

Optimal Orlicz domains in Sobolev embeddings

Vít Musil

Abstract: In this paper we present a method for determining whether there exists a largest Orlicz space $L^A(\Omega)$ satisfying the Sobolev embedding

$$W^m L^A(\Omega) \hookrightarrow Y(\Omega)$$

where $Y(\Omega)$ stands for an arbitrary so-called Marcinkiewicz endpoint space. The tool developed in this work enables us to investigate the optimality of Orlicz domain spaces in Sobolev embeddings and also in Sobolev trace embeddings on domains Ω in \mathbb{R}^n with various regularity.

1. Introduction and main results

For a given Banach function space $Y(\Omega)$, we study the question whether there exists an optimal (i.e. largest) Orlicz space $L^A(\Omega)$ satisfying the embedding

$$W^m L^A(\Omega) \hookrightarrow Y(\Omega),$$

where Ω stands for a bounded domain in \mathbb{R}^n , $n \geq 2$, and $W^m L^A(\Omega)$ is an Orlicz-Sobolev space (for the definition see the section below). By optimality we mean that the space $L^A(\Omega)$ cannot be replaced by any strictly bigger Orlicz space, i.e., every embedding of an Orlicz-Sobolev space to $Y(\Omega)$ factorizes through the space $W^m L^A(\Omega)$.

In the general setting of rearrangement-invariant (r.i.) Banach function spaces, such questions were investigated using the method of reducing the Sobolev embeddings to the boundedness of an appropriate modification of the weighted Hardy operator. In the setting of r.i. spaces, the optimal domain and the optimal target spaces are then explicitly described (see [10], [11], [16], [17]).

However, for certain specific applications such as to the solution of partial differential equations, it is often useful to investigate the optimality of spaces in Sobolev-type embeddings restricted to the context of Orlicz spaces. This creates a difficult and important problem that has been studied by several authors (see e.g. [3], [4], [5], [6], [8], [10], [12], [13]). In particular, the situation in this setting is significantly different than in the broader sense of r.i. spaces.

Consider, for instance, the well-known classical Sobolev embedding $W^1 L^p(\Omega) \hookrightarrow L^{p^*}(\Omega)$, where $1 < p < n$, $p^* = np/(n-p)$ and Ω has a Lipschitz boundary. Both the spaces $L^p(\Omega)$ and $L^{p^*}(\Omega)$ that appear in this embedding are clearly optimal in the context of Lebesgue spaces, the former as the domain and the latter as the range. It turns out that they are optimal even in the broader context of Orlicz spaces, but that is a deeper observation and more difficult to prove. The optimality of the range space $L^{p^*}(\Omega)$ follows from a general result of A. Cianchi [6]. On the other hand, the optimality of the domain space $L^p(\Omega)$ has not been known so far and will follow from our more general statement below (Example 5.2).

In the limiting case when $p = n$, the situation is different and more interesting. First, if we fix the domain space $L^n(\Omega)$, then there is no optimal range Lebesgue space $L^q(\Omega)$ that would render the embedding $W^1 L^n(\Omega) \hookrightarrow L^q(\Omega)$ true, because it holds for every $q < \infty$, but not for $q = \infty$. This discrepancy was remedied in the 1960s by a clever use of special Orlicz spaces of an exponential type. In particular, by now classic results of N. S. Trudinger, S. I. Pokhozhaev, and V. I. Yudovich (see [24], [25], [26]), one has

$$W^1 L^n(\Omega) \hookrightarrow \exp L^{n'}(\Omega),$$

where $n' = n/(n-1)$. Now, both the domain space $L^n(\Omega)$ and the range space $\exp L^{n'}(\Omega)$ are Orlicz spaces, and therefore we may ask, again, about their optimality. It turns out that, while the target space is the optimal (that means smallest) Orlicz space that renders this Sobolev embedding true (this was originally proved by J. A. Hempel, G. R. Morris and N. S. Trudinger [13] and it also follows from a general result of A. Cianchi [6]), the domain space is not. Rather surprisingly, it can even be shown that such an optimal Orlicz domain space does not exist at all. More

Keywords: Optimal Orlicz domain, Sobolev embedding, Sobolev trace embedding, Orlicz space, Marcinkiewicz space, fundamental function, rearrangement-invariant space, Hardy integral operator, supremum operator.

MSC: 46E30, 46E35, 47G10, 26D10.

This research was partly supported by the grant P201-13-14743S of the Grant Agency of the Czech Republic and by the Charles University in Prague, project GA UK No. 33315.

precisely, given an Orlicz space $L^A(\Omega)$ such that $W^1L^A(\Omega) \hookrightarrow \exp L^{n'}(\Omega)$, there always exists another Orlicz space $L^B(\Omega)$, strictly bigger than $L^A(\Omega)$ such that $W^1L^B(\Omega) \hookrightarrow \exp L^{n'}(\Omega)$. This result was shown in [22].

It is clear from these examples that even the very existence of an optimal Orlicz partner (either range or domain) is highly nontrivial and very interesting. However, the question of existence (and, possibly, characterization) of an optimal Orlicz domain partner, is of interest also in a more general situation when the given target space is not necessarily an Orlicz space.

