CONVOLUTION INEQUALITIES IN WEIGHTED LORENTZ SPACES

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Abstract. We characterize boundedness of a convolution operator with a fixed kernel between the weighted Lorentz spaces $\Lambda^p(v)$ and $\Gamma^q(w)$ for $0 < p \le q \le \infty$, $1 \le q < p < \infty$ and $0 < q \le p = \infty$. We provide corresponding weighted Young-type inequalities and also study basic properties of some new involved r.i. spaces.

1. Introduction

Methods involving convolution of a function f with a kernel function g, i.e.

$$(f * g)(t) = \int_{-\infty}^{\infty} f(x)g(t-x)\,\mathrm{d}x, \quad t \in \mathbb{R}, \tag{1}$$

have experienced a great attention and a widespread use in various important parts of analysis. By choosing a specific kernel in this general setting, we get many well-known operators, which themselves are of substantial importance. As examples here we can mention Newton, Riesz or Bessel potentials, Laplace, Stieltjes, Fourier, Hilbert transforms, mollifying operators, etc. One of the main questions in this field is the boundedness of the linear operator given by a fixed g and the formula

$$T_g : f \mapsto f * g$$

between certain function spaces. This problem is further related to convolution inequalities. The classic case is the well-known Young inequality stating that for $1 \le p, q, r \le \infty$ and $\frac{1}{p} + \frac{1}{r} = 1 + \frac{1}{q}$ it holds

$$\|f * g\|_q \le \|f\|_p \|g\|_r$$

for all measurable functions. Here $\|\cdot\|_p$ denotes the Lebesgue L^p-norm. The connection to the boundedness question is obvious: If X, Y, Z are given function spaces and the inequality

$$\|f * g\|_Z \le C\|f\|_X \|g\|_Y \tag{2}$$

is satisfied for all $f \in X$ and $g \in Y$, we get the boundedness $T_g : X \to Z$ for any $g \in Y$. On the other hand, if we have the estimate $\|T_g\|_{X \to Z} \le C\|g\|_Y$, then we retrieve (2).

The Young inequality was further developed for classical Lorentz spaces $L_{\alpha,\beta}$, $1 \le \alpha < \infty$, generated by

$$\|f\|_{L_{\alpha,\beta}} := \left(\int_0^{\infty} (f^*(x))^{\beta} x^{\frac{\beta}{\alpha}-1}\,\mathrm{d}x\right)^{\frac{1}{\beta}}, \quad 1 \le \beta < \infty,$$
$$\|f\|_{L_{\alpha,\infty}} := \sup_{x \in (0,\infty)} f^*(x) x^{\frac{1}{\alpha}},$$

and $L_{(\alpha,\beta)}$ generated by

$$\|f\|_{L_{(\alpha,\beta)}} := \|f^{**}\|_{L_{\alpha,\beta}}.$$

Here f^* stands for the nonincreasing rearrangement of f and f^{**} for the Hardy-Littlewood maximal function (see e.g. [1]).

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O'Neil [17] proved that, for $1 < a, b, c < \infty$ and $1 \le q < p \le \infty$ such that $1 + \frac{1}{a} = \frac{1}{b} + \frac{1}{c}$ and $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$, the inequality

$$\|f * g\|_{L_{a,q}} \le C\|f\|_{L_{b,p}}\|g\|_{L_{c,r}}, \quad f \in L_{b,p},\ g \in L_{c,r}, \tag{3}$$

is satisfied. This result was further improved in [11, 21] up to the range $0 < a, b, c < \infty$ and $1 \le q < p \le \infty$. Blozinski [2] showed that in a limit case of (3) with $a = b$ and $c = 1$, for an a.e. nonnegative g,

$$T_g : L_{p,b} \to L_{q,b}$$

holds if and only if $g = 0$ a.e. However, in a recent paper [15] Nursultanov and Tikhonov proved that the same problem has a nontrivial solution if we replace the interval of integration in (1) by $(0,1)$ and consider the convolution for 1-periodic functions. In that case the inequality

$$\|f * g\|_{L_{b,q}} \le C\|f\|_{L_{b,p}}\|g\|_{L_{(1,r)}}$$

was shown to be satisfied for all 1-periodic $f \in L_{b,p}$, $g \in L_{(1,r)}$. Here the functionals $\|\cdot\|_{L_{\alpha,\beta}}$, $\|\cdot\|_{L_{(\alpha,\beta)}}$ are naturally given just on $(0,1)$, as well.

In this paper, we provide necessary and sufficient conditions for the boundedness $T_g : \Lambda^p(v) \to \Gamma^q(w)$ for fixed weights v, w and various combinations of the parameters p, q. Moreover, we obtain Young-type inequalities (2) for $X = \Lambda^p(v)$, $Z = \Gamma^q(w)$ and characterize the largest rearrangement-invariant space Y for which these inequalities are valid.

To obtain these results we use the classical O'Neil inequality [17] and the weighted Hardy-type inequalities which have undergone a wide development in the last two decades. A survey of the classical cases may be found e.g. in [4], newer and more general results are developed and summarized in the upcoming article [10]. (For further related results see e.g. [13].) Our method enables us to obtain both the results for convolutions on $\mathbb{R}$ and on a finite interval.

Our paper proceeds in the following way: In Section 2 we present the definitions, state the problems and prove some preliminary results. Section 3 includes the main results, i.e. the weighted Young-type inequalities involving Λ and Γ spaces. In Section 4 we present some additional results and also verify that the results of [17, 2, 15] mentioned above follow as special cases of our theorems. Finally, Section 5 deals with some fundamental properties of function spaces which appear in the inequalities.

2. Preliminaries

Throughout the text we use the following notation: If Ω is a measurable subset of $\mathbb{R}$, we write $\mathscr{M}(\Omega) := \{f : \Omega \to \mathbb{R} \text{ measurable}\}$ and $\mathscr{M}_+(\Omega) := \{f \in \mathscr{M}(\Omega);\ f \ge 0 \text{ a.e.}\}$. In what follows, we will consider $m \in (0, \infty]$, unless specified else. We denote

$$\mathscr{P}_m := \begin{cases} \{f \in \mathscr{M};\ m\text{-periodic}\} & \text{if } m < \infty, \\ \mathscr{M} & \text{if } m = \infty, \end{cases}$$

and

$$\mathscr{E}_m := \left\{f \in \mathscr{P}_m;\ f \text{ even},\ f \ge 0 \text{ on } \mathbb{R},\ f \text{ on } \left(0, \tfrac{m}{2}\right)\right\}.$$

Notice that $\mathscr{P}_m, \mathscr{E}_m \subset \mathscr{M}\left(-\frac{m}{2}, \frac{m}{2}\right)$ in the sense of the restriction of f to $\left(-\frac{m}{2}, \frac{m}{2}\right)$. The usual notation $F \lesssim G$ means that $F \le CG$ where C is a constant independent of appropriate quantities in F and G. If $C^{-1}F \le G \le CF$ with such C, we write $F \simeq G$ and C is then called *the equivalence constant*. Next, a *weight* is a measurable nonnegative function on $(0, m)$. For such w we denote

$$W(t) := \int_0^t w(s)\,\mathrm{d}s, \quad t \in (0, m].$$

By L^1_{loc} we denote the set of all locally integrable functions on $\mathbb{R}$. For a weight w, the $L^q(w)$-norm of $f \in \mathscr{M}$ is given by

$$\|f\|_{L^q(w)} := \int_0^m |f(t)|^q w(t)\,\mathrm{d}t, \quad q < \infty,$$
$$\|f\|_{L^\infty(w)} := \operatorname*{ess\,sup}_{t \in (0,m)} |f(t)| w(t).$$

Let $f, g \in \mathscr{P}_m$. We define the *convolution* $f * g$ by

$$(f * g)(t) := \int_{-\frac{m}{2}}^{\frac{m}{2}} f(x) g(t-x) \, \mathrm{d}x, \tag{4}$$

if the right-hand side is well-defined for a.e. $t \in \left(-\frac{m}{2}, \frac{m}{2}\right)$. Notice that if $f * g$ is defined, then $f * g \in \mathscr{P}_m$.

For $f \in \mathscr{M}\left(-\frac{m}{2}, \frac{m}{2}\right)$ we define the *nonincreasing rearrangement* of f by

$$f^*(t) := \inf\left\{s \geq 0; \ \left|\left\{\tau \in \left(-\frac{m}{2}, \frac{m}{2}\right), \ |f(\tau)| > s\right\}\right| \leq t\right\}, \quad t \in (0, m), \tag{5}$$

and the *maximal function* f^{**} by

$$f^{**}(t) := \frac{1}{t} \int_0^t f^*(s) \, \mathrm{d}s, \quad t \in (0, m), \tag{6}$$

see e.g. [1]. Observe that, although the m-periodic function f (for $m < \infty$) is defined on $\mathbb{R}$, the above defined rearrangement of f represents just the rearrangement of f's restriction to the interval of periodicity. If $m = \infty$, we get the "standard" rearrangement and convolution on $\mathbb{R}$. This approach will allow us to cover the results for both finite and infinite m by a single theorem.

