WEIGHTED INTEGRAL INEQUALITIES

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I want to speak here about inequalities of the form

$$\left(\int_\Omega |f(x)|^q u(x)\,\mathrm{d}x\right)^{1/q} \leqq C\left(\int_\Omega |\nabla f(x)|^p v(x)\,\mathrm{d}x\right)^{1/p} \tag{1}$$

with Ω a domain in $\mathbb{R}^N$, with parameters $p \geqq 1$, $q > 0$, and with u, v given *weight functions* (i.e. functions measurable and positive a.e. in Ω). We want the inequality to hold for, say, all functions f from $C_0^\infty(\Omega)$ with a constant $C > 0$ independent of f.

It was Jindřich NEČAS who linked my attention to this inequality in the sixties (of the last century). At that time, he has held a lecture series at the Charles University, devoted to modern methods of solving (linear, elliptic) partial differential equations, and this topic involved the theory of (classical) Sobolev spaces, in particular imbedding theorems for these spaces. And inequality (1) - with special weight functions $u(x) = v(x) \equiv 1$ - expresses one of these imbeddings.

J. NEČAS investigated and used also weighted Sobolev spaces with special weights, namely powers of the distance to the boundary $\partial\Omega$ of Ω. He collected his results in the book [7], and his approach to the investigation of Sobolev spaces (weighted as well as nonweighted) was based on a certain "geometry"of the domain Ω (ore, more precisely, of the boundary $\partial\Omega$) which he explained in his famous, may be not enough evaluated, paper [6]. Here, he has described a class of domains which have shown to be a very useful tool in investigating function spaces of Sobolev type on domains and inequalities of the form (1). Among other, his approach allowed to reduce the investigation of such inequalities to the one-dimensional case. Finally, having in (1) weights of the form

$$u(x) = [\mathrm{dist}(x, \partial\Omega)]^\alpha, v(x) = [\mathrm{dist}(x, \partial\Omega)]^\beta, \quad \alpha, \beta \in \mathbb{R}, \tag{2}$$

inequality (1) can be handled - for $p = q$ - by the classical Hardy inequality

$$\int_0^\infty |f(t)|^p\ t^{\lambda-p}\,\mathrm{d}t \leqq C\int_0^\infty |f'(t)|^p t^\lambda\,\mathrm{d}t \tag{3}$$

valid

(i) for $\lambda < p-1$ provided $f(0) = 0$,

(ii) for $\lambda > p-1$ provided $f(\infty) = 0$,

with the (exact) constant

$$C = \left(\frac{p}{|p-\lambda-1|}\right)^p.$$

In fact, I want to speak here about the one-dimensional analogue of (1), i.e. about the inequality

$$\left(\int_a^b |f(t)|^q u(t)\,\mathrm{d}t\right)^{1/q} \leqq C\left(\int_a^b |f'(t)|^p v(t)\,\mathrm{d}t\right)^{1/p} \tag{4}$$

for functions f satisfying

$$f(a) = 0,$$

with (a, b) a fixed interval in $\mathbb{R}$ and u, v general weight functions, but first, let me say a few words about inequality (1), more precisely, about its special case:

$$1 < p = q,\ v(x) \equiv 1,\ u(x) = [\operatorname{dist}(x, \partial\Omega)]^{-p},$$

i.e. about the inequality

$$\int_\Omega \left|\frac{f(x)}{\operatorname{dist}(x, \partial\Omega)}\right|^p \mathrm{d}x \leqq C \int_\Omega |\nabla f(x)|^p \,\mathrm{d}x, \tag{5}$$

also commonly called *Hardy's inequality.*

Inequality (5), although it is simple and could be considered a counter part of (3) for $\lambda = 0$, is repeatedly up to now subject of many papers, the most recent being (probably) the paper of P. KOSKELA and X. ZHONG [4] from 2003. Of course, these papers are mainly dealing with question for what class of domains the inequality is still valid, since for "reasonable" domains, (5) can be derived easily for $f \in C_0^\infty(\Omega)$ with

$$C = \left(\frac{p}{p-1}\right)^p.$$

In this connection, let me mention a result of J. KADLEC (a gifted but unfortunately late student of J. NEČAS) and myself [3] from 1967 where a characterization of functions with zero traces is given : a function $f \in W^{1,p}(\Omega)$ belongs to $W_0^{1,p}(\Omega)$ iff $u/d \in L^p(\Omega)$ with $d(x) = \operatorname{dist}(x, \partial\Omega)$ (which means that inequality (5) holds).