For instance, one has the embedding

$$W^1L^p(\Omega) \hookrightarrow L^{p^*,p}(\Omega),$$

(see e.g. [14], [19], [20], [21]) in which the target space is a usual two-parameter Lorentz space. Moreover, it is known that $L^{p^*,p}(\Omega)$ is the optimal r.i. range space in this embedding and the space $L^p(\Omega)$ is the optimal r.i. domain space (see [11] or [10]). Therefore, $L^p(\Omega)$ is automatically also the optimal Orlicz space in this embedding.

On the other hand, when we start with the space $L^\infty(\Omega)$ at the position of the range space, then, again, as A. Cianchi and L. Pick showed in [9], an optimal Orlicz space does not exist at all. This situation resembles the above-mentioned embedding in which the target was the space $\exp L^{n'}(\Omega)$. Apart from these two very particular cases the question of the existence of an optimal Orlicz space has been open.

The general question of optimality among the Orlicz spaces has already been studied (see [5], [6], [7], [8], [10], [12]) however, all those papers focus on the optimality of target spaces. In the case of range, it turns out that the answer is always affirmative, and, furthermore, an explicit description of the optimal Orlicz space is available. The situation is however dramatically different when the target space is fixed and the optimality of the domain space is in question.

In this paper we study this question in the special case when the target space is chosen from the class of the so-called Marcinkiewicz endpoint spaces. This is not as restrictive as it may seem since the most customary cases including those given by the previous examples are covered.

An important ingredient of our approach is the use of known reduction theorems. This method will enable us to circumvent working with Sobolev spaces to consider instead the boundedness of the operator

$$(H_\alpha^\beta f)(t) := \int_{t^\beta}^1 f(s) s^{\alpha-1} ds, \quad t \in (0, 1)$$

in one dimension. Here $0 < \alpha < 1$, $0 < \beta < \infty$ and $\alpha + 1/\beta \geq 1$. Then, by using various special cases of α and β we obtain applications not only to Sobolev embeddings but also to the trace Sobolev embeddings of different orders and on various domains in \mathbb{R}^n at once.

Now we are in a position to state our main result which gives a complete characterization of when the optimal Orlicz domain exists, and also its explicit description. Simply put, to a given Marcinkiewicz endpoint space $M(\Omega)$ we construct an ‘‘optimal Orlicz candidate’’ $L^B(\Omega)$ in terms of the fundamental function. We exploit the fact that to a given fundamental function there always exists a uniquely defined Orlicz space. Next, we test whether the embedding $W^m L^B(\Omega) \hookrightarrow M(\Omega)$ holds. If so, then we show that $L^B(\Omega)$ is the optimal Orlicz domain. Otherwise, we can prove that an optimal Orlicz domain does not exist at all. The general result reads as follows.

Theorem A. *Let $0 < \alpha < 1$, $\beta > 0$, $\alpha + 1/\beta \geq 1$ and let $M(0, 1)$ be a Marcinkiewicz endpoint space with a fundamental function φ satisfying*

$$\sup_{0 < t < 1} \varphi(t^{\frac{1}{\beta}}) t^{\alpha-1} = \infty.$$

Let $X(0, 1)$ be the largest r.i. space satisfying

$$H_\alpha^\beta: X(0, 1) \rightarrow M(0, 1).$$

Denote by $L^B(0, 1)$ the Orlicz space having the same fundamental function as the space $X(0, 1)$. Define $G(t) = t^{1/(\alpha-1)} \tilde{B}(t)$, $t \in (0, \infty)$, where \tilde{B} is the complementary Young function to B . Then the following statements are equivalent.

(i) *There exists a largest Orlicz space $L^A(0, 1)$ satisfying the relation*

$$H_\alpha^\beta: L^A(0, 1) \rightarrow M(0, 1);$$

(ii)

$$H_\alpha^\beta: L^B(0, 1) \rightarrow M(0, 1);$$

(iii)

$$L^B(0, 1) \subseteq X(0, 1);$$

(iv)

$$S_\alpha: L^{\tilde{B}}(0, 1) \rightarrow L^{\tilde{B}}(0, 1),$$

where S_α is the operator given by

$$(S_\alpha f)(t) := t^{\alpha-1} \sup_{0 < s < t} s^{1-\alpha} f^*(s), \quad t \in (0, 1);$$

(v) there exists some $K \geq 1$ such that

$$\limsup_{t \rightarrow \infty} \frac{1}{G(Kt)} \int_1^t \frac{G(s)}{s} ds < \infty.$$

Moreover, if \tilde{B} satisfies the Δ_2 condition, then each of the conditions (i)–(v) is equivalent to the following statement:

(vi) there exists some $K \geq 1$ such that

$$\limsup_{t \rightarrow \infty} \frac{G(t)}{G(Kt)} < 1.$$

Note that the condition on φ causes no loss of generality, since otherwise $H_\alpha^\beta: L^1(0, 1) \rightarrow M(0, 1)$. The details are discussed in Remark 3.7.

The proof of Theorem A relies on the next result of independent interest, which provides us with a reduction theorem for Orlicz and Marcinkiewicz spaces.