Let $g \in \mathscr{P}_m$. We consider the operator T_g defined by

$$T_g : f \mapsto f * g, \tag{7}$$

acting on all functions $f \in \mathscr{P}_m$ for which $f * g$ is defined. We will study the boundedness

$$T_g : \Lambda^p(v) \to \Gamma^q(w),$$

where v, w are weights on $(0, m)$ and $\Lambda^p(v)$, $\Gamma^q(w)$ are the weighted Lorentz spaces defined as

$$\Lambda^p(v) := \left\{ f \in \mathscr{P}_m; \ \|f\|_{\Lambda^p(v)} := \left(\int_0^m (f^*(x))^p v(x) \, \mathrm{d}x \right)^{\frac{1}{p}} < \infty \right\},$$

$$\Gamma^q(w) := \left\{ f \in \mathscr{P}_m; \ \|f\|_{\Gamma^q(w)} := \left(\int_0^m (f^{**}(x))^q w(x) \, \mathrm{d}x \right)^{\frac{1}{q}} < \infty \right\}$$

for $p, q \in (0, \infty)$, and

$$\Lambda^\infty(v) := \left\{ f \in \mathscr{P}_m; \ \|f\|_{\Lambda^\infty(v)} := \operatorname*{ess\,sup}_{x \in (0,m)} f^*(x) v(x) < \infty \right\},$$

$$\Gamma^\infty(w) := \left\{ f \in \mathscr{P}_m; \ \|f\|_{\Gamma^\infty(w)} := \operatorname*{ess\,sup}_{x \in (0,m)} f^{**}(x) w(x) < \infty \right\}.$$

Of course, for $m < \infty$, the Λ or Γ norm of $f \in \mathscr{P}_m$ controls just the behavior of f on the periodical segment.

Our first aim is the following: Given weights v, w and exponents p, q, we want to find sufficient conditions on the kernel g under which $T_g : \Lambda^p(v) \to \Gamma^q(w)$ is bounded, i.e.

$$\|f * g\|_{\Gamma^q(w)} = \|T_g f\|_{\Gamma^q(w)} \leq C \|f\|_{\Lambda^p(v)}, \quad f \in \Lambda^p(v), \tag{8}$$

and to obtain estimates for the optimal constant $C = \|T_g\|_{\Lambda^p(v) \to \Gamma^q(w)}$ in terms of g. Recall that the operator norm of T_g is given by

$$\|T_g\|_{X \to Z} := \sup_{\|f\|_X \leq 1} \|T_g f\|_Z.$$

Let us formally put $\|T_g\|_{X \to Z} := \infty$ if there exists a function $f \in X$ such that $T_g f$ is not defined.

In addition to this, it will be shown that if $g \in \mathscr{E}_m$, then the sufficient conditions are also necessary for the boundedness $T_g : \Lambda^p(v) \to \Gamma^q(w)$.

Later on, we will see that $\|T_g\|_{\Lambda^p(v) \to \Gamma^q(w)}$ is estimated from above by a norm of g in an r.i. space Y. (In case of $g \in \mathscr{E}_m$, it will even hold $\|T_g\|_{\Lambda^p(v) \to \Gamma^q(w)} \simeq \|g\|_Y$.) This will allow us to write the result in the form of a Young-O'Neil inequality

$$\|f * g\|_{\Gamma^q(w)} \lesssim \|f\|_{\Lambda^p(v)} \|g\|_Y, \quad f \in \Lambda^p(v), \ g \in Y. \tag{9}$$

Moreover, the space Y will be optimal in the following sense:

Definition 2.1. Let X, Y, Z be r.i. spaces. We say that Y is *optimal for the pair* (X, Z) if the following is satisfied: If Y' is an r.i. space such that $Y \hookrightarrow Y'$ and

$$\|f * g\|_Z \lesssim \|f\|_X \|g\|_{Y'}, \quad f \in X, \ g \in Y', \tag{10}$$

then $Y = Y'$ with $\|\cdot\|_Y \simeq \|\cdot\|_{Y'}$.

In other words, the optimal space for (X, Z) is the essentially largest one for which (10) is satisfied.

The key result in our method is the O'Neil inequality:

Lemma 2.2. *Let $m \in (0, \infty]$ and $f, g \in \mathscr{P}_m \cap L^1_{\mathrm{loc}}$. Then, for every $t \in (0, m)$ it holds*

$$(f * g)^{**}(t) \le t f^{**}(t) g^{**}(t) + \int_t^m f^*(s) g^*(s) \, \mathrm{d}s. \tag{11}$$

Proof. See [17, Lemma 2.5]. □

Observe that for convolutions both on a bounded and unbounded interval we get the same estimate (11) which allows us to treat the two cases at once, as mentioned before.

Furthermore, we are going to use the fact that the O'Neil inequality is sharp in the following sense:

Lemma 2.3. *Let $m \in (0, \infty]$. Let $f, g \in \mathscr{E}_m \cap L^1_{\mathrm{loc}}$. Then for every $t \in (0, m)$ it holds*

$$t f^{**}(t) g^{**}(t) + \int_t^m f^*(y) g^*(y) \, \mathrm{d}y \le 12 (f * g)^{**}(t). \tag{12}$$

Proof. The result was mentioned in [17] without proof. A part of the proof is sketched e.g. in [19, Remark, p. 145]. For the convenience of the reader, we present the whole proof here.

Let $m \in (0, \infty]$ and $f, g \in \mathscr{E}_m \cap L^1_{\mathrm{loc}}$. According to the symmetry, we observe that $f(t) = f^*(2t)$ and $g(t) = g^*(2t)$ for all $t \in \left(0, \frac{m}{2}\right)$. Now let $t \in \left(0, \frac{m}{2}\right)$ be fixed. Then

$$\begin{aligned} \int_0^t f(t-x) g(x) \, \mathrm{d}x \ge g(t) \int_0^t f(t-x) \, \mathrm{d}x = g(t) \int_0^t f(x) \, \mathrm{d}x = \\ = g^*(2t) \int_0^t f^*(2x) \, \mathrm{d}x = \frac{g^*(2t)}{2} \int_0^{2t} f^*(x) \, \mathrm{d}x. \end{aligned}$$

Next,

$$\begin{aligned} \int_t^{\frac{m}{2}} f(t-x) g(x) \, \mathrm{d}x = \int_t^{\frac{m}{2}} f(x-t) g(x) \, \mathrm{d}x \ge \int_t^{\frac{m}{2}} f(x) g(x) \, \mathrm{d}x = \\ = \int_t^{\frac{m}{2}} f^*(2x) g^*(2x) \, \mathrm{d}x = \frac{1}{2} \int_{2t}^m f^*(x) g^*(x) \, \mathrm{d}x. \end{aligned}$$

Thus it holds

$$\begin{aligned} (f * g)(t) \ge \int_0^t f(t-x) g(x) \, \mathrm{d}x + \int_t^{\frac{m}{2}} f(t-x) g(x) \, \mathrm{d}x \ge \\ \ge \frac{1}{2} \left(g^*(2t) \int_0^{2t} f^*(x) \, \mathrm{d}x + \int_{2t}^m f^*(x) g^*(x) \, \mathrm{d}x \right). \end{aligned}$$

Hence, we get $g^*(2t) \int_0^{2t} f^*(x) \, \mathrm{d}x + \int_{2t}^m f^*(x) g^*(x) \, \mathrm{d}x \le 2(f * g)(t)$ and since the left side is nonincreasing, we obtain

$$g^*(2t) \int_0^{2t} f^*(x) \, \mathrm{d}x + \int_{2t}^m f^*(x) g^*(x) \, \mathrm{d}x \le 2(f * g)^*(t). \tag{13}$$

Now, using Fubini theorem and (13) (once as it is and once with f and g having changed places), we write

$$\begin{aligned}2tg^{**}(t)f^{**}(t) &= \frac{1}{2t}\int_0^{2t} g^*(y)\,\mathrm{d}y \int_0^{2t} f^*(x)\,\mathrm{d}x = \\ &= \frac{1}{2t}\int_0^{2t} g^*(y)\int_0^{y} f^*(x)\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{2t}\int_0^{2t} g^*(y)\int_y^{2t} f^*(x)\,\mathrm{d}x\,\mathrm{d}y = \\ &= \frac{1}{2t}\int_0^{2t} g^*(y)\int_0^{y} f^*(x)\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{2t}\int_0^{2t} f^*(x)\int_0^{x} g^*(y)\,\mathrm{d}y\,\mathrm{d}x \le \\ &\le \frac{2}{t}\int_0^{2t} (f*g)^*\left(\frac{y}{2}\right)\mathrm{d}y = \\ &= \frac{4}{t}\int_0^{t} (f*g)^*(y)\,\mathrm{d}y = \\ &= 4(f*g)^{**}(t).\end{aligned}$$

Combining this and (13), we finally proceed to

$$\begin{aligned}2tg^{**}(2t)f^{**}(2t) + \int_{2t}^{m} f^*(x)g^*(x)\,\mathrm{d}x &\le 4(f*g)^{**}(t) + 2(f*g)^*(t) \le \\ &\le 6(f*g)^{**}(t) \le 12(f*g)^{**}(2t).\end{aligned}$$

Since $t \in \left(0, \frac{m}{2}\right)$, we have proved (12). □

Remark 2.4. Let $a, b \in \mathbb{R}$ and $\widetilde{f}, \widetilde{g} \in \mathscr{E}_m \cap L^1_{\mathrm{loc}}$. Then the inequality (12) is actually satisfied for any $f, g \in L^1_{\mathrm{loc}}$ such that $f(t) = \widetilde{f}(t+a)$ and $g(t) = \widetilde{g}(t+b)$ for all $t \in \mathbb{R}$. It follows from the fact that $(f*g)^* = (\widetilde{f}*\widetilde{g})^*$.

