mathbb{R}^n)} \leq t_\mu} J_\mu(u) < 0 \qquad \text{and thus} \qquad c_0 < 0 \;.$$

Proof. Since h is continuous and nontrivial, we obtain an open set $G \subset \mathbb{R}^n$ such that h is bounded away from zero on G. We can easily construct a non-trivial $X(\mathbb{R}^n)$-function $\tilde{v}$ supported on G with the same sign as h has on G. Further we can suppose that $\tilde{v}$ and $\nabla\tilde{v}$ are bounded. Normalizing suitably, we obtain $v \in X(\mathbb{R}^n)$ such that $||v||_{X(\mathbb{R}^n)} = 1$ and

$$\int_{\mathbb{R}^n} hv = \int_G hv = C_1 > 0 \;.$$

Finally, as F is non-negative, using above construction and (1.6) we obtain for $t > 0$ small enough

$$\begin{aligned} J_\mu(tv) &= \int_{\mathbb{R}^n} \Phi\big(t|\nabla v|\big) + V(x)\Phi(t|v|) - F(x,tv) - \mu t h(x) v \, dx \\ &\le Ct^n \int_{\mathbb{R}^n} |\nabla v|^n + V(x)|v|^n \, dx - \mu C_1 t = Ct^n - C_1 \mu t \end{aligned}$$

and we conclude the proof easily. □

Finally, we need a suitable estimate of the Palais-Smale level.

Lemma 6.4. *If $\mu_0 > 0$ is small enough, then there is $w \in X(\mathbb{R}^n)$ such that*

$$J_\mu(\theta w) < c_0 + \Big(\frac{K_{\ell,n,\alpha}}{b}\Big)^{\frac{n}{\gamma}} \qquad \text{for every } \theta \in [0,\infty) \text{ and } \mu \in (0,\mu_0) \ .$$

Proof. By [8, Lemma 5.3], there is $\varepsilon > 0$ and a compactly supported non-negative function $w \in X(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} \Phi\big(|\nabla tw|\big) + V(x)\Phi\big(|tw|\big) - F(x,\theta w) \le \Big(\frac{K_{\ell,n,\alpha}}{b}\Big)^{\frac{n}{\gamma}} - 2\varepsilon \ .$$

Hence, since $\int_{\mathbb{R}^n} |hw| \le C$, using Lemma 6.1 and (1.20) we obtain for $\mu > 0$ small enough

$$\max_{\theta \ge 0} J_\mu(\theta) \le \Big(\frac{K_{\ell,n,\alpha}}{b}\Big)^{\frac{n}{\gamma}} - \varepsilon \ .$$

Moreover, by Lemma 6.2 we can guarantee that $c_0 > -\varepsilon$ providing $\mu > 0$ is small enough, and the result follows. □

7. Properties of the Palais-Smale sequence

In this section we study the properties of a Palais-Smale sequence corresponding to the functional J_μ. Our main aim is to show that it contains a subsequence with the gradients converging a.e. in $\mathbb{R}^n$ (see Lemma 7.3) and that the limit (in the sense of (7.9)) is a weak solution to the problem (1.10) (see Lemma 7.4).

Let $\{u_k\}$ be a Palais-Smale sequence from $X(\mathbb{R}^n)$, that is by (2.7),

$$J(u_k) = \int_{\mathbb{R}^n} \Phi\big(|\nabla u_k|\big) + V(x)\Phi\big(|u_k|\big) - F(x,u_k) - \mu h(x) u_k \overset{k\to\infty}{\to} c \ , \tag{7.1}$$

and by (1.21) there are $\varepsilon_k \to 0$ such that for every $v \in X(\mathbb{R}^n)$ we have

$$|\langle J'(u_k), v\rangle| = \Big|\int_{\mathbb{R}^n} \Phi'\big(|\nabla u_k|\big)\frac{\nabla u_k}{|\nabla u_k|} \cdot \nabla v + V(x)\Phi'\big(|u_k|\big)\frac{u_k}{|u_k|} v - f(x,u_k)v - \mu h(x) v\Big| \le \varepsilon_k \|v\|_{X(\mathbb{R}^n)} \ . \tag{7.2}$$

Lemma 7.1. *There is a constant $C > 0$ independent of $k \in \mathbb{N}$ such that*

$$\|\nabla u_k\|_{L^\Phi(\mathbb{R}^n)} \le C \ , \qquad \int_{\mathbb{R}^n} \Phi\big(|\nabla u_k|\big) \le C \ , \tag{7.3}$$

$$\|u_k\|_{L^\Phi(\mathbb{R}^n, V(x)dx)} \le C \ , \qquad \int_{\mathbb{R}^n} V(x)\Phi\big(|u_k|\big) \le C \ , \tag{7.4}$$

and

$$0 \le \int_{\mathbb{R}^n} f(x,u_k) u_k \le C \ . \tag{7.5}$$

Proof. Using (1.5) and (1.6) it can be easily shown that there is $\lambda_0 > 0$ large enough so that

$$\Phi(\lambda t) \ge \lambda^{n-\frac{1}{2}} \Phi(t) \qquad \text{for every } t \ge 0, \lambda \ge \lambda_0 \ . \tag{7.6}$$

We obtain from (1.15), (7.1), (7.2) with $v = u_k$ and (1.9)