Theorem B. Let $0 < \alpha < 1$, $\beta > 0$, $\alpha + 1/\beta \geq 1$ and let $L^A(0, 1)$ be an Orlicz space with a Young function A and $M(0, 1)$ be a Marcinkiewicz endpoint space with a fundamental function φ satisfying

$$\sup_{0 < t < 1} \varphi(t^{\frac{1}{\beta}}) t^{\alpha-1} = \infty.$$

Then the relation

$$H_\alpha^\beta: L^A(0, 1) \rightarrow M(0, 1)$$

holds if and only if there exists $C > 0$ such that

$$\int_1^t \frac{\tilde{A}(s)}{s^{1/(1-\alpha)+1}} ds \lesssim \frac{\tilde{B}(Ct)}{t^{1/(1-\alpha)}}, \quad t \in (2, \infty),$$

where B is a Young function described in Theorem A and \tilde{A} and \tilde{B} are complementary Young functions to A and B respectively.

Our final principal result describes the fundamental function of the optimal r.i. domain space.

Theorem C. Let $0 < \alpha < 1$, $\beta > 0$, $\alpha + 1/\beta \geq 1$. Suppose that $M(0, 1)$ is the Marcinkiewicz endpoint space with fundamental function φ . Then the fundamental function φ_X of the largest r.i. space $X(0, 1)$ having the property

$$H_\alpha^\beta: X(0, 1) \rightarrow M(0, 1)$$

satisfies

$$\varphi_X(t) \simeq t \sup_{t < s < 1} \varphi(s^{\frac{1}{\beta}}) s^{\alpha-1}, \quad t \in (0, 1).$$

The paper is structured as follows. In Chapter 2 we collect all the necessary basic background material. In Chapter 3 we prove Theorems B and C. In Chapter 4 we prove Theorem A. Finally, Chapter 5 contains various applications and examples of the main result.

2. Function spaces

Let us now fix the notation which will be used in this paper.

By $A \lesssim B$ and $A \gtrsim B$ we mean that $A \leq CB$ and $A \geq CB$, respectively, where C is a positive constant independent of the appropriate quantities involved in A and B . We shall write $A \simeq B$ when both of the estimates $A \lesssim B$ and $A \gtrsim B$ are satisfied. We shall use the convention $0 \cdot \infty = 0$, $\frac{0}{0} = 0$ and $\frac{\infty}{\infty} = 0$.

When X and Y are Banach spaces, we say that X is *embedded* into Y , and write $X \hookrightarrow Y$, if $X \subseteq Y$ and there exists a positive constant C , such that $\|f\|_Y \leq C \|f\|_X$ for every $f \in X$. We say that a linear operator T defined on X with values in Y is *bounded* if there exists a constant $C > 0$ such that $\|Tf\|_Y \leq C \|f\|_X$ for every $f \in X$. We write $T: X \rightarrow Y$ in this case.

We say that a function $G: [0, \infty) \rightarrow (0, \infty)$ satisfies the Δ_2 condition at infinity if there exists $K > 0$ and $T \geq 0$ such that $G(2t) \leq KG(t)$ for every $t \geq T$. We will use only Δ_2 condition at infinity, hence we shall shortly say Δ_2 condition and write $G \in \Delta_2$.

For a nonnegative function f we shall write $\int_0^c f < \infty$ when there exists some $c > 0$ such that the integral $\int_0^c f$ converges. By integral we always mean the Lebesgue integral.

2.1 Rearrangement-invariant spaces

In this section we recall definitions and some basic facts concerning the rearrangement-invariant spaces, which we will need in the following text. We shall not prove well-known results; all proofs and further details can be found in the monograph by C. Bennett and R. Sharpley [1].

Suppose Ω is a domain in \mathbb{R}^n . Let $\mathcal{M}(\Omega)$ be a class of real-valued measurable functions on Ω and $\mathcal{M}^+(\Omega)$ the class of nonnegative functions in $\mathcal{M}(\Omega)$. Given $f \in \mathcal{M}$ we define its *nonincreasing rearrangement* on $(0, |\Omega|)$ as

$$f^*(t) := \inf\{\lambda > 0, \mu_f(\lambda) \leq t\}, \quad 0 < t < |\Omega|,$$

where μ_f is the *distribution function* of f , i.e.,

$$\mu_f(\lambda) := |\{x \in \Omega, |f(x)| > \lambda\}|, \quad \lambda > 0,$$

where the $|\cdot|$ stands for the Lebesgue measure. The *Hardy average* f^{**} is defined on $(0, |\Omega|)$ as

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) ds, \quad 0 < t < |\Omega|.$$

Let $f, g \in \mathcal{M}^+(\Omega)$. Then we have the *Hardy-Littlewood inequality*

$$\int_{\Omega} f(x) g(x) dx \leq \int_0^{|\Omega|} f^*(t) g^*(t) dt.$$

When $E \subseteq \Omega$ is measurable, we denote by χ_E the *characteristic function* of E . A *simple function* is a finite sum $\sum_j \lambda_j \chi_{E_j}$, where $\lambda_j \neq 0$ is a real number and $E_j \subseteq \Omega$ has finite measure for every index j .