3. Main results

We start this section with the general theorem below. It treats the boundedness of the operator T_g between an r.i. lattice X and $\Gamma^q(w)$. The term "r.i. lattice" (cf. e.g. [4]) is used here because $\Lambda^p(v)$, which is the choice of X we are interested in, is not necessarily a normed (not even quasi-normed) linear space (see e.g. [7] and the references therein).

Theorem 3.1. *Let $m \in (0, \infty]$. Let X be an r.i. lattice over $\left(-\frac{m}{2}, \frac{m}{2}\right)$ and let $g \in \mathscr{P}_m$. Let w be a weight and $q \in (0, \infty]$. For $f \in \mathscr{P}_m$, $t \in (0, m)$ put*

$$R^1_g f(t) := t f^{**}(t) g^{**}(t), \quad R^2_g f(t) := \int_t^m f^*(s) g^*(s)\,\mathrm{d}s, \quad R_g f(t) := R^1_g f(t) + R^2_g f(t).$$

Then

(i) *If $R_g : X \to L^q(w)$ is bounded, then $T_g : X \to \Gamma^q(w)$ is bounded and*
$$\|T_g\|_{X\to\Gamma^q(w)} \lesssim \|R_g\|_{X\to L^q(w)} < \infty.$$
(ii) *Let $g \in \mathscr{E}_m$. If $T_g : X \to \Gamma^q(w)$ is bounded, then $R_g : X \to L^q(w)$ is bounded and*
$$\|R_g\|_{X\to L^q(w)} \lesssim \|T_g\|_{X\to\Gamma^q(w)} < \infty.$$
(iii) *If there exists an r.i. space Y over $\left(-\frac{m}{2}, \frac{m}{2}\right)$ such that, for all $g \in \mathscr{P}_m$, it holds $\|R_g\|_{X\to L^q(w)} \simeq \|g\|_Y$, then Y is optimal for $(X, \Gamma^q(w))$.*

Proof. (i) It holds $\|R_{|g|}\|_{X\to L^q(w)} = \|R_g\|_{X\to L^q(w)} < \infty$. Thus, for any $f \in X$, it holds $R_{|g|}|f|(t) < \infty$ for $t \in (0, m)$. From (11) we get $(T_{|g|}|f|)^{**}(t) \le R_{|g|}|f|(t) < \infty$ for $t \in (0, m)$, therefore $T_{|g|}|f|(t) < \infty$ for a.e. $t \in (0, m)$. Thus, $|T_g f(t)| \le T_{|g|}|f|(t)$ for a.e. $t \in (0, m)$, so T_g is well-defined on X. Next, we get

$$\|T_g\|_{X\to\Gamma^q(w)} = \sup_{\|f\|_X \le 1} \|(T_g f)^{**}\|_{L^q(w)} \le \sup_{\|f\|_X \le 1} \|R_g f\|_{L^q(w)} = \|R_g\|_{X\to L^q(w)}.$$

(ii) Let $g \in \mathscr{E}_m$ and $T_g : X \to \Gamma^q(w)$ be bounded. By definition of the operator norm, there exists a sequence $\{f_n\}_{n\in\mathbb{N}}$ of functions such that $\|f_n\|_X \le 1$ for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \|R_g f_n\|_{L^q(w)} = \|R_g\|_{X\to L^q(w)}.$$

Since $R_g f = R_g \widetilde{f}$ if $f^* = \widetilde{f}^*$, we may assume that $f_n \in \mathscr{E}_m$, $n \in \mathbb{N}$. Thus, by Lemma 2.3 we obtain $\|R_g f_n\|_{L^q(w)} \le 12 \|f_n * g\|_{\Gamma^q(w)}$, hence

$$\tfrac{1}{12} \|R_g\|_{X \to L^q(w)} = \tfrac{1}{12} \lim_{n\to\infty} \|R_g f_n\|_{L^q(w)} \le \liminf_{n\to\infty} \|f_n * g\|_{\Gamma^q(w)} \le \|T_g\|_{X\to\Gamma^q(w)},$$

so the proof of this part is finished.

(iii) If $g \in Y$, we get

$$\|f * g\|_{\Gamma^q(w)} = \|T_g f\|_{\Gamma^q(w)} \lesssim \|f\|_X \|T_g\|_{X\to\Gamma^q(w)} \simeq \|f\|_X \|R_g\|_{X\to L^q(w)} \lesssim \|f\|_X \|g\|_Y,$$

hence (10) holds with the given Y. Now let Y' be an r.i. space such that $Y \hookrightarrow Y'$ but $Y' \not\hookrightarrow Y$. Then for every $n \in \mathbb{N}$ there exists a function $g_n \in Y'$ such that $n\|g_n\|_{Y'} < \|g_n\|_Y$. Since Y, Y' are r.i., we may assume that all $g_n \in \mathscr{E}_m$, $n \in \mathbb{N}$. For each g_n we find $f_n \in X$ such that

$$\|T_{g_n} f_n\|_{\Gamma^q(w)} \ge \tfrac{1}{2} \|f_n\|_X \|T_{g_n}\|_{X\to\Gamma^q(w)} \simeq \|f_n\|_X \|R_{g_n}\|_{X\to L^q(w)} \simeq \|f_n\|_X \|g_n\|_Y,$$

where the second equivalence follows from (i) and (ii). Hence, we now obtain

$$\|f_n * g_n\|_{\Gamma^q(w)} = \|T_{g_n} f_n\|_{\Gamma^q(w)} \gtrsim \|f_n\|_X \|g_n\|_Y > n\|f_n\|_X \|g_n\|_{Y'}.$$

Thus, if we replace Y with Y' in (10), the inequality ceases to hold, so Y is optimal for $(X, \Gamma^q(w))$. □

Now we are ready to bring the desired results about the convolution operator between $\Lambda^p(v)$ and $\Gamma^q(w)$. We are going to characterize the norm $\|\cdot\|_Y$ of the r.i. space $Y := \{h \in \mathscr{P}_m;\ \|h\|_Y < \infty\}$ which is optimal for $(\Lambda^p(v), \Gamma^q(w))$ in (9). The form of the results varies depending on the mutual relation of p and q. We need to find estimates on $\|R_g^1\|_{\Lambda^p(v)\to L^q(w)}$, $\|R_g^2\|_{\Lambda^p(v)\to L^q(w)}$. The norm $\|R_g^1\|_{\Lambda^p(v)\to L^q(w)}$ equals the best constant C_1 such that

$$\left(\int_0^m (f^{**}(t))^q t^q (g^{**}(t))^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} \le C_1 \left(\int_0^m (f^*(t))^p v(t)\,\mathrm{d}t\right)^{\frac{1}{p}}, \quad f \in \mathscr{M}\left(-\tfrac{m}{2}, \tfrac{m}{2}\right), \tag{14}$$

holds, while $\|R_g^2\|_{\Lambda^p(v)\to L^q(w)}$ equals the best C_2 in

$$\left(\int_0^m \left(\int_t^m f^*(s) g^*(s)\,\mathrm{d}s\right)^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} \le C_2 \left(\int_0^m (f^*(t))^p v(t)\,\mathrm{d}t\right)^{\frac{1}{p}}, \quad f \in \mathscr{M}\left(-\tfrac{m}{2}, \tfrac{m}{2}\right). \tag{15}$$

Both (14) and (15) are Hardy-type inequalities for monotone functions and the optimal constants C_1, C_2 have been fully characterized. The inequality (14) represents the embedding $\Lambda \hookrightarrow \Gamma$ (see e.g. [4, 3]). A similar survey of (15) may be found e.g. in [10]. Direct references are given in the proof of Theorem 3.2 below.

In what follows, we will use the fact that for any $m \in (0,\infty]$ and any $\varphi, \psi \in \mathscr{M}_+(\mathbb{R})$ it holds

$$\sup_{x\in(0,m)} \varphi(x) + \sup_{x\in(0,m)} \psi(x) \simeq \sup_{x\in(0,m)} [\varphi(x) + \psi(x)].$$

We also apply the convention "$\frac{\infty}{\infty} := 0$".