But now, let us consider the general one-dimensional Hardy inequality (4). We can assume that $a = 0$, hence we will deal with the inequality

$$\left(\int_0^b |f(t)|^q u(t)\,\mathrm{d}t\right)^{1/q} \leqq C\left(\int_0^b |f'(x)|^p v(x)\,\mathrm{d}t\right)^{1/p} \tag{6}$$

with a fixed $b, 0 < b \leqq \infty$, for functions f satisfying

$$f(a) = 0$$

and for parameters p, q satisfying

$$1 < p \leqq q < \infty.$$

[The case $p > q$ must be handled separately.]

There are several criteria for the validity of (6). First, let us introduce some auxiliary functions. Suppose that

$$V(x){:}\ = \int_0^x v^{1-p'}(t)\,\mathrm{d}t < \infty \quad \text{for every } x \in (0, b)$$

where $p' = p/(p-1)$, and denote

$$
\begin{aligned}
A_M(x)\colon &= \left(\int_x^b u(t)\,\mathrm{d}t\right)^{1/q}\left(\int_0^x v^{1-p'}(t)\,\mathrm{d}t\right)^{1/p'} \\
&= \left(\int_x^b u(t)\,\mathrm{d}t\right)^{1/q} V^{1/p'}(x); \\
A_T(x)\colon &= \left(\int_0^x u(t)V^q(t)\,\mathrm{d}t\right)^{1/q} V^{-1/p}(x): \\
A_B(x,g)\colon &= \left(\frac{1}{g(x)}\int_0^x u(t)[g(t)+V(t)]^{q/p'+1}\,\mathrm{d}t\right)^{1/q}, \quad g(x)>0; \\
A_W(x,s)\colon &= \left(\int_x^b u(t)V^{q(p-s)/p}(t)\,\mathrm{d}t\right)^{1/q} V^{(s-1)/p}(x), \quad 1<s<p.
\end{aligned}
\tag{7}
$$

If we further denote

$$
\begin{aligned}
A_1\colon &= \sup_{x\in(0,b)} A_M(x) \\
A_2\colon &= \sup_{x\in(0,b)} A_T(x) \\
A_3\colon &= \inf_{g(x)>0}\ \sup_{x\in(0,b)} A_B(x,g) \\
A_4(s)\colon &= \sup_{x\in(0,b)} A_W(x,s),
\end{aligned}
\tag{8}
$$

then the following important result holds:

Theorem 1. *The Hardy inequality (6) holds with $1 < p \leqq q < \infty$ for all function f such that $f(0) = 0$ if and only if any of the numbers A_1, A_2, A_3 and $A_4(s)$ $(1 < s < p)$ is finite.*

Hence, we have four criteria for the validity of (6), four necessary and sufficient conditions. The letters M, T, B and W are due to B. MUCKENHOUPT, G. TOMASELLI, P. R. BEESACK and A. WEDESTIG who have been (probably) the first who introduced the corresponding functions, as least for the case $p = q$. More precisely, G. TOMASELLI [9] used $A_M(x)$ and $A_T(x)$ for $p = q$ in 1969, B. MUCKENHOUPT [5] published the function $A_M(x)$ for $p \leqq q$ in 1972, while the version of $A_T(x)$ for $p \leqq q$ is due to L. E. PERSSON and V. STRPANOV [8] in 2002. The oldest result is due to BEESACK [1] (1961) with $A_B(x)$ for $p = q$, which the form given here for $p \leqq q$ is due to D. GURKA [2] (1984). The results of A. WEDESTIG is the most recent and appeared up to now only in her Thesis [10].