Denote by I the interval $(0, 1)$. A mapping $\varrho: \mathcal{M}^+(I) \rightarrow [0, \infty]$ is called a *rearrangement-invariant (r.i.) Banach function norm* on $\mathcal{M}^+(I)$, if for all f, g, f_n ($n \in \mathbb{N}$) in $\mathcal{M}^+(I)$, for all constants $a \geq 0$ and for every measurable subset E of I , the following properties hold:

- (P1) $\varrho(f) = 0 \Leftrightarrow f = 0$ a.e.; $\varrho(af) = a\varrho(f)$; $\varrho(f+g) \leq \varrho(f) + \varrho(g)$;
- (P2) $0 \leq f \leq g$ a.e. implies $\varrho(f) \leq \varrho(g)$;
- (P3) $0 \leq f_n \uparrow f$ a.e. implies $\varrho(f_n) \uparrow \varrho(f)$;
- (P4) $\varrho(\chi_I) < \infty$;
- (P5) $\int_0^1 f(x) dx \lesssim \varrho(f)$;
- (P6) $\varrho(f) = \varrho(f^*)$.

The *associate norm* of an r.i. norm ϱ is another such norm ϱ' defined as

$$\varrho'(g) := \sup_{\varrho(f) \leq 1} \int_0^1 g(t) f(t) dt, \quad f, g \in \mathcal{M}^+(I).$$

It obeys the *Principle of Duality*; that is,

$$\varrho'' := (\varrho')' = \varrho.$$

Furthermore, the *Hölder inequality*

$$\int_0^1 f(t) g(t) dt \leq \varrho(f) \varrho'(g)$$

holds for every $f, g \in \mathcal{M}^+(I)$.

Given the r.i. norm ϱ , the corresponding *rearrangement-invariant Banach function space* or, for short, *r.i. space*, is the collection

$$L_{\varrho}(I) := \{f \in \mathcal{M}(I), \varrho(|f|) < \infty\},$$

endowed with the norm

$$\|f\|_{L_{\varrho}(I)} := \varrho(|f|), \quad f \in L_{\varrho}(I).$$

Next, given a bounded domain Ω in \mathbb{R}^n , we define the r.i. space

$$L_{\varrho}(\Omega) := \{f \in \mathcal{M}(\Omega), \varrho(f^*(t|\Omega|)) < \infty\}$$

with

$$\|f\|_{L_{\varrho}(\Omega)} := \varrho(f^*(t|\Omega|)), \quad f \in L_{\varrho}(\Omega).$$

If ϱ_1 and ϱ_2 are two r.i. norms, then $L_{\varrho_1}(\Omega) \subseteq L_{\varrho_2}(\Omega)$ implies $L_{\varrho_1}(\Omega) \hookrightarrow L_{\varrho_2}(\Omega)$.

Let φ be a nonnegative function defined on the interval $[0, \infty)$. If

- (i) $\varphi(t) = 0$ iff $t = 0$,
- (ii) $\varphi(t)$ is nondecreasing on $(0, \infty)$,
- (iii) $\varphi(t)/t$ is nonincreasing on $(0, \infty)$,

then φ is said to be *quasiconcave*. We also say that a function φ defined on bounded interval $[0, R]$, for $R \in (0, \infty)$, is quasiconcave if the continuation by constant value $\varphi(R)$ is quasiconcave on $[0, \infty)$.

The *fundamental function* of an r.i. norm ϱ on $\mathcal{M}^+(I)$ is defined by

$$\varphi_{\varrho}(t) := \varrho(\chi_{(0,t)}), \quad t \in I, \quad \varphi_{\varrho}(0) = 0.$$

The fundamental function is quasiconcave on $[0, 1)$, continuous except perhaps at the origin and satisfies

$$\varphi_{\varrho}(t) \varphi_{\varrho'}(t) = t, \quad t \in I.$$

Quasiconcave functions need not be concave, however, every r.i. space can be equivalently renormed so that its fundamental function is concave.

Let φ be a concave function. We define the *Lorentz endpoint space* $\Lambda_{\varphi}(\Omega)$ by the function norm

$$\varrho_{\Lambda_{\varphi}}(f) := \int_0^1 f^*(t) d\varphi(t), \quad f \in \mathcal{M}^+(I),$$

where $d\varphi$ stands for the Lebesgue-Stieltjes measure associated with φ . We define the *Marcinkiewicz endpoint space* $M_{\varphi}(\Omega)$ by the function norm

$$\varrho_{M_{\varphi}}(f) := \sup_{0 < t < 1} f^{**}(t) \varphi(t), \quad f \in \mathcal{M}^+(I).$$

The endpoint spaces $\Lambda_{\varphi}(\Omega)$ and $M_{\varphi}(\Omega)$ are r.i. spaces with the fundamental function φ . If $X(\Omega)$ is an r.i. space with the fundamental function φ , then

$$\Lambda_{\varphi}(\Omega) \hookrightarrow X \hookrightarrow M_{\varphi}(\Omega).$$

In other words, $\Lambda_{\varphi}(\Omega)$ and $M_{\varphi}(\Omega)$ are respectively the smallest and the largest r.i. spaces having the fundamental function equivalent to φ .