Theorem 3.2. *Let $m \in (0,\infty]$ and let v, w be weights. For $g \in \mathscr{P}_m$ let $\|g\|_Y$ be given by the following:*

(i) *If $0 < p \le 1$, $p \le q < \infty$, let*

$$\|g\|_Y := \sup_{x\in(0,m)} x V^{-\frac{1}{p}}(x) \left[(g^{**}(x))^q W(x) + \int_x^m (g^{**}(t))^q w(t)\,\mathrm{d}t\right]^{\frac{1}{q}}.$$

(ii) *If $1 < p \le q < \infty$, let*

$$\begin{aligned}\|g\|_Y := \sup_{x\in(0,m)} &\left(\int_x^m (g^{**}(t))^{p'} t^{p'} V^{-p'}(t) v(t)\,\mathrm{d}t\right)^{\frac{1}{p'}} W^{\frac{1}{q}}(x) + \\ &+ (g^{**}(x))^q x^q W^{\frac{1}{q}}(x) V^{-\frac{1}{p}}(x) + \\ &+ \left(\int_0^x t^{p'} V^{-p'}(t) v(t)\,\mathrm{d}t\right)^{\frac{1}{p'}} \left(\int_x^m (g^{**}(t))^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}}.\end{aligned}$$

(iii) *If* $1 < q < p < \infty$*, let*

$$\begin{aligned}\|g\|_Y := \Bigg[\int_0^m \left(\int_x^m (g^{**}(t))^q w(t)\,\mathrm{d}t\right)^{\frac{r}{q}} \left(\int_0^x t^{p'} V^{-p'}(t) v(t)\,\mathrm{d}t\right)^{\frac{r}{q'}} x^{p'} V^{-p'}(x) v(x) + \\ + (g^{**}(x))^r x^r W^{\frac{r}{q}}(x) V^{-\frac{r}{q}}(x) v(x) + \\ + W^{\frac{r}{p}}(x) w(x) \left(\int_x^m (g^{**}(t))^{p'} t^{p'} V^{-p'}(t) v(t)\,\mathrm{d}t\right)^{\frac{r}{p'}} \,\mathrm{d}x\Bigg]^{\frac{1}{r}} + \\ + \left(\int_0^m x^q (g^{**}(x))^q w(x)\,\mathrm{d}x\right)^{\frac{1}{q}} V^{-\frac{1}{p}}(m).\end{aligned}$$

(iv) *If* $1 = q < p < \infty$*, let*

$$\begin{aligned}\|g\|_Y := \left[\int_0^m \left(g^{**}(x) W(x) + \int_x^m g^{**}(t) w(t)\,\mathrm{d}t\right)^{p'} x^{p'} V^{-p'}(x) v(x)\,\mathrm{d}x\right]^{\frac{1}{p'}} + \\ + \int_0^m x g^{**}(x) w(x)\,\mathrm{d}x \; V^{-\frac{1}{p}}(m).\end{aligned}$$

Then, for each combination of p, q *from the previous, the inequality* (9) *is satisfied. If* $g \in \mathscr{E}_m$*, then* $\|T_g\|_{\Lambda^p(v)\to\Gamma^q(w)} \simeq \|g\|_Y$*. The space* $(Y, \|\cdot\|_Y)$ *is optimal for the pair* $(\Lambda^p(v), \Gamma^q(w))$*.*

Proof. As for checking that Y generated by $\|\cdot\|_Y$ in each of the cases is a (quasi-)normed r.i. space, we refer to Theorem 5.2.

Now let us focus on the main part of the proof: At first, obviously it holds $\|R_g\|_{\Lambda^p(v)\to L^q(w)} \simeq \|R_g^1\|_{\Lambda^p(v)\to L^q(w)} + \|R_g^2\|_{\Lambda^p(v)\to L^q(w)}$. In each case (i)-(iv), we will use the known equivalent estimates of $\|R_g^1\|_{\Lambda^p(v)\to L^q(w)}$, $\|R_g^2\|_{\Lambda^p(v)\to L^q(w)}$. They have a form of certain functionals of g and we will show that, when added together, they actually form a norm of g in Y, i.e. $\|g\|_Y \simeq \|R_g^1\|_{\Lambda^p(v)\to L^q(w)} + \|R_g^2\|_{\Lambda^p(v)\to L^q(w)}$ for every $g \in \mathscr{P}_m$.

Then the results will follow from Theorem 3.1: By its (i) part, if $g \in Y$, then $T_g : \Lambda^p(v) \to \Gamma^q(w)$ is bounded and $\|T_g\|_{\Lambda^p(v)\to\Gamma^q(w)} \lesssim \|g\|_Y$, hence (9) is satisfied. By Theorem 3.1(ii), if $g \in \mathscr{E}_m$, then we get even $\|T_g\|_{\Lambda^p(v)\to\Gamma^q(w)} \simeq \|g\|_Y$. Theorem 3.1(iii) then implies the optimality of Y.

So, in each case we just need to check that $\|R_g^1\|_{\Lambda^p(v)\to L^q(w)} + \|R_g^2\|_{\Lambda^p(v)\to L^q(w)}$, obtained from the appropriate Hardy-type inequalities, are equivalent to $\|g\|_Y$ for any $g \in \mathscr{P}_m$.

(i) By [20, Theorem 3(b)] and [14, Theorem 2.1(a)] we get

$$\begin{aligned}\|R_g^1\|_{\Lambda^p(v)\to L^q(w)} &\simeq \sup_{x\in(0,m)} V^{-\frac{1}{p}}(x)\left[x\left(\int_x^m (g^{**}(t))^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} + \left(\int_0^x t^q (g^{**}(t))^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}}\right], \\ \|R_g^2\|_{\Lambda^p(v)\to L^q(w)} &\simeq \sup_{x\in(0,m)} V^{-\frac{1}{p}}(x)\left(\int_0^x \left(\int_t^x g^*(s)\,\mathrm{d}s\right)^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}}.\end{aligned}$$

Obviously, $\|R_g^1\|_{\Lambda^p(v)\to L^q(w)} + \|R_g^2\|_{\Lambda^p(v)\to L^q(w)} \simeq \|g\|_Y$.

(ii) From [18, Theorem 2] and the dual version of [16, Theorem 1.1] it follows:

$$\begin{aligned}\|R_g^1\|_{\Lambda^p(v)\to L^q(w)} &\simeq \sup_{x\in(0,m)} \left(\int_x^m (g^{**}(t))^q w(t)\,\mathrm{d}t\right)^{\frac1q}\left(\int_0^x t^{p'}V^{-p'}(t)v(t)\,\mathrm{d}t\right)^{\frac{1}{p'}} + \\ &+ \sup_{x\in(0,m)} \left(\int_0^x t^q (g^{**}(t))^q w(t)\,\mathrm{d}t\right)^{\frac1q} V^{-\frac1p}(x) \\ &=: A_1 + A_2, \\ \|R_g^2\|_{\Lambda^p(v)\to L^q(w)} &\simeq \sup_{x\in(0,m)} \left(\int_x^m \left(\int_x^t g^*(s)\,\mathrm{d}s\right)^{p'} V^{-p'}(t)v(t)\,\mathrm{d}t\right)^{\frac{1}{p'}} W^{\frac1q}(x) + \\ &+ \sup_{x\in(0,m)} \left(\int_0^x \left(\int_t^x g^*(s)\,\mathrm{d}s\right)^q w(t)\,\mathrm{d}t\right)^{\frac1q} V^{-\frac1p}(x) \\ &=: A_3 + A_4.\end{aligned}$$

Since for every $x\in(0,m)$ it holds

$$V^{-\frac1p}(x) \geq \left(V^{1-p'}(x) - V^{1-p'}(m)\right)^{\frac{1}{p'}} = \left(\int_x^m \left(-V^{1-p'}\right)'(t)\,\mathrm{d}t\right)^{\frac{1}{p'}} \simeq \left(\int_x^m V^{-p'}(t)v(t)\,\mathrm{d}t\right)^{\frac{1}{p'}}, \tag{16}$$

we get

$$\left(\int_0^x \left(\int_0^x g^*(s)\,\mathrm{d}s\right)^q w(t)\,\mathrm{d}t\right)^{\frac1q}\left(\int_x^m V^{-p'}v(t)\,\mathrm{d}t\right)^{\frac{1}{p'}} \lesssim \left(\int_0^x \left(\int_0^x g^*(s)\,\mathrm{d}s\right)^q w(t)\,\mathrm{d}t\right)^{\frac1q} V^{-\frac1p}(x)$$

and so

$$A_5 := \sup_{x\in(0,m)} \left(\int_x^m (g^{**}(t))^{p'} t^{p'} V^{-p'}(t)v(t)\,\mathrm{d}t\right)^{\frac{1}{p'}} W^{\frac1q}(x) \lesssim A_2 + A_3 + A_4.$$

Observe also that $A_3 \lesssim A_5$. Hence

$$\begin{aligned}\|R_g^1\|_{\Lambda^p(v)\to L^q(w)} + \|R_g^2\|_{\Lambda^p(v)\to L^q(w)} &\lesssim A_1 + A_2 + A_3 + A_4 \leq A_1 + A_2 + A_3 + A_4 + A_5 \lesssim \\ &\lesssim A_1 + A_2 + A_4 + A_5 \lesssim \\ &\lesssim A_1 + A_2 + A_3 + A_4 \lesssim \|R_g^1\|_{\Lambda^p(v)\to L^q(w)} + \|R_g^2\|_{\Lambda^p(v)\to L^q(w)}.\end{aligned}$$

Since $A_1 + A_2 + A_4 + A_5 \simeq \|g\|_Y$, we have obtained $\|R_g^1\|_{\Lambda^p(v)\to L^q(w)} + \|R_g^2\|_{\Lambda^p(v)\to L^q(w)} \simeq \|g\|_Y$.