(7.7)

$$\begin{aligned}
&\int_{\mathbb{R}^n} \Phi(|\nabla u_k|) + V(x)\Phi(|u_k|) \\
&\quad \le C + \int_{\mathbb{R}^n} F(x,u_k) + \int_{\mathbb{R}^n} \mu h(x) u_k \le C + A\int_{\mathbb{R}^n} f(x,u_k)u_k + \int_{\mathbb{R}^n} \mu h(x) u_k \\
&\quad \le C + A\Big(\int_{\mathbb{R}^n} \Phi'(|\nabla u_k|)|\nabla u_k| + V(x)\Phi'(|u_k|)|u_k| - \mu h(x)u_k + \varepsilon_k ||u_k||_{X(\mathbb{R}^n)}\Big) + \int_{\mathbb{R}^n} \mu h(x) u_k \\
&\quad \le C + Ac_\Phi \int_{\mathbb{R}^n} \Phi(|\nabla u_k|) + V(x)\Phi(|u_k|) + A\varepsilon_k ||u_k||_{X(\mathbb{R}^n)} + (1-A)\int_{\mathbb{R}^n} \mu h(x) u_k \ .
\end{aligned}$$

Next, as $h \in L^\Psi(\mathbb{R}^n)$ and $||u_k||_{L^\Phi(\mathbb{R}^n)} \le C||u_k||_{X(\mathbb{R}^n)}$, the generalized Hölder's inequality gives

$$\int_{\mathbb{R}^n} h(x)u_k \le 2||h||_{L^\Psi(\mathbb{R}^n)} ||u_k||_{L^\Phi(\mathbb{R}^n)} \le C||u_k||_{X(\mathbb{R}^n)} \ .$$

Thus, $Ac_\Phi < 1$ and (7.7) imply

$$\int_{\mathbb{R}^n} \Phi(|\nabla u_k|) + V(x)\Phi(|u_k|) \le C + C||u_k||_{X(\mathbb{R}^n)} \ . \tag{7.8}$$

Now, from the definition of the norm on $X(\mathbb{R}^n)$ and (2.4) together with (7.6) we can easily see that all terms in (7.8) have to be bounded. This is (7.3) and (7.4).

The upper estimate in (7.5) now follows from (7.2) (with $v = u_k$, see also (1.9)). The integral in (7.5) is non-negative by (1.13). □

By (7.3), (7.4) and Proposition 2.9 there is a function $u \in X(\mathbb{R}^n)$ (passing to a suitable subsequence of $\{u_k\}$ if necessary) such that

$$\begin{aligned}
&u_k \rightharpoonup u && \text{in } WL^\Phi(\mathbb{R}^n) \ , \\
&u_k \to u && \text{in } L^\Phi(\mathbb{R}^n) \ , \\
&u_k \to u && \text{in } L^r(\mathbb{R}^n) \text{ for every } r \in [n,\infty) \ , \\
&u_k \to u && \text{a.e. in } \mathbb{R}^n \ .
\end{aligned} \tag{7.9}$$

The function u has the following property by [8, Proposition 6.4].

Lemma 7.2. *The function $u \in X(\mathbb{R}^n)$ given by* (7.9) *satisfies*

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} F(x,u_k) = \int_{\mathbb{R}^n} F(x,u) \ .$$

Lemma 7.3. *Passing to a subsequence we have*

$$\nabla u_k \to \nabla u \quad \textit{a.e. on } \mathbb{R}^n \ . \tag{7.10}$$

Sketch of proof. The proof is almost the same as the proof of [8, Lemma 6.2]. The only difference (corresponding to the fact that paper [8] deals with the case $h \equiv 0$) is the following. When proving that

$$\int_{\mathbb{R}^n} \Big(\Phi'(|\nabla u_k|)\frac{\nabla u_k}{|\nabla u_k|} - \Phi'(|\nabla u|)\frac{\nabla u}{|\nabla u|}\Big) \cdot (\nabla u_k - \nabla u)\psi_\varepsilon \stackrel{k\to\infty}{\to} 0 \ ,$$

there occurs the additional term

$$I_7 = \mu \int_{\mathbb{R}^n} \psi_\varepsilon h(x)(u_k - u).$$

We need to show that $I_7 \to 0$. But, this plainly follows from $h \in L^\Psi(\mathbb{R}^n)$ and (7.9). □

Lemma 7.4. *The function $u \in X(\mathbb{R}^n)$ given by* (7.9) *is a weak solution to problem* (1.10), *i.e. we have* (1.22).

Sketch of proof. We can use the proof of [8, Lemma 6.2] with a minor modification. Indeed, the only difference is that in the second step we need to show in addition that for $\psi_k \to v$ in $X(\mathbb{R}^n)$ we have

$$\mu \int_{er^n} h(x)(v - \psi_k) \overset{k\to\infty}{\to} 0 \ .$$

But, this plainly follows from $h \in L^\Psi(\mathbb{R}^n)$ and the continuous embedding of $X(\mathbb{R}^n)$ into $L^\Phi(\mathbb{R}^n)$. □

Lemma 7.5. *If the Palais-Smale sequence $\{u_k\} \subset X(\mathbb{R}^n)$ satisfies*

$$\liminf_{k\to\infty} ||u_k||_{X(\mathbb{R}^n)} < \frac{1}{2}\Big(\frac{K_{\ell,n,\alpha}}{b}\Big)^{\frac{1}{\gamma}} \ , \tag{7.11}$$

then passing to a subsequence we obtain $u_k \to u$ in $X(\mathbb{R}^n)$ (the strong convergence).

Proof. We can write $u_k = u + w_k$. Further we can suppose that

$$\liminf_{k\to\infty} ||u_k||_{X(\mathbb{R}^n)} = \lim_{k\to\infty} ||u_k||_{X(\mathbb{R}^n)} \ .$$

Our aim is to show that $w_k \to 0$ in $X(\mathbb{R}^n)$ after passing to a suitable subsequence.

STEP 1.