The associate space of a Lorentz endpoint space Λ_{φ} is the Marcinkiewicz endpoint space M_{ψ} where both φ and ψ are concave and $\varphi(t) \psi(t) = t$ on I .

If $|\Omega| < \infty$, then for every r.i. space $X(\Omega)$

$$L^{\infty}(\Omega) \hookrightarrow X(\Omega) \hookrightarrow L^1(\Omega).$$

Assume either $1 < p, q < \infty$ or $p = q = 1$ or $p = q = \infty$. The *Lorentz space* $L^{p,q}(\Omega)$ is defined by the functional

$$\varrho_{p,q}(f) = \varrho_q \left(t^{\frac{1}{p} - \frac{1}{q}} f^*(t) \right), \quad f \in \mathcal{M}^+(I),$$

where

$$\varrho_q(f) = \begin{cases} \left(\int_0^1 f(t)^q dt \right)^{\frac{1}{q}}, & 1 \leq q < \infty, \\ \text{ess sup}_{0 < t < 1} f(t), & q = \infty, \end{cases}$$

stands for the Banach function norm of the *Lebesgue space* $L^q(\Omega)$. The functional $\varrho_{p,q}$ is a Banach function norm if and only if $1 \leq q \leq p$. However, for $1 < p < \infty$, $\varrho_{p,q}$ can be equivalently replaced by Banach function norm

$$\varrho_{(p,q)}(f) = \varrho_q \left(t^{\frac{1}{p} - \frac{1}{q}} f^{**}(t) \right).$$

The fundamental function of the norm $\varrho_{(p,q)}$ satisfies

$$\varphi_{\varrho_{(p,q)}}(t) \simeq t^{\frac{1}{p}}, \quad t \in [0, 1).$$

The spaces $L^{p,1}(\Omega)$ and $L^{p,\infty}(\Omega)$ are equal to the Lorentz and Marcinkiewicz endpoint spaces $\Lambda_{\varphi}(\Omega)$ and $M_{\varphi}(\Omega)$, respectively, with $\varphi(t) = t^{1/p}$. If the first parameter is fixed, then the Lorentz spaces are nested, i.e., we have $L^{p,q}(\Omega) \hookrightarrow L^{p,r}(\Omega)$ whenever $1 < p < \infty$ and $1 \leq q \leq r \leq \infty$.

2.2 Orlicz Spaces

We also need to know definitions and all the basic facts about Young functions and Orlicz Spaces. All of these can be found for instance in the book by L. Pick, A. Kufner, O. John and S. Fučík [23].

We shall say that A is a *Young function* if there exists a function $a: [0, \infty) \rightarrow [0, \infty)$ such that

$$A(t) = \int_0^t a(s) \, ds, \quad t \in [0, \infty),$$

and a has the following properties:

- (i) $a(s) > 0$ for $s > 0$, $a(0) = 0$;
- (ii) a is right-continuous;
- (iii) a is nondecreasing;
- (iv) $\lim_{s \rightarrow \infty} a(s) = \infty$.

Every Young function is continuous, nonnegative, strictly increasing, convex on $[0, \infty)$ and satisfies

$$\lim_{t \rightarrow 0^+} \frac{A(t)}{t} = \lim_{t \rightarrow \infty} \frac{t}{A(t)} = 0.$$

Furthermore, one has

$$A(\alpha t) \leq \alpha A(t), \quad \alpha \in [0, 1], \quad t \geq 0,$$

and

$$A(\beta t) \geq \beta A(t), \quad \beta \in (1, \infty), \quad t \geq 0.$$

Moreover $A(t)/t$ is increasing on $(0, \infty)$ and we have the estimates

$$A(t) \leq a(t)t \leq A(2t), \quad t \in (0, \infty).$$

For a Young function A and a domain $\Omega \subseteq \mathbb{R}^n$, the *Orlicz space* $L^A = L^A(\Omega)$ is the collection of all functions $f \in \mathcal{M}(\Omega)$ for which there exists a $\lambda > 0$ such that

$$\int_{\Omega} A\left(\frac{|f(x)|}{\lambda}\right) \, dx < \infty.$$

The Orlicz space $L^A(\Omega)$ is endowed with the *Luxemburg norm*

$$\|f\|_{L^A} := \inf \left\{ \lambda > 0, \int_{\Omega} A\left(\frac{|f(x)|}{\lambda}\right) \, dx \leq 1 \right\}.$$