(iii) In this case [18, Theorem 2] and the dual version of [16, Theorem 1.2] (cf. also [4, Theorem 4.1] and [10, Theorem 5.1]) yield

$$\begin{aligned}&\|R_g^1\|_{\Lambda^p(v)\to L^q(w)} \simeq \\ &\simeq \left(\int_0^m \left(\int_t^\infty (g^{**}(x))^q w(x)\,\mathrm{d}x\right)^{\frac rq}\left(\int_0^t x^{p'}V^{-p'}(x)v(x)\,\mathrm{d}x\right)^{\frac{r}{q'}} t^{p'}V^{-p'}(t)v(t)\,\mathrm{d}t\right)^{\frac1r} + \\ &+ \left(\int_0^m \left(\int_0^x t^q (g^{**}(t))^q w(t)\,\mathrm{d}t\right)^{\frac rq} V^{-\frac rq}(x)v(x)\,\mathrm{d}x\right)^{\frac1r} + \\ &+ \left(\int_0^m x^q (g^{**}(x))^q w(x)\,\mathrm{d}x\right)^{\frac1q} V^{-\frac1p}(m) \\ &=: A_1 + A_2 + A_3,\end{aligned}$$

$$\begin{aligned}\|R_g^2\|_{\Lambda^p(v)\to L^q(w)} &\simeq \left(\int_0^m \left(\int_0^x \left(\int_t^x g^*(s)\,\mathrm{d}s\right)^q w(t)\,\mathrm{d}t\right)^{\frac{r}{q}} V^{-\frac{r}{q}}(x)v(x)\,\mathrm{d}x\right)^{\frac{1}{r}} + \\ &+ \left(\int_0^\infty \left(\int_x^m \left(\int_x^t g^*(s)\,\mathrm{d}s\right)^{p'} V^{-p'}(t)v(t)\,\mathrm{d}t\right)^{\frac{r}{p'}} W^{\frac{r}{p}}(x)w(x)\,\mathrm{d}x\right)^{\frac{1}{r}} \\ &=: A_4 + A_5.\end{aligned}$$

Clearly it holds

$$A_2 + A_4 \simeq \left(\int_0^m \left(\int_0^x g^*(s)\,\mathrm{d}s\right)^r W^{\frac{r}{q}}(x)V^{-\frac{r}{q}}(x)v(x)\,\mathrm{d}x\right)^{\frac{1}{r}} =: A_6.$$

Integration by parts and (16) provides that for all $t \in (0,m)$ we have

$$\int_t^m W^{\frac{r}{p}}(x)\left(\int_x^m V^{-p'}v\right)^{\frac{r}{p'}} w(x)\,\mathrm{d}x \lesssim \int_t^m W^{\frac{r}{q}}(x)V^{-\frac{r}{q}}(x)v(x)\,\mathrm{d}x.$$

The function $x \mapsto \left(\int_0^x g^*(s)\,\mathrm{d}s\right)^r$ is nondecreasing, so by Hardy's lemma (an analogue of [1, Proposition 3.6, p. 56]) we obtain

$$A_7 := \left(\int_0^m \left(\int_0^x g^*(s)\,\mathrm{d}s\right)^r W^{\frac{r}{p}}(x)\left(\int_x^m V^{-p'}v\right)^{\frac{r}{p'}} w(x)\,\mathrm{d}x\right)^{\frac{1}{r}} \lesssim A_6,$$

thus also $A_5 + A_7 \lesssim A_2 + A_4 + A_5$. Next, we can write

$$A_5 + A_7 \simeq \left(\int_0^m W^{\frac{r}{p}}(x)w(x)\left(\int_x^m (g^{**}(t))^{p'}t^{p'}V^{-p'}(t)v(t)\,\mathrm{d}t\right)^{\frac{r}{p'}}\,\mathrm{d}x\right)^{\frac{1}{r}} =: A_8,$$

hence putting all the estimates together yields

$$A_2 + A_4 + A_5 \lesssim A_6 + A_8 \lesssim A_2 + A_4 + A_5$$

and so finally $\|R_g^1\|_{\Lambda^p(v)\to L^q(w)} + \|R_g^2\|_{\Lambda^p(v)\to L^q(w)} \simeq A_1 + A_3 + A_6 + A_8 \simeq \|g\|_Y$.

(iv) By [4, Theorem 4.1(iv)] and [10, Theorem 5.1(v)] we have

$$\begin{aligned}\|R_g^1\|_{\Lambda^p(v)\to L^q(w)} &\simeq \left(\int_0^m \left(\int_0^x t g^{**}(t)w(t)\,\mathrm{d}t\right)^{p'} V^{-p'}(x)v(x)\,\mathrm{d}x\right)^{\frac{1}{p'}} + \\ &+ \left(\int_0^m \left(\int_x^\infty g^{**}(t)w(t)\,\mathrm{d}t\right)^{p'} x^{p'}V^{-p'}(x)v(x)\,\mathrm{d}x\right)^{\frac{1}{p'}} + \\ &+ \int_0^m x g^{**}(x)w(x)\,\mathrm{d}x\ V^{-\frac{1}{p}}(\infty) \\ &=: A_1 + A_2 + A_3,\end{aligned}$$

$$\|R_g^2\|_{\Lambda^p(v)\to L^q(w)} \simeq \left(\int_0^m \left(\int_0^x \int_t^x g^*(y)\,\mathrm{d}y\ w(t)\,\mathrm{d}t\right)^{p'} V^{-p'}(x)v(x)\,\mathrm{d}x\right)^{\frac{1}{p'}}.$$

Clearly,

$$A_1 + \|R_g^2\|_{\Lambda^p(v)\to L^q(w)} \simeq \left(\int_0^m (g^{**}(x))^{p'}x^{p'}W^{p'}(x)V^{-p'}(x)v(x)\,\mathrm{d}x\right)^{\frac{1}{p'}} =: A_4,$$

hence $\|R_g^1\|_{\Lambda^p(v)\to L^q(w)} + \|R_g^2\|_{\Lambda^p(v)\to L^q(w)} \simeq A_2 + A_3 + A_4 \simeq \|g\|_Y$. ☐

For a given combination of weights v, w and exponents p, q in Theorems 3.2-3.6 we got the optimal space $(Y, \|\cdot\|_Y)$. However, this space may consist only of a.e. zero functions. In such case we have the following observation:

Corollary 3.3. *Let $m \in (0,\infty]$, $p, q \in (0,\infty]$, let v, w be weights. Let the optimal space Y for $(\Lambda^p(v), \Gamma^q(w))$ in (9) satisfy $Y = \{0\}$. Let $g \in \mathscr{P}_m$ be nonnegative a.e. and such that $T_g : \Lambda^p(v) \to \Gamma^q(w)$ is bounded. Then $g = 0$ a.e.*

Proof. Let $g \in \mathscr{P}_m$ be nonnegative and $g \not\equiv 0$ in measure. Then there exist $\varepsilon > 0$, $a, b \in \left(-\frac{m}{2}, \frac{m}{2}\right)$ and $h = \varepsilon\chi_{(a,b)}$ such that $h \le g$ a.e. Since $h \not\equiv 0$, it holds $\|h\|_Y = \infty$ and therefore, by Theorem 3.1(ii) and Remark 2.4, T_h is not bounded between $\Lambda^p(v)$ and $\Gamma^q(w)$. Since $0 \le h \le g$, for every nonnegative $f \in \Lambda^p(v)$ we get $0 \le T_h f \le T_g f$. Thus also $(T_h f)^* \le (T_g f)^*$ and it follows that T_g is not bounded between $\Lambda^p(v)$ and $\Gamma^q(w)$. □

Remark 3.4. In general, functions from $\Lambda^p(v)$ do not have to be locally integrable. In particular, for $p \in (0, \infty)$, we know that $\Lambda^p(v) \subset L^1_{\mathrm{loc}}$ if and only if one of the following conditions is satisfied (cf. [4, 18, 20]):

(a) $p \in (0, 1]$ and $\limsup_{t\to 0+} tV^{-\frac{1}{p}}(t) < \infty$,

(b) $p \in (1, \infty)$ and there exists $\varepsilon > 0$ such that $\int_0^\varepsilon t^{p'-1}V^{1-p'}(t)\,\mathrm{d}t < \infty$.

Let $\Lambda^p(v) \not\subset L^1_{\mathrm{loc}}$. Then T_g is well-defined on $\Lambda^p(v)$ if and only if $g = 0$ a.e. One may directly check that $Y = \{0\}$ in all cases of Theorem 3.2(i)-(iv). Hence, this theorem (trivially) holds even for $\Lambda^p(v) \not\subset L^1_{\mathrm{loc}}$, thus we do not assume (a) or (b) in its statement.

Now we state the results for the weak-type spaces. The way of proving them is the same as in Theorem 3.2. Analogues of Corollary 3.3 and Remark 3.4 hold for these cases as well.