First, let us prove that

$$\int_{\mathbb{R}^n} f(x, u_k)u \overset{k\to\infty}{\to} \int_{\mathbb{R}^n} f(x, u)u \ . \tag{7.12}$$

Fix $\varepsilon > 0$. By the density of $C_0^\infty(\mathbb{R}^n)$-functions in $X(\mathbb{R}^n)$ we can find $\psi \in C_0^\infty(\mathbb{R}^n)$ such that $||u - \psi||_{X(\mathbb{R}^n)} < \varepsilon$. We have

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^n} (f(x, u_k) - f(x, u))u\Big| \\
&\quad\le \Big|\int_{\mathbb{R}^n} f(x, u_k)(u - \psi)\Big| + \Big|\int_{\mathbb{R}^n} f(x, u)(u - \psi)\Big| + \Big|\int_{\mathbb{R}^n} (f(x, u_k) - f(x, u))\psi\Big| \\
&\quad= I_1 + I_2 + I_3 \ .
\end{aligned}$$

Using (7.2) with $v = u - \psi$ we obtain

$$\begin{aligned}
|I_1| \le& \int_{\mathbb{R}^n} \Big|\Phi'(\nabla u_k)\frac{\nabla u_k}{|\nabla u_k|} \cdot (\nabla u - \nabla\psi)\Big| + \int_{\mathbb{R}^n} \Big|V(x)\Phi'(u_k)\frac{u_k}{|u_k|}(u - \psi)\Big| \\
&+ \mu \int_{\mathbb{R}^n} |h(x)(u - \psi)| + \varepsilon_k ||u - \psi||_{X(\mathbb{R}^n)} \\
=& J_1 + J_2 + J_3 + J_4 \ .
\end{aligned}$$

By Lemma 2.2 and (7.3) we know that $||\Phi'(\nabla u_k)\frac{\nabla u_k}{|\nabla u_k|}||_{L^\Psi(\mathbb{R}^n)} \le C$ and the definition of the norm on $X(\mathbb{R}^n)$ gives $||\nabla u - \nabla\psi||_{L^\Phi(\mathbb{R}^n)} < \varepsilon$. Thus the generalized Hölder's inequality yields $|J_1| \le C\varepsilon$. Similar way we obtain $|J_2| \le C\varepsilon$ (notice that all the norms are with respect to the measure $V(x)dx$). Using $h \in L^\Psi(\mathbb{R}^n)$ and $||u_k - \psi||_{L^\Phi(\mathbb{R}^n)} < C\varepsilon$ for $k \in \mathbb{N}$ large (it follows from (7.9) and $||u - \psi||_{X(\mathbb{R}^n)} < \varepsilon$) we obtain $|J_3| < C\varepsilon$ for $k \in \mathbb{N}$ large. The estimate $|J_4| < \varepsilon$ for $k \in \mathbb{N}$ large follows from the fact that $\varepsilon_k \to 0$. Hence we have $|I_1| < C\varepsilon$ for k large.

The same way we obtain that $|I_2| < C\varepsilon$.

Let us estimate I_3. Since $\psi \in C_0^\infty(\mathbb{R}^n)$, it is enough to show

$$\int_{\operatorname{supp}\psi} f(x, u_k) - f(x, u) \overset{k\to\infty}{\to} 0 \ .$$

But this easily follows from Lemma 2.6, (1.13), (7.5), convergence a.e. (see (7.9)) and the boundedness of $\operatorname{supp}\psi$. Thus, we have proved (7.12).

STEP 2.

Fix $\varepsilon > 0$. By the Brezis-Lieb Lemma (Lemma 2.3), Lemma 2.1, (7.2) (with $v = u$ and $v = u_k$, respectively), (7.3), (7.4), (7.9) and (7.12) we have for $k \in \mathbb{N}$ large enough

(7.13)
$$\begin{aligned}
&\int_{\mathbb{R}^n} \Phi(|\nabla w_k|) + V(x)\Phi(|w_k|) \\
&\quad \le \varepsilon + \int_{\mathbb{R}^n} \Phi(|\nabla u_k|) - \Phi(|\nabla u|) + V(x)\Big(\Phi(|u_k|) - \Phi(|u|)\Big) \\
&\quad \le \varepsilon + C\int_{\mathbb{R}^n} \Phi'(|\nabla u_k|)|\nabla u_k| - \Phi'(|\nabla u|)|\nabla u| + V(x)\Big(\Phi'(|u_k|)|u_k| - \Phi'(|u|)|u|\Big) \\
&\quad \le \varepsilon + C\int_{\mathbb{R}^n} f(x,u_k)u_k - f(x,u)u + \mu h(x)(u_k - u) + C\varepsilon_k||u_k||_{X(\mathbb{R}^n)} + C\varepsilon_k||u||_{X(\mathbb{R}^n)} \\
&\quad \le \varepsilon + C\int_{\mathbb{R}^n} f(x,u_k)w_k + \int_{\mathbb{R}^n} f(x,u_k)u - f(x,u)u + 2C\mu||h||_{L^{\Psi}(\mathbb{R}^n)}||u_k - u||_{L^{\Phi}(\mathbb{R}^n)} + 2\varepsilon \\
&\quad \le C\varepsilon + C\int_{\mathbb{R}^n} f(x,u_k)w_k \ .
\end{aligned}$$

Notice that the Brezis-Lieb Lemma also gives that

$$||\nabla w_k||_{L^{\Phi}(\mathbb{R}^n)} \le ||\nabla u_k||_{L^{\Phi}(\mathbb{R}^n)} + \mathrm{o}(1) \ . \tag{7.14}$$