The *complementary function* \tilde{A} of a Young function A is given by

$$\tilde{A}(t) := \sup_{s>0} (st - A(s)), \quad t \in [0, \infty).$$

The complementary function \tilde{A} is a Young function as well and the complementary function of \tilde{A} is once more A . For any Young function A and its complementary function \tilde{A} there is the relation

$$t \leq A^{-1}(t) \tilde{A}^{-1}(t) \leq 2t, \quad t \in [0, \infty).$$

With the help of the complementary function, we can define an alternative *Orlicz norm* on an Orlicz space by

$$\|f\|_{(L^A)} := \sup \left\{ \int_{\Omega} |f(x)g(x)| \, dx \right\},$$

where the supremum is taken over all functions $g \in \mathcal{M}(\Omega)$ such that

$$\int_{\Omega} \tilde{A}(|g(x)|) \, dx < \infty.$$

The Luxemburg and Orlicz norms are equivalent, namely,

$$\|f\|_{L^A} \leq \|f\|_{(L^A)} \leq 2 \|f\|_{L^A}.$$

When $L^A(\Omega)$ is an Orlicz space endowed with the Luxemburg norm, then the associate space is $L^{\tilde{A}}(\Omega)$ with the Orlicz norm. In particular, the sharp Hölder inequality for Orlicz spaces has the form

$$\int_{\Omega} |f(x)g(x)| \, dx \leq \|f\|_{L^A} \|g\|_{(L^{\tilde{A}})}.$$

The Orlicz space $L^A(\Omega)$ is an r.i. space and

$$\|\chi_E\|_{L^A} = \frac{1}{A^{-1}\left(\frac{1}{|E|}\right)}$$

for every measurable $E \subseteq \Omega$ of positive measure, thus, for a bounded domain Ω , the fundamental function for the Luxemburg norm is

$$\varphi_{L^A}(t) = \frac{1}{A^{-1}\left(\frac{1}{t|\Omega|}\right)}, \quad t \in I, \quad \varphi_{L^A}(0) = 0.$$

An Orlicz space $L^A(I)$ with fundamental function φ coincides with the Marcinkiewicz endpoint space $M_{\varphi}(I)$ if and only if there exists $\delta \in (0, 1)$ such that

$$\int_0^1 A\left(\delta A^{-1}\left(\frac{1}{t}\right)\right) dt < \infty. \quad (2.1)$$

(see also [18]).

For $|\Omega| < \infty$, the inclusion relation between Orlicz spaces is governed by inequalities involving the corresponding Young functions. If A and B are Young functions then $L^A(\Omega) \hookrightarrow L^B(\Omega)$ if and only if there exist $c > 0$ and $T \geq 0$ such that

$$B(t) \leq A(ct), \quad t \geq T,$$

which we denote by $B \prec A$ or $A \succ B$. If both $A \prec B$ and $A \succ B$ hold, we say that A and B are equivalent and write $A \approx B$. When $|\Omega| < \infty$, the inclusion $L^A(\Omega) \subseteq L^B(\Omega)$ is proper if and only if

$$\limsup_{t \rightarrow \infty} \frac{B(t)}{A(\lambda t)} = 0$$

for every $\lambda > 0$. In such case we write $B \ll A$ or $A \gg B$.

If $A \prec B$ or $A \ll B$ then $\tilde{A} \succ \tilde{B}$ or $\tilde{A} \gg \tilde{B}$, respectively.

3. Proofs of Theorems B and C

Lemma 3.1. *Let A be a Young function and let ξ be a nonzero real number. Assuming*

$$\int_0^{\infty} A(s) s^{\frac{1}{\xi}-1} \, ds < \infty, \quad (3.1)$$

we define

$$E_{\xi}(t) = |\xi|^{-1} t^{-\frac{1}{\xi}} \int_0^t A(s) s^{\frac{1}{\xi}-1} \, ds, \quad t \in (0, \infty).$$

Such E_{ξ} is an increasing mapping of $(0, \infty)$ onto itself. Moreover, if $R \in (0, \infty]$, then the following relations hold.

$$\|t^{\xi} \chi_{(0,a)}(t)\|_{L^A(0,R)} = \frac{a^{\xi}}{E_{\xi}^{-1}\left(\frac{1}{a}\right)}, \quad a \in (0, R), \quad \xi > 0, \quad (3.2)$$

$$\|t^{\xi} \chi_{(a,\infty)}(t)\|_{L^A(0,\infty)} = \frac{a^{\xi}}{E_{\xi}^{-1}\left(\frac{1}{a}\right)}, \quad a \in (0, \infty), \quad \xi < 0. \quad (3.3)$$

If, in addition, $\varepsilon \in (0, R)$ and if $\xi < 0$ then

$$\|t^{\xi} \chi_{(a,R)}(t)\|_{L^A(0,R)} \simeq \|t^{\xi} \chi_{(a,\infty)}(t)\|_{L^A(0,\infty)}, \quad a \in (0, R - \varepsilon). \quad (3.4)$$

Proof. Assume (3.1). By change of variables $s \mapsto ts$ we have

$$E_\xi(t) = |\xi|^{-1} \int_0^1 A(ts) s^{\frac{1}{\xi}-1} ds, \quad t \in (0, \infty),$$

hence E_ξ is increasing.