Theorem 3.5. *Let $m \in (0, \infty]$. Let v, w be weights. For $g \in \mathscr{P}_m$ let $\|g\|_Y$ be given by what follows:*

(i) *If $0 < p \le 1$, then*

$$\|g\|_Y := \operatorname*{ess\,sup}_{0<x<y<m} \left[g^{**}(y)w(y)xV^{-\frac{1}{p}}(x) + g^{**}(y)w(x)yV^{-\frac{1}{p}}(y)\right].$$

(ii) *If $1 < p < \infty$, then*

$$\|g\|_Y := \operatorname*{ess\,sup}_{0<x<m} w(x)\left[\left(\int_x^m (g^{**}(t))^{p'}t^{p'}V^{-p'}(t)v(t)\,\mathrm{d}t\right)^{\frac{1}{p'}} + g^{**}(x)\left(\int_0^x t^{p'-1}V^{1-p'}(t)\,\mathrm{d}t\right)^{\frac{1}{p'}}\right].$$

Then, for $p \in (0, \infty)$, it holds

$$\|f * g\|_{\Gamma^\infty(w)} \lesssim \|f\|_{\Lambda^p(v)}\|g\|_Y, \quad f \in \Lambda^p(v),\ g \in Y. \tag{17}$$

Moreover, if $g \in \mathscr{E}_m$, then $\|T_g\|_{\Lambda^p(v)\to\Gamma^\infty(w)} \simeq \|g\|_Y$. The space $(Y, \|\cdot\|_Y)$ is optimal for the pair $(\Lambda^p(v), \Gamma^\infty(w))$.

Proof. We will again show that $\|g\|_Y \simeq \|R_g^1\|_{\Lambda^p(v)\to L^\infty(w)} + \|R_g^2\|_{\Lambda^p(v)\to L^\infty(w)}$ and apply Theorem 3.1. (For more details see the proof of Theorem 3.2.)

(i) From [5, Theorem 3.3] (see also [4, Theorem 4.2]) and [10, Theorem 5.3] it follows:

$$\|R_g^1\|_{\Lambda^p(v)\to L^\infty(w)} \simeq \operatorname*{ess\,sup}_{0<x<y<m} g^{**}(y)w(y)xV^{-\frac{1}{p}}(x),$$

$$\|R_g^2\|_{\Lambda^p(v)\to L^\infty(w)} \simeq \operatorname*{ess\,sup}_{0<x<y<m} \int_x^y g^*(s)\,\mathrm{d}s\ w(x)V^{-\frac{1}{p}}(y).$$

In the definition of $\|g\|_Y$ we observe that $\|g\|_Y \simeq \|R_g^1\|_{\Lambda^p(v)\to L^\infty(w)} + \|R_g^2\|_{\Lambda^p(v)\to L^\infty(w)} + B$, where

$$B := \operatorname*{ess\,sup}_{x\in(0,m)} xg^{**}(x)w(x)V^{-\frac{1}{p}}(x).$$

However, it is easy to see that $B \le \|R_g^1\|_{\Lambda^p(v)\to L^\infty(w)}$, therefore $\|g\|_Y \simeq \|R_g^1\|_{\Lambda^p(v)\to L^\infty(w)} + \|R_g^2\|_{\Lambda^p(v)\to L^\infty(w)}$.

(ii) From the same sources as in (i) we obtain the following characterizations:

$$\|R_g^1\|_{\Lambda^p(v)\to L^\infty(w)} \simeq \operatorname*{ess\,sup}_{x\in(0,m)} g^{**}(x)w(x)\left(\int_0^x t^{p'-1}V^{1-p'}(t)\,\mathrm{d}t\right)^{\frac{1}{p'}},$$

$$\|R_g^2\|_{\Lambda^p(v)\to L^\infty(w)} \simeq \operatorname*{ess\,sup}_{x\in(0,m)} w(x)\left(\int_x^m \left(\int_x^t g^*(s)\,\mathrm{d}s\right)^{p'} V^{-p'}(t)v(t)\,\mathrm{d}t\right)^{\frac{1}{p'}}.$$

Since

$$\left(\int_x^m V^{-p'}(t)v(t)\,\mathrm{d}t\right)^{\frac{1}{p'}} \le V^{-\frac{1}{p}}(x) = x^{-1}V^{-\frac{1}{p}}(x)\left(\int_0^x t^{p'-1}\,\mathrm{d}t\right)^{\frac{1}{p'}} \le x^{-1}\left(\int_0^x V^{1-p'}(t)t^{p'-1}\,\mathrm{d}t\right)^{\frac{1}{p'}},$$

we get

$$B := \operatorname*{ess\,sup}_{x>0}\ xg^{**}(x)w(x)\left(\int_x^\infty V^{-p'}(t)v(t)\,\mathrm{d}t\right)^{\frac{1}{p'}} \le \|R_g^1\|_{\Lambda^p(v)\to L^\infty(w)}.$$

Thus, $\|g\|_Y \simeq \|R_g^1\|_{\Lambda^p(v)\to L^\infty(w)} + \|R_g^2\|_{\Lambda^p(v)\to L^\infty(w)} + B \simeq \|R_g^1\|_{\Lambda^p(v)\to L^\infty(w)} + \|R_g^2\|_{\Lambda^p(v)\to L^\infty(w)}$ and the proof is finished. □

Theorem 3.6. *Let $m \in (0,\infty]$. Let v, w be weights. For $g \in \mathscr{P}_m$ let $\|g\|_Y$ be given by what follows:*

(i) *If $0 < q < \infty$, then*

$$\|g\|_Y := \left(\int_0^m \left(g^{**}(x)\int_0^x \frac{\mathrm{d}t}{\operatorname{ess\,sup}_{s\in(0,t)} v(s)} + \int_x^m \frac{g^*(t)\,\mathrm{d}t}{\operatorname{ess\,sup}_{s\in(0,t)} v(s)}\right)^q w(x)\,\mathrm{d}x\right)^{\frac{1}{q}}.$$

(ii) *If $q = \infty$, then*

$$\|g\|_Y := \operatorname*{ess\,sup}_{x\in(0,m)} \left(g^{**}(x)\int_0^x \frac{\mathrm{d}t}{\operatorname{ess\,sup}_{s\in(0,t)} v(s)} + \int_x^m \frac{g^*(t)\,\mathrm{d}t}{\operatorname{ess\,sup}_{s\in(0,t)} v(s)}\right) w(x).$$

Then, for $q \in (0,\infty]$, it holds

$$\|f * g\|_{\Gamma^q(w)} \lesssim \|f\|_{\Lambda^\infty(v)}\|g\|_Y, \quad f \in \Lambda^\infty(v),\ g \in Y. \tag{18}$$

Moreover, if $g \in \mathscr{E}_m$, then $\|T_g\|_{\Lambda^\infty(v)\to\Gamma^q(w)} \simeq \|g\|_Y$. The space $(Y, \|\cdot\|_Y)$ is optimal for the pair $(\Lambda^\infty(v), \Gamma^q(w))$.

Proof. Once again, let us show $\|g\|_Y \simeq \|R_g^1\|_{\Lambda^\infty(v)\to L^q(w)} + \|R_g^2\|_{\Lambda^\infty(v)\to L^q(w)}$ and apply Theorem 3.1. (See details in the analogous proof of Theorem 3.2.)

(i) From [10, Theorem 5.5] it follows

$$\|R_g^1\|_{\Lambda^\infty(v)\to L^q(w)} = \left(\int_0^m \left(g^{**}(x)\int_0^x \frac{\mathrm{d}t}{\operatorname{ess\,sup}_{s\in(0,t)} v(s)}\right)^q w(x)\,\mathrm{d}x\right)^{\frac{1}{q}},$$

$$\|R_g^2\|_{\Lambda^\infty(v)\to L^q(w)} = \left(\int_0^m \left(\int_x^m \frac{g^*(t)\,\mathrm{d}t}{\operatorname{ess\,sup}_{s\in(0,t)} v(s)}\right)^q w(x)\,\mathrm{d}x\right)^{\frac{1}{q}}.$$

One clearly sees that $\|g\|_Y \simeq \|R_g^1\|_{\Lambda^\infty(v)\to L^q(w)} + \|R_g^2\|_{\Lambda^\infty(v)\to L^q(w)}$.

(ii) Here, by [10, Theorem 5.5] as well, we get

$$\|R_g^1\|_{\Lambda^\infty(v)\to L^\infty(w)} = \operatorname*{ess\,sup}_{x\in(0,m)}\ g^{**}(x)\int_0^x \frac{\mathrm{d}t}{\operatorname{ess\,sup}_{s\in(0,t)} v(s)} w(x),$$

$$\|R_g^2\|_{\Lambda^\infty(v)\to L^\infty(w)} = \operatorname*{ess\,sup}_{x\in(0,m)} \int_x^m \frac{g^*(t)\,\mathrm{d}t}{\operatorname{ess\,sup}_{s\in(0,t)} v(s)} w(x),$$

and thus obviously $\|g\|_Y \simeq \|R_g^1\|_{\Lambda^\infty(v)\to L^\infty(w)} + \|R_g^2\|_{\Lambda^\infty(v)\to L^\infty(w)}$. □

4. Further results and applications

At first, here we present two additional results of independent interest. The theorem below provides an alternative expression for the right-hand side of O'Neil inequality:

Theorem 4.1. *Let $m \in (0,\infty]$ and let $f, g \in \mathscr{P}_m \cap L^1_{\mathrm{loc}}$. Then for every $t \in (0,m)$ it holds:*

$$tf^{**}(t)g^{**}(t) + \int_t^m f^*(s)g^*(s)\,\mathrm{d}s = \lim_{s\to m-} sf^{**}(s)g^{**}(s) + \int_t^m (f^{**}(s) - f^*(s))(g^{**}(s) - g^*(s))\,\mathrm{d}s.$$