Remark 1. *Notice that $A_W(x,p) = A_M(x)$, so that A_1 is an extension of $A_4(s)$ involving a new parameter s. Moreover, since $V(t)$ is increasing and $q(p-s)/p > 0$,* *we immediately have that*

$$
\begin{aligned}
A_w(x,s) &= \left(\int_x^b u(t)V^{q(p-s)/p}(t)\,dt\right)^{1/q} V^{(s-1)/p}(x)\,dx \\
&\geqq \left(\int_x^b u(t)V^{q(p-s)/p}(x)\,dt\right)^{1/q} V^{(s-1)/p}(x) \\
&= \left(\int_x^b u(x)\,dt\right)^{1/q} V^{(p-s)/p}(x)V^{(s-1)/p}(x) \\
&= \left(\int_x^b u(x)\,dt\right)^{1/q} V^{(p-1)/p}(x) = A_M(x).
\end{aligned}
$$

The numbers A_1, A_2, A_3 and $A_4(s)$ can be used to estimate the best constant C in (6). It is

$$1 \leqq \frac{C}{A_1} \leqq k(p,q)$$

with

$$k(p,q) = \left(1+\frac{q}{p'}\right)^{1/q}\left(1+\frac{p'}{q}\right)^{1/p'}, \tag{9}$$

and

$$1 \leqq \frac{C}{A_2} \leqq p',$$

$$\frac{(p'/q)^{1/q}}{k(p,q)} \leqq \frac{C}{A_3} \leqq \left(\frac{p'}{q}\right)^{1/q},$$

and

(10)

$$\sup_{1<s<p}\left\{\left[1+\frac{1}{s-1}+\left(\frac{p-s}{p}\right)^{1/p}\right]^{-1/p} A_4(s)\right\} \leqq C \leqq \inf_{1<s<p}\left[\left(\frac{p-1}{p-s}\right)^{1/p'} A_4(s)\right].$$

In fact, we can also obtain other criteria for the validity of (6). Using duality arguments, we can introduce the following analogue of $A_W(x,s)$:

$$
\begin{aligned}
A_5(x,s) =& \left(\int_0^x v^{1-p'}(t)\left(\int_t^b u(s)\,\mathrm{d}s\right)^{p'(q'-s)/q'} \mathrm{d}t\right)^{1/p'} \\
& \left(\int_x^b u(s)\,\mathrm{d}s\right)^{(s-1)/q'}, \quad 1<s<q',
\end{aligned}
$$

and the analogue of $A_T(x)$:

$$A_6(x) = \left(\int_x^b v^{1-p'}(t)\left(\int_t^b u(s)\,\mathrm{d}t\right)^{p'} \mathrm{d}s\right)^{1/p'} \left(\int_x^b u(s)\,\mathrm{d}s\right)^{-1/q'}.$$

Again, inequality (6) holds if and only if

$$A_5(s)\colon = \sup_{0<x<b} A_5(x,s) < \infty$$

or

$$A_6 \colon = \sup_{0<x<b} A_6(x) < \infty$$

and we have the estimate

$$1 \leqq \frac{C}{A_6} \leqq q$$

and an estimate analogous to (10) with $A_5(s)$.

Remark 2. *Analogously as in Remark 1, it is*

$$A_5(x, q') = A_M(x)$$

and, since the function $U(t) = \int_t^b u(s)\, ds$ *is decreasing,*

$$A_5(x, s) \geqq A_M(x).$$

Moreover, the dual analogue of $A_B(x, g)$ *reads*

$$A_7(x; h) = \left(\frac{1}{h(x)} \int_x^b v^{1-p'}(t) \left[h(t) + \int_t^\infty u(s)\, ds \right]^{p'/q+1} dt \right)^{1/p'}, \quad h(x) > 0,$$

with the estimate

$$\frac{(q/p')^{1/p'}}{k(q', p')} \leqq \frac{C}{A_7} \leqq (\frac{q}{p'})^{1/p'}$$

where

$$A_7 \colon = \inf_{h(x)>0} \sup_{x\in(0,b)} A_7(x; h).$$

Obviously, using the fact that the (best) constant C in (6) satisfies $C \approx A_i$ for $i = 1, 2, \ldots, 7$, we can now easily estimate any of the constants A_i with help of any A_j, $j \neq i$. But it would be useful to estimate mutually not only the *suprema* of the functions $A_i(x)$, but also the functions themselves. One reason for this claim is based on the fact that the mapping