In view of (2.5) it is enough to show that the last integral in (7.13) tends to zero. From (1.16) and the fact that $t \mapsto \exp_{[\ell]}(t) - S_{\ell,n,\alpha}(t)$ is increasing we have

$$\begin{aligned}
I := \Big|\int_{\mathbb{R}^n} f(x,u_k)w_k\Big| &\le \tilde{C}_b\int_{\mathbb{R}^n} |u_k|^{n-1}|w_k| + C_b\int_{\mathbb{R}^n}\Big(\exp_{[\ell]}(b|u+w_k|^{\gamma}) - S_{\ell,n,\alpha}(b|u+w_k|^{\gamma})\Big)|w_k| \\
&\le \tilde{C}_b\int_{\mathbb{R}^n} |u_k|^{n-1}|w_k| + C_b\int_{\mathbb{R}^n}\Big(\exp_{[\ell]}(b2^{\gamma}|u|^{\gamma}) - S_{\ell,n,\alpha}(b2^{\gamma}|u|^{\gamma})\Big)|w_k| \\
&\qquad + C_b\int_{\mathbb{R}^n}\Big(\exp_{[\ell]}(b2^{\gamma}|w_k|^{\gamma}) - S_{\ell,n,\alpha}(b2^{\gamma}|w_k|^{\gamma})\Big)|w_k| \\
&= I_1 + I_2 + I_3 \ .
\end{aligned}$$

Now, as u_k are bounded in $L^n(\mathbb{R}^n)$ and $w_k \to 0$ in $L^n(\mathbb{R}^n)$ (see (7.9)), we can use Hölder's inequality to show that $I_1 \to 0$. Next, choosing arbitrary $q > 1$, from Theorem 3.3(i) with $K = qb2^{\gamma}$ and (2.1) we see that

$$\Big(\exp_{[\ell]}(b2^{\gamma}|u|^{\gamma}) - S_{\ell,n,\alpha}(b2^{\gamma}|u|^{\gamma})\Big)^q \le \exp_{[\ell]}(qb2^{\gamma}|u|^{\gamma}) - S_{\ell,n,\alpha}(qb2^{\gamma}|u|^{\gamma}) \in L^1(\mathbb{R}^n) \ .$$

Therefore (7.9) and Hölder's inequality with powers q and $\frac{q}{q-1}$ give $I_2 \to 0$. Finally, we use a similar method to estimate I_3. This time we use Theorem 3.3(ii). The assumption concerning the $L^{\Phi}(\mathbb{R}^n)$-norm of the gradients turns to

$$qb2^{\gamma}||\nabla w_k||^{\gamma}_{L^{\Phi}(\mathbb{R}^n)} < K_{\ell,n,\alpha} \ .$$

However, this is satisfied for $q > 1$ sufficiently close to 1 and $k \in \mathbb{N}$ large enough, thanks to (7.11) and (7.14). Hence we have $I \to 0$ and thus $||u_k - u||_{X(\mathbb{R}^n)} \to 0$. □

8. Existence results

In this section we show that the Ekeland Variational Principle (Theorem 2.11) and Mountain Pass Theorem (Theorem 2.10) give us two different nontrivial weak solutions to (1.10).

Proposition 8.1. *There is $\mu_0 > 0$ such that if $\mu \in (0,\mu_0)$, then* (1.10) *has a nontrivial minimum-type solution $u_0 \in X(\mathbb{R}^n)$ with $J_\mu(u_0) = c_0 < 0$, where c_0 is given in Lemma 6.2. Moreover, there is a corresponding Palais-Smale sequence $\{u_k\} \subset X(\mathbb{R}^n)$ converging to u_0 in the sense of* (7.9) *and strongly in $X(\mathbb{R}^n)$.*

Proof. Let $\varrho_\mu > 0$ be the same as in Lemma 6.2. We can suppose that μ_0 is so small that

$$\varrho_\mu < \frac{1}{2}\Big(\frac{K_{\ell,n,\alpha}}{b}\Big)^{\frac{1}{\gamma}} \qquad \text{for all } \mu \in (0,\mu_0) \ .$$

Since $Y(\mathbb{R}^n) := \{v \in X(\mathbb{R}^n) : ||v||_{X(\mathbb{R}^n)} \le \varrho_\mu\}$ is a complete metric space, the functional J_μ is a C^1-functional and bounded from bellow (see Lemma 6.2), we can use Ekeland Variational Principle (Theorem 2.11) to obtain a sequence $\{u_k\} \subset Y(\mathbb{R}^n)$ such that

$$J_\mu(u_k) \overset{k\to\infty}{\to} c_0 \qquad \text{and} \qquad ||J'_\mu(u_k)||_{C(Y(\mathbb{R}^n),\mathbb{R})} \overset{k\to\infty}{\to} 0 \ .$$

These are conditions (7.1) and (7.2) (up to the fact that we have (7.2) with the test-functions from $Y(\mathbb{R}^n)$ only, but using the linearity of (7.2) in v we obtain (7.2) also for the test-functions from $X(\mathbb{R}^n)$). Therefore we can use all the results from Section 7 for the sequence $\{u_k\}$. By Lemma 7.4, Lemma 7.5 and continuity of J_μ we obtain that u_0 is a weak solution to (1.10) satisfying $J_\mu(u_0) = c_0$. Since μ and h are nontrivial, u_0 has to be nontrivial (see (1.22)). ☐

Proposition 8.2. *There is $\mu_0 > 0$ such that if $\mu \in (0, \mu_0)$, then* (1.10) *has a Mountain Pass-type solution $u_M \in X(\mathbb{R}^n)$. Moreover, there is a corresponding Palais-Smale sequence $\{v_k\} \subset X(\mathbb{R}^n)$ converging to u_M in the sense of* (7.9).