Proof. We may assume that $f^{**}, g^{**} < \infty$ on $(0,\infty)$, otherwise the identity holds trivially. Recall that $(g^{**})'(t) = \frac{g^*(t)-g^{**}(t)}{t}$ for all $t > 0$. Assume first $m < \infty$ and take a fixed $t \in (0,m)$. Then integration by parts yields

$$\int_t^m f^{**}(s)(g^{**}(s) - g^*(s))\,\mathrm{d}s = \big[-sf^{**}(s)g^{**}(s)\big]_{s=t}^m + \int_t^m f^*(s)g^{**}(s)\,\mathrm{d}s.$$

Subtracting $\int_t^m f^*(g^{**}-g^*)$ from both sides, we get

$$\int_t^m (f^{**}(s)-f^*(s))(g^{**}(s)-g^*(s))\,\mathrm{d}s = \big[-sf^{**}(s)g^{**}(s)\big]_{s=t}^m + \int_t^m f^*(s)g^*(s)\,\mathrm{d}s,$$

hence

$$tf^{**}(t)g^{**}(t) + \int_t^m f^*g^* = mf^{**}(m)g^{**}(m) + \int_t^m (f^{**}-f^*)(g^{**}-g^*). \tag{19}$$

Notice that since all integrals involved in the procedure exist and are finite, all performed steps were correct. Now, consider $f,g\in\mathscr{M}$ and suppose that f^*, f^{**}, g^*, g^* are rearrangements on $\mathbb{R}$ (given by (5) and (6) with $m=\infty$). By the previous part, (19) holds for any parameter $m\in(0,\infty)$, thus passing $m\to\infty$ on both sides and using the monotone convergence theorem gives (19) for $m=\infty$. □

We now get the following corollary:

Corollary 4.2. *Let $m\in(0,\infty)$, $f,g\in\mathscr{P}_m\cap L^1_{\mathrm{loc}}$ and let w be a weight. Denote $\|\cdot\|_1 := \|\cdot\|_{L^1(0,m)}$. Then*

$$\int_0^m (f*g)^{**}w \le \frac{\|f\|_1\|g\|_1\|w\|_1}{m} + \int_0^m (f^{**}(t)-f^*(t))(g^{**}(t)-g^*(t))W(t)\,\mathrm{d}t. \tag{20}$$

Proof. Following Theorems 2.2 and 4.1 we get

$$\begin{aligned}
&\int_0^m (f*g)^{**}(t)w(t)\,\mathrm{d}t \le\\
&\le \int_0^m \left[tf^{**}(t)g^{**}(t) + \int_t^m f^*(s)g^*(s))\,\mathrm{d}s\right]w(t)\,\mathrm{d}t \le\\
&\le \int_0^m \left[mf^{**}(m)g^{**}(m) + \int_t^m (f^{**}(s)-f^*(s))(g^{**}(s)-g^*(s))\,\mathrm{d}s\right]w(t)\,\mathrm{d}t \le\\
&\le \frac{\|f\|_1\|g\|_1\|w\|_1}{m} + \int_0^m\int_t^m (f^{**}(s)-f^*(s))(g^{**}(s)-g^*(s))w(t)\,\mathrm{d}s\,\mathrm{d}t =\\
&= \frac{\|f\|_1\|g\|_1\|w\|_1}{m} + \int_0^m (f^{**}(t)-f^*(t))(g^{**}(t)-g^*(t))W(t)\,\mathrm{d}t.
\end{aligned}$$

□

This improves the result of [15, Lemma 2.1], in which a weaker version of it is proved, namely with g^{**} instead of $g^{**}-g^*$ in the integrand on the right-hand side of (20).

Next, let us show that our theorems cover the classical convolution-related results which we thus can obtain by applying the inequalities from Section 2 to special choices of weights.

Remark 4.3. O'Neil's result [17, Theorem 2.6] says that for $1<a,b,c<\infty$ and $1\le q<p<\infty$ such that $1+\frac{1}{a}=\frac{1}{b}+\frac{1}{c}$ and $\frac{1}{r}=\frac{1}{q}-\frac{1}{p}$ the inequality (3) holds for all $f,g\in\mathscr{P}_m$, where m may be both finite or infinite and the functionals $\|\cdot\|_{L_{\alpha,\beta}}$ are defined on a corresponding interval $(0,m)$. Let us show that this result now follows as a special case of Theorem 3.2(iii)/(iv):

Consider $q>1$. Recall that since $a,b,c>1$, it holds $\|\cdot\|_{L_{a,q}}\simeq\|\cdot\|_{L_{(a,q)}}$ and analogously for $L_{c,r}$ (see e.g. [1, p. 219]). Hence, it suffices to confirm the inequality

$$\|f*g\|_{\Gamma^q(w)} \lesssim \|f\|_{\Lambda^p(v)}\|g\|_{\Gamma^r(u)} \tag{21}$$

with $v(x):=x^{\frac{p}{b}-1}$, $w(x):=x^{\frac{q}{a}-1}$ and $u(x):=x^{\frac{r}{c}-1}$. By application of Theorem 3.2(iii) and a direct calculation involving the given weights, we obtain that $\|f*g\|_{\Gamma^q(w)}\lesssim\|f\|_{\Lambda^p(v)}\|g\|_Y$ holds with

$$\begin{aligned}
\|g\|_Y \simeq \|g\|_{\Gamma^r(u)} + V^{-\frac{1}{p}}(m)\left(\int_0^m (g^{**}(t))^q t^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} +\\
+\left(\int_0^m\left(\int_x^m (g^{**}(t))^q t^{\frac{q}{a}-1}\,\mathrm{d}t\right)^{\frac{r}{q}} x^{\frac{r(b-1)}{b}-1}\,\mathrm{d}x + \int_0^m\left(\int_x^m (g^{**}(t))^{p'} t^{\frac{b-p}{b(p-1)}}\,\mathrm{d}t\right)^{\frac{r}{p'}} x^{\frac{r}{a}-1}\,\mathrm{d}x\right)^{\frac{1}{r}}.
\end{aligned}$$

Since g^{**} is nonincreasing, the Hardy-type inequality [10, Theorem 5.1(iii)] implies

$$\left(\int_0^m \left(\int_x^m (g^{**}(t))^q t^{\frac{q}{a}-1}\,\mathrm{d}t\right)^{\frac{r}{q}} x^{\frac{r(b-1)}{b}-1}\,\mathrm{d}x\right)^{\frac{1}{r}} \lesssim \|g\|_{\Gamma^r(u)},$$
$$\left(\int_0^m \left(\int_x^m (g^{**}(t))^{p'} t^{\frac{b-p}{b(p-1)}}\,\mathrm{d}t\right)^{\frac{r}{p'}} x^{\frac{r}{a}-1}\,\mathrm{d}x\right)^{\frac{1}{r}} \lesssim \|g\|_{\Gamma^r(u)}.$$

If $m=\infty$, we obtain that $V^{-\frac{1}{p}}(m)\left(\int_0^m (g^{**}(t))^q t^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} = 0$ since $V(\infty)=\infty$ (by the convention "$\frac{\infty}{\infty}=0$"). For $m<\infty$, from [18, Remark (i), p. 148] it follows

$$V^{-\frac{1}{p}}(m)\left(\int_0^m (g^{**}(t))^q t^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} \le \|g\|_{\Gamma^r(u)}.$$

Verifying the requirements of all the used theorems is yet again done by a direct calculation of the weights. We got $\|\cdot\|_Y \simeq \|\cdot\|_{\Gamma^r(u)}$ and it shows that (21) holds.

The case $q=1$ follows analogously using Theorem 3.2(iv) and the same sources. Therefore (3) is recovered.

Remark 4.4. Furthermore, we can investigate the limit case of (3) with $a=b$ and $c=1$. Using exactly the same method as above, we reach the inequality

$$\|f*g\|_{L_{b,q}} \lesssim \|f\|_{L_{b,p}}\|g\|_{L_{(1,r)}}, \quad f\in L_{b,p},\ g\in L_{(1,r)}.$$

For $m<\infty$ we recover the result of [15, Theorem 2.1(a)] so. Unlike the case of a finite m, for $m=\infty$ the space $L_{(1,r)}$, which we obtained as the optimal one, consists only of the a.e. zero function. Thus, Corollary 3.3 yields: If $g\in\mathscr{P}_m$ is nonnegative, then T_g is bounded from $L_{p,b}$ to $L_{q,b}$ if and only if $g=0$ a.e. Hence, we recovered the result of [2, Theorem 2] for convolution operators.

5. Properties of related function spaces

In this part we introduce a new type of function spaces based on the optimal space Y we got in the previous and list some basic properties of these structures. We define them as systems of functions over the domain $\left(-\frac{m}{2},\frac{m}{2}\right)$, where m is, without loss of generality, taken from $[1,\infty]$.