By definition of the Luxemburg norm, we have

$$\|t^\xi \chi_{(0,a)}(t)\|_{L^A(0,R)} = \inf \left\{ \lambda > 0, \int_0^a A\left(\frac{t^\xi}{\lambda}\right) dt \leq 1 \right\}.$$

Next, by change of variables we get for $\xi > 0$

$$\begin{aligned} \|t^\xi \chi_{(0,a)}(t)\|_{L^A(0,R)} &= \inf \left\{ \lambda > 0, \frac{\lambda^{\frac{1}{\xi}}}{\xi} \int_0^{\frac{a^\xi}{\lambda}} A(s) s^{\frac{1}{\xi}-1} ds \leq 1 \right\} \\ &= \inf \left\{ \lambda > 0, a E_\xi\left(\frac{a^\xi}{\lambda}\right) \leq 1 \right\} \\ &= \frac{a^\xi}{E_\xi^{-1}\left(\frac{1}{a}\right)}. \end{aligned}$$

This proves the part (3.2). The proof of the relation (3.3) can be done in an analogous way and we omit it.

It remains to prove the (3.4). Clearly,

$$\|t^\xi \chi_{(a,\infty)}(t)\|_{L^A(0,\infty)} \geq \|t^\xi \chi_{(a,R)}(t)\|_{L^A(0,\infty)} = \|t^\xi \chi_{(a,R)}(t)\|_{L^A(0,R)}$$

by the monotonicity of the norm. On the other hand, we have by the triangle inequality

$$\|t^\xi \chi_{(a,\infty)}(t)\|_{L^A(0,\infty)} \leq \|t^\xi \chi_{(a,R)}(t)\|_{L^A(0,R)} + \|t^\xi \chi_{(R,\infty)}(t)\|_{L^A(0,\infty)}.$$

Using (3.3), the term $\|t^\xi \chi_{(R,\infty)}(t)\|_{L^A(0,\infty)}$ equals $R^\xi/E_\xi^{-1}(1/R)$ since $\xi < 0$. Thanks to the assumptions, this quantity is finite, say K . The term $\|t^\xi \chi_{(a,R)}(t)\|_{L^A(0,R)}$ is a decreasing function of the variable a , positive on $(0, R)$ and vanishing at R . Hence for every $\varepsilon \in (0, R)$ there exists a constant C such that

$$K \leq C \|t^\xi \chi_{(a,R)}(t)\|_{L^A(0,R)}, \quad a \in (0, R - \varepsilon).$$

For those a we conclude that

$$\|t^\xi \chi_{(a,\infty)}(t)\|_{L^A(0,\infty)} \leq (C + 1) \|t^\xi \chi_{(a,R)}(t)\|_{L^A(0,R)}.$$

□

Lemma 3.2. Let $0 < \alpha < 1$, $\beta > 0$, $\alpha + 1/\beta \geq 1$ and let φ be a quasiconcave function on $(0, 1)$. We define

$$\bar{\varphi}(t) = t^{\beta(1-\alpha)} \sup_{t < s < 1} \varphi(s) s^{\beta(\alpha-1)}, \quad t \in (0, 1), \quad \bar{\varphi}(0) = 0.$$

Then $\bar{\varphi}(t)$ and $\bar{\varphi}(t^{1/\beta}) t^\alpha$ are quasiconcave.

Proof. Since φ is nondecreasing, we have for every $t \in (0, 1)$

$$\begin{aligned} \bar{\varphi}(t) &= t^{\beta(1-\alpha)} \sup_{t < s < 1} s^{\beta(\alpha-1)} \sup_{0 < r < s} \varphi(r) \\ &= t^{\beta(1-\alpha)} \sup_{0 < r < 1} \varphi(r) \sup_{\max\{r,t\} < s < 1} s^{\beta(\alpha-1)} \\ &= t^{\beta(1-\alpha)} \sup_{0 < r < 1} \varphi(r) \min \left\{ t^{\beta(\alpha-1)}, r^{\beta(\alpha-1)} \right\} \\ &= \sup_{0 < r < 1} \varphi(r) \min \left\{ 1, \left(\frac{t}{r}\right)^{\beta(1-\alpha)} \right\}, \end{aligned}$$

hence $\bar{\varphi}$ is nondecreasing. Next, by definition, we have

$$\frac{\bar{\varphi}(t)}{t} = t^{\beta(1-\alpha)-1} \sup_{t < s < 1} \varphi(s) s^{\beta(\alpha-1)}, \quad t \in (0, 1).$$

The function $t^{\beta(1-\alpha)-1}$ is nonincreasing since the exponent $\beta(\alpha - 1) + 1$ is nonnegative by the assumptions of the lemma. Hence $\bar{\varphi}(t)/t$ is nonincreasing on $(0, 1)$.

The function $\bar{\varphi}(t^{1/\beta}) t^\alpha$ is increasing as a composition of nondecreasing functions multiplied by increasing function t^α . Next the expression

$$\frac{\bar{\varphi}(t^{\frac{1}{\beta}}) t^\alpha}{t} = \sup_{t < s < 1} \varphi(s^{\frac{1}{\beta}}) s^{\alpha-1}, \quad t \in (0, 1),$$

is nonincreasing. The rest is trivial. □

Lemma 3.3. *Let u be a quasiconcave, right continuous at origin and strictly increasing function on $[0, 1)$ such that*

$$\lim_{t \rightarrow 0^+} \frac{u(t)}{t} = \infty.$$

Then there exists a Young function B such that the fundamental function of the Orlicz space $L^B(0, 1)$ is equivalent to u on $[0, 1)$. Moreover

$$\tilde{B}^{-1}(t) \simeq t u\left(\frac{1}{t}\right) \quad t \in (0, 1),$$

where \tilde{B} is the complementary Young function to B .