Proof. Since we have $J(0) = 0$, Lemmata 6.1, 6.2 and Proposition 5.1, we can apply the Mountain Pass Theorem (Theorem 2.10) which together with Lemma 6.4 gives us a Palais-Smale sequence $\{u_k\} \subset X(\mathbb{R}^n)$. Passing to a subsequence we can further suppose that we have (7.9). Finally, if we set $u_M = u$, where $u \in X(\mathbb{R}^n)$ is given by (7.9), then u_M is a weak solution to (1.10) by Lemma 7.4. Since μ and h are nontrivial, u_M has to be nontrivial (see (1.22)). ☐

Proposition 8.3. *If $\mu_0 > 0$ is small enough, then the functions u_0 and u_M given by Proposition 8.1 and Proposition 8.2 are distinct.*

Proof. By Proposition 8.1 and Proposition 8.2 we have $\{u_k\}, \{v_k\} \subset X(\mathbb{R}^n)$ such that

$$\begin{aligned} &u_k \to u_0 \text{ in } X(\mathbb{R}^n) \qquad \text{and} \qquad v_k \rightharpoonup u_M \text{ in } WL^\Phi(\mathbb{R}^n),\ v_k \to u_M \text{ in } L^\Phi(\mathbb{R}^n)\ ,\\ &J_\mu(u_k) \to c_0 \qquad \text{and} \qquad J_\mu(v_k) \to c_M\ ,\\ &\langle J'_\mu(u_k), u_k\rangle \to 0 \qquad \text{and} \qquad \langle J'_\mu(v_k), v_k\rangle \to 0\ . \end{aligned} \tag{8.1}$$

Moreover, by Lemmata 6.2, 6.3 and 6.4 we have

$$c_0 < 0 < c_M \qquad \text{and} \qquad c_M - c_0 < \Big(\frac{K_{\ell,n,\alpha}}{b}\Big)^{\frac{n}{\gamma}} \tag{8.2}$$

For the sake of contradiction suppose that $u_0 = u_M$. As both sequences converge to $u_0 = u_M$ in $L^\Phi(\mathbb{R}^n)$, $h \in L^\Psi(\mathbb{R}^n)$ and we have Lemma 7.2, we see that

$$\begin{aligned} J_\mu(u_k) &= \int_{\mathbb{R}^n} \Phi\big(|\nabla u_k|\big) + V(x)\Phi(|u_k|) - F(x,u_0) - \mu h(x)u_0 + \mathrm{o}(1) \overset{k\to\infty}{\to} c_0\\ J_\mu(v_k) &= \int_{\mathbb{R}^n} \Phi\big(|\nabla v_k|\big) + V(x)\Phi(|v_k|) - F(x,u_0) - \mu h(x)u_0 + \mathrm{o}(1) \overset{k\to\infty}{\to} c_M \end{aligned}$$

and subtracting one from another we obtain

$$\int_{\mathbb{R}^n} \Phi\big(|\nabla u_k|\big) + V(x)\Phi(|u_k|) - \int_{\mathbb{R}^n} \Phi\big(|\nabla v_k|\big) + V(x)\Phi(|v_k|) \overset{k\to\infty}{\to} c_0 - c_M < 0\ . \tag{8.3}$$

Next, $\langle J'_\mu(u_k), u_k\rangle \to 0$ and $\langle J'_\mu(v_k), v_k\rangle \to 0$ read by (1.21)

$$\begin{aligned} &\int_{\mathbb{R}^n} \Phi'\big(|\nabla u_k|\big)|\nabla u_k| + V(x)\Phi'\big(|u_k|\big)|u_k| - f(x,u_k)u_k - \mu h(x)u_k \overset{k\to\infty}{\to} 0\\ &\int_{\mathbb{R}^n} \Phi'\big(|\nabla v_k|\big)|\nabla v_k| + V(x)\Phi'\big(|v_k|\big)|v_k| - f(x,v_k)v_k - \mu h(x)v_k \overset{k\to\infty}{\to} 0 \end{aligned}$$

and thus

$$\begin{aligned} &\Big(\int_{\mathbb{R}^n} \Phi'\big(|\nabla u_k|\big)|\nabla u_k| + V(x)\Phi'\big(|u_k|\big)|u_k| - \int_{\mathbb{R}^n} \Phi'\big(|\nabla v_k|\big)|\nabla v_k| + V(x)\Phi'\big(|v_k|\big)|v_k|\Big)\\ &\quad - \int_\Omega f(x,u_k)u_k - f(x,v_k)v_k - \mu\int_\Omega h(x)(u_k - v_k) \overset{k\to\infty}{\to} 0\ . \end{aligned} \tag{8.4}$$

As both sequences converge to u_0 in $L^\Phi(\mathbb{R}^n)$ and $h \in L^\Psi(\mathbb{R}^n)$, the last integral tends to zero.

Further, since $u_k \to u_0$ in $X(\mathbb{R}^n)$ by (8.1), which implies the convergence in $WL^\Phi(\mathbb{R}^n)$, passing to a subsequence we can construct a common majorant $g \in WL^\Phi(\mathbb{R}^n)$.