$$H \colon L^p(0, b; v) \to L^q(0, b; u) \tag{11}$$

with H the Hardy operator,

$$(Hf)(x) \colon = \int_0^x f(t)\, \mathrm{d}t,$$

which is *continuous* due to the Hardy inequality

$$\left(\int_0^b |(Hf)(x)|^q u(x)\, \mathrm{d}x \right)^{1/q} \leqq C \left(\int_0^b |f(t)|^p v(t)\, \mathrm{d}s \right)^{1/p} \tag{12}$$

if and only if any of the numbers A_i is finite [notice that the last inequality and inequality (6) for functions vanishing at zero are *equivalent*], is, moreover, *compact* if and only if

$$\lim_{x\to 0+} A_M(x) = 0, \ \lim_{x\to b-} A_M(x) = 0. \tag{13}$$

Consequently, if we would for example know, that

$$A_M(x) \approx A_T(x),$$

i.e., that there exist positive constants C_1, C_2 such that

$$C_1 A_M(x) \leqq A_T(x) \leqq C_2 A_M(x) \quad \text{for } x \in (0,b), \tag{14}$$

we could conclude from the properties of the function $A_M(x)$ to the properties of $A_T(x)$.

But unfortunately, estimates of the type (14) need not to hold:

Example 1. Let us take $u(t) = t^\alpha$, $v(t) = t^\beta$, $\alpha, \beta \in \mathbb{R}$. The condition $V(t) = \int_0^x v^{1-p'}(t)\,\mathrm{d}s < \infty$ leads to the claim

$$\beta < p - 1. \tag{15}$$

(i) Let us first consider the case $b = \infty$, i.e. investigate the validity of (6) [or (12)] on the interval $(0, \infty)$. Then all functions $A_M(x)$, $A_T(x)$, $A_W(x)$ and also $A_B(x, g)$ for $g = V$ are of the form

$$A_i(x) = c_i x^{\frac{\alpha+1}{q} - \frac{\beta+1}{p} + 1}$$

with some positive constants c_i. Consequently, the numbers $A_i\colon = \sup_{x>0} A_i(x)$ are finite if and only if the pair α, β satisfies

$$\frac{\alpha+1}{q} - \frac{\beta+1}{p} + 1 = 0.$$

For this pair, inequality (6) holds and the mapping (11) is continuous, but cannot be compact, since the conditions (13) cannot be satisfied.

(ii) Let us now consider the interval $(0, b)$ with $b < \infty$, say $b = 1$. Then we have again

$$A_T(x) = C_T x^\lambda, \quad \lambda = \frac{\alpha+1}{q} - \frac{\beta+1}{p} + 1$$

and the mapping M will be continuous provided

$$\frac{\alpha+1}{q} - \frac{\beta+1}{p} + 1 \geqq 0. \tag{16}$$

On the other hand, we have that

$$A_M(x) = C_M \left(\frac{1 - x^{\alpha+1}}{\alpha+1} \right)^{1/q} x^{-(\beta+1)/p+1}$$

with $\alpha \neq -1$ and $\beta < p - 1$. Consequently, the mapping H will be continuous if (14) holds, and moreover, it will be compact since $A_M(0) = A_M(1) = 0$.

But a comparison of the formulas of $A_M(x)$ and $A_T(x)$ shows that we never can expect that it would be $A_T(x) \approx A_M(x)$ for $x \in (0,1)$ due to the behaviour in the neighbourhood of $x = 1$ where $A_M(x)$ vanishes but $A_T(1) = C_T > 0$.

Up to now, we have only the estimates

$$A_W(x) \geqq A_M(x) \text{ and } A_5(x) \geqq A_M(x)$$

(see Remark 1 and Remark 2). Hence let us formulate two open questions:

Question 1. Is it possible to estimate at least some if the function $A_i(x)$ from above and from below by some other function $A_j(x)$, $j \neq i$?

Question 2. Is it possible to give conditions for the compactness of the mapping H in terms of some of the functions $A_i(x)$ (except $A_M(x)$)?

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