Definition 5.1. Let $m\in[1,\infty]$, $p,q\in(0,\infty)$ and let u,v be weights. For $g\in\mathscr{P}_m$ we define

$$\|g\|_{K^{p,q}(u,v)} := \left(\int_0^m \left(\int_x^m (g^{**}(t))^p u(t)\,\mathrm{d}t\right)^{\frac{q}{p}} v(x)\,\mathrm{d}x\right)^{\frac{1}{q}},$$
$$\|g\|_{K^{p,\infty}(u,v)} := \operatorname*{ess\,sup}_{x\in(0,m)} \left(\int_x^m (g^{**}(t))^p u(t)\,\mathrm{d}t\right)^{\frac{1}{p}} v(x),$$
$$\|g\|_{K^{\infty,q}(u,v)} := \left(\int_0^m \operatorname*{ess\,sup}_{t\in(x,m)} \left(g^{**}(t)u(t)\right)^q v(x)\,\mathrm{d}x\right)^{\frac{1}{q}}.$$

Then we put $K^{p,q}(u,v) := \left\{f\in\mathscr{P}_m;\ \|f\|_{K^{p,q}(u,v)}<\infty\right\}$, analogously we define $K^{p,\infty}(u,v)$ and $K^{\infty,q}(u,v)$.

We could also consider the norm

$$\|g\|_{K^{\infty,\infty}(u,v)} := \operatorname*{ess\,sup}_{t>x>0} g^{**}(t)u(t)v(x).$$

However, this would bring no innovation since $\|\cdot\|_{K^{\infty,\infty}(u,v)}$ then coincides with $\|\cdot\|_{\Gamma^\infty(\omega)}$ for $\omega(t) := u(t)\operatorname{ess\,sup}_{x\in(0,t)} v(x)$.

Function spaces which actually are special cases of these have already been sporadically mentioned before. For example, in [6], the space $K^{1,\infty}(u,v)$ with a special choice of u,v appears as the optimal space for a certain Sobolev embedding into a Morrey-type space.

Let us now justify our use of the word "space" in connection with these structures:

Theorem 5.2. *Let $m \in [1,\infty]$. Suppose that $p,q \in (0,\infty)$ and u,v are weights. Then $\|\cdot\|_{K^{p,q}(u,v)}$ is a quasi-norm. If $p,q \geq 1$, then $\|\cdot\|_{K^{p,q}(u,v)}$ is a norm.*

Proof. The quasi-norm property follows from the fact that $(f+g)^{**}(t) \leq f^{**}(t) + g^{**}(t)$ for all $t \in (0,m)$ (see e.g. [1, p. 54]). For the case $p,q \geq 1$ Minkowski inequality is used to prove that $\|\cdot\|_{K^{p,q}(u,v)}$ is then even a norm. □

We continue with showing the conditions under which a K space is nontrivial.

Theorem 5.3. *Let $m \in [1,\infty]$ and let u,v be weights. Then:*

(i) *If $0 < p,q < \infty$, then $K^{p,q}(u,v) \neq \{0\}$ if and only if*

$$\int_0^m \left(\int_x^m \frac{u(t)}{(t+1)^p} \,\mathrm{d}t \right)^{\frac{q}{p}} v(x) \,\mathrm{d}x < \infty. \tag{22}$$

(ii) *If $0 < p < \infty$, then $K^{p,\infty}(u,v) \neq \{0\}$ if and only if*

$$\operatorname*{ess\,sup}_{x \in (0,m)} \left(\int_x^m \frac{u(t)}{(t+1)^p} \,\mathrm{d}t \right)^{\frac{1}{p}} v(x) < \infty.$$

(iii) *If $0 < q < \infty$, then $K^{\infty,q}(u,v) \neq \{0\}$ if and only if*

$$\int_0^m \operatorname*{ess\,sup}_{t \in (x,m)} \frac{u^q(t)}{(t+1)^q} v(x) \,\mathrm{d}x < \infty.$$

Proof. (i) At first, one sees that for all $t > 0$ it holds

$$\frac{1}{2(t+1)^p} \leq \chi_{[0,1)}(t) + \frac{\chi_{[1,m)}(t)}{t^p} \leq \frac{2^p}{(t+1)^p}. \tag{23}$$

Assume that there exists $0 \neq f \in \mathscr{M}\left(-\frac{m}{2}, \frac{m}{2}\right)$ such that $\|f\|_{K^{p,q}(u,v)} < \infty$. Then $0 < f^{**}(1) < \infty$ and by (23) we get

$$\infty > \|f\|^q_{K^{p,q}(u,v)} \geq \|f^{**}(1)\chi_{[0,1]}\|^q_{K^{p,q}(u,v)} \geq \frac{(f^{**}(1))^q}{2^q} \int_0^m \left(\int_x^m \frac{u(t)}{(t+1)^p} \,\mathrm{d}t \right)^{\frac{q}{p}} v(x).$$

Now assume that (22) holds. Then by the other part of (23) we obtain that $\chi_{[0,1]} \in K^{p,q}(u,v)$. Cases (ii) and (iii) are proved analogously. □

Recall (see e.g. [1, p. 73]) the spaces $L^1 \cap L^\infty$ and $L^1 + L^\infty$ generated by the norms

$$\|f\|_{L^1+L^\infty} := \inf_{f=f_1+f_2} \{\|f_1\|_1 + \|f_2\|_\infty\}, \quad \|f\|_{L^1 \cap L^\infty} := \max\{\|f\|_1, \|f\|_\infty\},$$

where $L^1 = L^1(0,m)$ and $L^\infty = L^\infty(0,m)$.

Proposition 5.4. *Let $m \in [1,\infty]$. Let $0 < p,q \leq \infty$ and let u,v be weights such that $K^{p,q}(u,v) \neq \{0\}$. Then*

$$L^1 \cap L^\infty \hookrightarrow K^{p,q}(u,v) \hookrightarrow L^1 + L^\infty.$$

Proof. This is proved directly by exactly the same method as in [12, Proposition 1.4(2)] where analogous result for Γ spaces is shown. □

We see that if $\|\cdot\|_{K^{p,q}(u,v)} \lesssim \|\cdot\|_{L^1+L^\infty}$, then $K^{p,q}(u,v) = L^1 + L^\infty$ in the sense of equivalence of norms. This is considered to be another type of triviality. We characterize it by what follows:

Theorem 5.5. *Let $m \in [1,\infty]$ and let u,v be weights. Then:*

(i) *If $0 < p,q < \infty$, then $L^1 + L^\infty \hookrightarrow K^{p,q}(u,v)$ if and only if*

$$C := \int_0^m \left(\int_x^m \left(\frac{1}{t} + 1 \right)^p u(t) \,\mathrm{d}t \right)^{\frac{q}{p}} v(x) \,\mathrm{d}x < \infty.$$

(ii) *If* $0 < p < \infty$*, then* $L^1 + L^\infty \hookrightarrow K^{p,\infty}(u,v)$ *if and only if*

$$\operatorname*{ess\,sup}_{x\in(0,m)} \left(\int_x^m \left(\frac{1}{t} + 1 \right)^p u(t)\,\mathrm{d}t \right)^{\frac{1}{p}} v(x) < \infty.$$

(iii) *If* $0 < q < \infty$*, then* $L^1 + L^\infty \hookrightarrow K^{\infty,q}(u,v)$ *if and only if*

$$\int_0^m \operatorname*{ess\,sup}_{t\in(x,m)} \left(\frac{1}{t} + 1 \right)^q u^q(t) v(x)\,\mathrm{d}x < \infty.$$

Proof. (i) First suppose that $C = \infty$. For each $n \in \mathbb{N}$ define $f_n := n\chi_{[0,\frac{1}{n}]} + 1$. Then $\|f_n\|_{L^1+L^\infty} \le 1$ for all $n \in \mathbb{N}$ but by the monotone convergence theorem it holds $\|f_n\|^q_{K^{p,q}(u,v)} \uparrow C = \infty$. Thus, $L^1 + L^\infty \not\hookrightarrow K^{p,q}(u,v)$.

Now assume that $C < \infty$. Let $f \in L^1 + L^\infty$ be arbitrary. Let $f_1 \in L^1$ and $f_2 \in L^\infty$ be functions such that $f = f_1 + f_2$ and $\|f\|_{L^1+L^\infty} \ge \frac{1}{2}(\|f_1\|_1 + \|f_2\|_\infty)$. Then $f^{**}(t) \le \frac{\|f_1\|_1}{t} + \|f_2\|_\infty$, $t \in (0,m)$, and thus it holds

$$\|f\|^q_{K^{p,q}(u,v)} = \|f_1 + f_2\|^q_{K^{p,q}(u,v)} \le \int_0^m \left(\int_x^m \left(\frac{\|f_1\|_1}{t} + \|f_2\|_\infty \right)^p u(t)\,\mathrm{d}t \right)^{\frac{q}{p}} v(x)\,\mathrm{d}x \le 2^q C \|f\|^q_{L^1+L^\infty},$$

hence $L^1 + L^\infty \hookrightarrow K^{p,q}(u,v)$. The proofs of (ii) and (iii) are analogous. □

Remark 5.6. Notice that if $m < \infty$, the conditions may be slightly simplified: In Theorem 5.3(i), the factor $\frac{u(t)}{(t+1)^p}$ in (22) may be replaced just by $u(t)$ and analogously in Theorem 5.3(ii),(iii). In Theorem 5.5(i) we may replace $\left(1 + \frac{1}{t}\right)^p$ by $\frac{1}{t^p}$ and similarly in (ii) and (iii).

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