Next, from (1.16) we infer

$$|f(x,u_k)u_k| \le \tilde{C}_b|u_k|^n + C_b\Big(\exp_{[\ell]}(b|u_k|^\gamma) - S_{\ell,n,\alpha}(b|u_k|^\gamma)\Big)|u_k|$$
$$\le \tilde{C}_b|g|^n + C_b\Big(\exp_{[\ell]}(b|g|^\gamma) - S_{\ell,n,\alpha}(b|g|^\gamma)\Big)|g| \ .$$

Since the right hand side is an $L^1(\mathbb{R}^n)$-function (the first term is plainly an $L^1(\mathbb{R}^n)$-function, while for the second term we can use Hölder's inequality with powers n' and n together with Theorem 3.3(i) and (2.1)), we can use the Lebesgue Dominated Convergence Theorem to obtain

$$\int_{\mathbb{R}^n} f(x,u_k)u_k \overset{k\to\infty}{\to} \int_{\mathbb{R}^n} f(x,u_0)u_0 \ . \tag{8.5}$$

Further, thanks to Lemma 2.1, we see that if we also prove that

$$\int_\Omega f(x,v_k)v_k - f(x,u_0)u_0 \overset{k\to\infty}{\to} 0 \ , \tag{8.6}$$

then (8.4) would give us a contradiction with (8.3). Hence, it remains to show (8.6).

Since $\int_{\mathbb{R}^n}\Phi(|\nabla v_k|)$ are bounded by Lemma 7.1, passing to a subsequence we can suppose that these modulars converge. Notice that by Fatou's lemma the limit is larger or equal to $\int_{\mathbb{R}^n}\Phi(|\nabla u_0|)$. In the rest of the proof we distinguish two cases.

Case 1.: $\int_{\mathbb{R}^n}\Phi(|\nabla v_k|) \to \int_{\mathbb{R}^n}\Phi(|\nabla u_0|)$.
In this case we have

$$\nabla v_k \overset{k\to\infty}{\to} \nabla u_0 \qquad \text{in } L^\Phi(\mathbb{R}^n)$$

(indeed, we can use the Brezis-Lieb lemma to show that the modular of $\nabla(v_k - u_0)$ tends to zero and so does the norm by (2.5)). Since we also have $v_k \to u_0$ in $L^\Phi(\mathbb{R}^n)$, we obtain $v_k \to u_0$ in $WL^\Phi(\mathbb{R}^n)$. Hence, we can prove (8.6) the same way as we proved (8.5). Thus, we are done in the first case.

Case 2.: $\lim_{k\to\infty}\int_{\mathbb{R}^n}\Phi(|\nabla v_k|) - \int_{\mathbb{R}^n}\Phi(|\nabla u_0|) > 0$.
In this case, our first step is to prove that there is $q \in (1,n')$ such that

$$\int_{\mathbb{R}^n}\Big(\exp_{[\ell]}(b|v_k|^\gamma) - S_{\ell,n,\alpha}(b|v_k|^\gamma)\Big)^q \le C \ . \tag{8.7}$$

By the Brezis-Lieb lemma, $u_k \to u_0$ in $X(\mathbb{R}^n)$ and (8.3) we see that

$$\lim_{k\to\infty}\int_{\mathbb{R}^n}\Phi(|\nabla v_k|) + \lim_{k\to\infty}\int_{\mathbb{R}^n}V(x)\Phi(|v_k|) - \int_{\mathbb{R}^n}\Phi(|\nabla u_0|) - \int_{\mathbb{R}^n}V(x)\Phi(|u_0|) = c_M - c_0 \ .$$

Further, from Fatou's lemma and $v_k \to u_0$ a.e. on $\mathbb{R}^n$ we obtain

$$\lim_{k\to\infty}\int_{\mathbb{R}^n}V(x)\Phi(|v_k|) \ge \int_{\mathbb{R}^n}V(x)\Phi(|u_0|) \ .$$

Thus, (8.2) yields

$$\lim_{k\to\infty}\int_{\mathbb{R}^n}\Phi(|\nabla v_k|) - \int_{\mathbb{R}^n}\Phi(|\nabla u_0|) \le c_M - c_0 < \Big(\frac{K_{\ell,n,\alpha}}{b}\Big)^{\frac{n}{\gamma}} \ .$$

Therefore there is $q \in (1,n')$ such that

$$\lim_{k\to\infty}\int_{\mathbb{R}^n}\Phi(|\nabla v_k|) - \int_{\mathbb{R}^n}\Phi(|\nabla u_0|) < \Big(\frac{K_{\ell,n,\alpha}}{bq^2}\Big)^{\frac{n}{\gamma}} \ .$$

This is

$$bq \le \frac{K_{\ell,n,\alpha}}{q}\Big(\frac{1}{\lim_{k\to\infty}\int_{\mathbb{R}^n}\Phi(|\nabla v_k|) - \int_{\mathbb{R}^n}\Phi(|\nabla u_0|)}\Big)^{\frac{\gamma}{n}} \ . \tag{8.8}$$

Next, by (2.1) we have

$$\int_{\mathbb{R}^n}\Big(\exp_{[\ell]}(b|v_k|^\gamma) - S_{\ell,n,\alpha}(b|v_k|^\gamma)\Big)^q \le \int_{\mathbb{R}^n}\exp_{[\ell]}(bq|v_k|^\gamma) - S_{\ell,n,\alpha}(bq|v_k|^\gamma) \ .$$

Now, the integral on the right hand side is uniformly bounded by Proposition 4.3 and thus we have (8.7).

Next, we can use (1.16), (8.7) and the continuous embedding of $X(\mathbb{R}^n)$ into $L^{nq}(\mathbb{R}^n)$ together with Lemma 7.1 to obtain

$$\int_{\mathbb{R}^n} |f(x,v_k)|^q \le \int_{\mathbb{R}^n} C|v_n|^{nq} + C\Big(\exp_{[\ell]}(b|v_k|^\gamma) - S_{\ell,n,\alpha}(b|v_k|^\gamma)\Big)^q \le C \ . \tag{8.9}$$

Similarly we obtain

$$\int_{\mathbb{R}^n} |f(x,u_0)|^q \le C \ . \tag{8.10}$$

Next, we are going to apply estimates (8.9) and (8.10) to

$$\begin{aligned}\int_{\mathbb{R}^n} &|f(x,v_k)v_k - f(x,u_0)u_0| \\ &\le \int_{\mathbb{R}^n} |(f(x,v_k)-f(x,u_0))u_0| + \int_{\mathbb{R}^n} |f(x,v_k)(v_k-u_0)| = I_1 + I_2 \ .\end{aligned}$$

First, let us deal with I_2. Estimate (8.9), Hölder's inequality and $v_k \to u_0$ in $L^{q'}(\mathbb{R}^n)$ yield

$$I_2 = \int_{\mathbb{R}^n} |f(x,v_k)(v_k-u_0)| \le ||f(x,v_k)||_{L^q(\mathbb{R}^n)} ||v_k-u_0||_{L^{q'}(\mathbb{R}^n)} \overset{k\to\infty}{\to} 0 \ .$$

In the rest of the proof we deal with I_1. Fix $\varepsilon > 0$. From Fatou's lemma, (1.13) and (7.5) we have

$$\int_{\mathbb{R}^n} |f(x,u_0)u_0| < \infty \ .$$

Hence there is $M_1 > \frac{1}{\varepsilon}$ such that

$$J_2 := \int_{\{u_0 > M_1\}} |f(x,u_0)u_0| < \varepsilon \ .$$

Let $\delta > 0$ be such that $q = 1 + 2\delta$. By Assumption (1.19) we can find $M_2 \ge M_1$ so that

$$f(x,t)^\delta \ge t \qquad \text{for all } t \ge M_2 \ .$$

Hence from (8.9) we obtain

$$J_3 := \int_{\{v_k > M_2\}} |f(x,v_k)u_0| \le \frac{1}{M_2}\int_{\mathbb{R}^n} |f(x,v_k)|^{1+\delta}|u_0| \le C\varepsilon$$

where the last inequality follows from Hölder's inequality with powers $\frac{q}{1+\delta}$ and $(\frac{q}{1+\delta})' > n$, (8.9) and $u_0 \in L^r(\mathbb{R}^n)$ for all $r \in [n,\infty)$.

Finally, by (1.16) we see that we have

$$|f(x,t)| \le Ct^{n-1} \qquad \text{for all } t \in [0, M_2] \tag{8.11}$$

and moreover, as v_k converge in $L^n(\mathbb{R}^n)$, there is their common majorant $U \in L^n(\mathbb{R}^n)$. We have

$$I_1 \le J_1 + J_2 + J_3 \quad \text{where} \quad J_1 = \int_{\mathbb{R}^n} |f(x,v_k)\chi_{\{v_k \le M_2\}} - f(x,u_0)\chi_{\{u_0 \le M_2\}}||u_0|$$

and J_2, J_3 are defined above. Hence, it is enough to show that $J_1 \to 0$. We observe that the integrand converges to zero a.e. in $\mathbb{R}^n$, therefore it suffices to find an integrable majorant so that we could conclude the proof using the Lebesgue Dominated Convergence Theorem. But we have plainly from (8.11)

$$|f(x,v_k)\chi_{\{v_k \le M_2\}}||u_0| \le C|v_k|^{n-1}|u_0| \le C|U|^{n-1}|u_0| \in L^1(\mathbb{R}^n)$$

and

$$|f(x,u_0)\chi_{\{u_0 \le M_2\}}||u_0| \le C|u_0|^{n-1}|u_0| = C|u_0|^n \in L^1(\mathbb{R}^n) \ .$$

Hence we have proved (8.6) also in the second case and we are done. □

Finally, we see that Theorem 1.1 follows from Propositions 8.1, 8.2 and 8.3.

9. Concluding remarks

Sub-critical case. Similarly as in papers [11], [8] and [9], we can use our methods to obtain the existence of at least two non-trivial weak solutions to (1.10) also in the sub-critical case. It is, instead of (1.16) we have

$$\text{for every } b > 0 \text{ there is } C_b > 0 \text{ such that}$$
$$|f(x,t)| \le C|t|^{n-1} + C_b \left(\exp_{[\ell]}(b|t|^{\gamma}) - S_{\ell,n,\alpha}(b|t|^{\gamma})\right) \quad \text{whenever } t \in \mathbb{R} \text{ and } x \in \mathbb{R}^n .$$

In this case we do not need to assume (1.8) and (1.19).

Case of a bounded domain. Quasilinear nonhomogenous problems are often studied on bounded domains. It is, in our case, we consider (1.10) for the functions from $W_0L^{\Phi}(\Omega)$ where $\Omega \subset \mathbb{R}^n$ is a bounded domain.

Recall that the space $W_0L^{\Phi}(\mathbb{R}^n)$ is compactly embedded into $L^{\Phi}(\Omega)$ and $L^r(\Omega)$ for all $r \in [1,\infty)$ and we have the equivalence of the $W_0L^{\Phi}(\Omega)$-norm of a function and the $L^{\Phi}(\Omega)$-norm of its gradient. Hence, we do not have to construct the auxiliary space $X(\Omega)$ as we had to in the case of $WL^{\Phi}(\mathbb{R}^n)$. This time we suppose that the potential V is non-negative, continuous and bounded.

Acknowledgement. The author was supported by the research project MSM 0021620839 of the Czech Ministry MŠMT.

References

[1] Adimurthi, Existence of positive solutions of the semilinear Dirichlet problem with critical growth for the n-Laplacian, *Ann. Sc. Norm. Sup. Pisa* **17** (1990), 393-413.

[2] Adimurthi, Sandeep K., A singular Moser-Trudinge inequality and its application, *Nonlinear. differ. equ. appl.* **13** (2007), 585-603.

[3] Ambrosetti A., Rabinowitz P. H., Dual variational methods in critical point theory and applications, *Funct. Anal.* **14** (1973), 349-381.

[4] Berestycki H., Lions P.-L., Nonlinear scalar field equations, I. Existence of ground state, *Arch. Rational. Mech. Anal.* **82** (1983), 313-346.

[5] Brezis H., Lieb E., A relation between pointwise convergence of functions and convergence of functionals, *Proc. Amer. Math. Soc.* **88** no. 3 (1983), 486-490.

[6] Brezis H., Nirenberg L., Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, *Comm. Pure Appl. Math.* **36** no. 4 (1983), 437-477.

[7] Černý R., *Concentration-Compactness Principle for embedding into multiple exponential spaces*, to appear in Math. Inequal. Appl. (preprint is available at www.karlin.mff.cuni.cz/kma-preprints/).

[8] Černý R., *Generalized Moser-Trudinger inequality for unbounded domains and its application*, submitted (preprint is available at www.karlin.mff.cuni.cz/kma-preprints/).

[9] Černý R., *On the Dirichlet problem for the generalized n-Laplacian: singular nonlinearity with the exponential and multiple exponential critical growth range*, submitted (preprint is available at www.karlin.mff.cuni.cz/kma-preprints/).

[10] Černý R., Gurka P., Hencl S., *Concentration-compactness principle for generalized Trudinger inequalities*, to appear in Z. Anal. Anwendungen (preprint is available at www.karlin.mff.cuni.cz/kma-preprints/).

[11] Černý R., Gurka P., Hencl S., *On the Dirichlet problem for the generalized n,α-Laplacian with the nonlinearity in the critical growth range*, submitted (preprint is available at www.karlin.mff.cuni.cz/kma-preprints/).

[12] Černý R., Mašková S., A sharp form of an embedding into multiple exponential spaces, *Czechoslovak Math. J.* **60** no. 3 (2010), 751-782.

[13] Černý R., Mašková S., *On generalization of Moser's theorem in the critical case*, to appear in Math. Inequal. Appl. (preprint is available at www.karlin.mff.cuni.cz/kma-preprints/).

[14] Edmunds D.E., Gurka P., Opic B., Double exponential integrability of convolution operators in generalized Lorentz-Zygmund spaces, *Indiana Univ. Math. J.* **44** (1995), 19-43.

[15] Edmunds D.E., Gurka P., Opic B., Double exponential integrability, Bessel potentials and embedding theorems, *Studia Math.* **115** (1995), 151-181.

[16] Edmunds D.E., Gurka P., Opic B., On embeddings of logarithmic Bessel potential spaces, *J. Functional Analysis* **146** (1997), 116-150.

[17] Ekeland I., On the variational principle, *J. Math. Anal. Appl.* **47** (1974), 324-353.

[18] de Figueiredo D.G., Miyagaki O. H., Ruf B., Elliptic equations in $\mathbb{R}^2$ with nonlinearities in the critical growth range, *Calc. Var.* **3** (1995), 139-153.

[19] Fusco N., Lions P. L., Sbordone C., Sobolev imbedding theorems in borderline cases, *Proc. Amer. Math. Soc.* **124** (1996), 561-565.

[20] Hencl S., A sharp form of an embedding into exponential and double exponential spaces, *J. Funct. Anal.* **204,** no. 1 (2003), 196–227.
[21] Lions P. L., On the existence of positive solutions of semilinear elliptic equations, *SIAM review* **24** (1982), 441-467.
[22] Lions P. L., The concentration-compactness principle in the calculus of variations. The limit case. I., *Rev. Mat. Iberoamericana* **1,** no. 1 (1985), 145-201.
[23] Moser J., A sharp form of an inequality by N. Trudinger, *Indiana Univ. Math. J.* **20** (1971), 1077-1092.
[24] do Ó J. M., N-Laplacian equations in $\mathbb{R}^N$ with critical growth, *Abstr. Appl. Anal.* **2,** no. 3-4 (1997), 301-315.
[25] do Ó J. M., Medeiros E., Severo U., On a quasilinear nonhomogenous elliptic equation with critical growth in $\mathbb{R}^N$, *J. Differential Equations* **246** (2009), 1363-1386.
[26] do Ó J. M., Semilinear Dirichlet problems for the N-laplacian in $\mathbb{R}^n$ with nonlinearities in the critical growth range, *Differential Integral Equations* **9,** no. 5 (1996), 967-979.
[27] Rana I. K., *An introduction to measure and integration*, Second edition. Graduate Studies in Mathematics, vol. **45**, American Mathematical Society, Providence, RI.
[28] Talenti G., Inequalities in rearrangement invariant function spaces, *Nonlinear Analysis, Function Spaces and Applications* **5** (1994), 177-230.
[29] Tonkes E., Solutions to a perturbed critical semilinear equation concerning the N-Laplacian in $\mathbb{R}^N$, *Comment. Math. Univ. Carolin.* **40,** no. 4 (1999), 679-699.
[30] Trudinger N. S., On imbeddings into Orlicz spaces and some applications, *J. Math. Mech.* **17** (1967), 473-484.

Department of Mathematical Analysis, Charles University, Sokolovská 83, 186 00 Prague 8, Czech Republic
E-mail address: rcerny@karlin.mff.cuni.cz