Proof. We can assume without loss of generality that $u(1) = 1$. Then by continuity of u we have $u(0, 1) = (0, 1)$. Let us define

$$b(s) = \begin{cases} \frac{1}{s u^{-1}\left(\frac{1}{s}\right)}, & s \in (1, \infty), \\ s, & s \in [0, 1]. \end{cases}$$

and

$$B(t) = \int_0^t b(s) \, ds, \quad t \in [0, \infty).$$

We claim that B is a Young function. The properties (i) and (ii) from the definition of Young function are clear. Let us prove that b is nondecreasing. The function $u(t)/t$ is nonincreasing and u itself is increasing, hence $s/u^{-1}(s)$ is nonincreasing and therefore $b(s) = \frac{1}{s u^{-1}(1/s)}$ is nondecreasing on $(1, \infty)$ and also (trivially) on $[0, 1]$. It remains to show that $\lim_{s \rightarrow \infty} b(s) = \infty$. Indeed,

$$\lim_{s \rightarrow \infty} b(s) = \lim_{t \rightarrow 0^+} \frac{t}{u^{-1}(t)} = \lim_{t \rightarrow 0^+} \frac{u(t)}{t} = \infty.$$

Now, since B is a Young function, we have that

$$B(t) \leq b(t) t \leq B(2t), \quad t \in [0, \infty).$$

It follows by definition of b that

$$B(t) \leq \frac{1}{u^{-1}\left(\frac{1}{t}\right)} \leq B(2t), \quad t \in (1, \infty).$$

Applying the increasing function B^{-1} , we get

$$t \leq B^{-1}\left(\frac{1}{u^{-1}\left(\frac{1}{t}\right)}\right) \leq 2t, \quad t \in (1, \infty),$$

that is, taking reciprocal values and $t \mapsto 1/s$,

$$\frac{s}{2} \leq \frac{1}{B^{-1}\left(\frac{1}{u^{-1}(s)}\right)} \leq s, \quad s \in (0, 1).$$

Finally, since u is increasing on $(0, 1)$ and $u(0, 1) = (0, 1)$, this implies

$$\frac{u(y)}{2} \leq \frac{1}{B^{-1}\left(\frac{1}{y}\right)} \leq u(y), \quad y \in (0, 1).$$

Hence by the definition of the fundamental function for the Luxemburg norm we conclude that

$$\varphi_{L^B}(t) \simeq u(t), \quad t \in (0, 1).$$

In addition

$$\tilde{B}^{-1}(t) \simeq \frac{t}{B^{-1}(t)} \simeq t u\left(\frac{1}{t}\right) \quad t \in (0, 1).$$

The proof is complete. □

The following proposition enables us to reduce an embedding to a Lorentz endpoint spaces only to testing on characteristic functions. The idea of this statement is based on [2, Theorem 7], where the Lorentz space $L^{p,1}(\Omega)$ occurs as a target space, nonetheless the proof also works for any Lorentz endpoint space. For the sake of completeness, we show also the proof here.

Proposition 3.4. *Let $Y(0, 1)$ be a Banach function space and $\Lambda(0, 1)$ be a Lorentz endpoint space over $(0, 1)$. Suppose that T is a sublinear operator mapping $\Lambda(0, 1)$ to $Y(0, 1)$ and satisfying*

$$\|T\chi_E\|_{Y(0,1)} \lesssim \|\chi_E\|_{\Lambda(0,1)} \quad (3.5)$$

for every measurable set $E \subseteq (0, 1)$. Then

$$\|Tf\|_{Y(0,1)} \lesssim \|f\|_{\Lambda(0,1)}$$

for every $f \in \Lambda(0, 1)$.

Proof. Let f be a simple nonnegative function on $(0, 1)$. Thus f can be written as a finite sum $f = \sum_j \lambda_j \chi_{E_j}$, where λ_j are positive real numbers and the sets E_j are measurable subsets of $(0, 1)$ satisfying $E_1 \subseteq E_2 \subseteq \dots$. Then, as readily seen, we have $f^* = \sum_j \lambda_j \chi_{E_j}^*$. Let φ be a fundamental function of $\Lambda(0, 1)$. By the definition of the Lorentz norm we have

$$\|f\|_{\Lambda(0,1)} = \int_0^1 f^* d\varphi = \int_0^1 \sum_j \lambda_j \chi_{E_j}^* d\varphi = \sum_j \lambda_j \int_0^1 \chi_{E_j}^* d\varphi = \sum_j \lambda_j \|\chi_{E_j}\|_{\Lambda(0,1)}.$$

On account of the sublinearity of T we have $|Tf| \leq \sum_j \lambda_j |T\chi_{E_j}|$, and consequently by (3.5) and by axioms (P1) and (P2) we obtain

$$\|Tf\|_{Y(0,1)} \leq \sum_j \lambda_j \|T\chi_{E_j}\|_{Y(0,1)} \lesssim \sum_j \lambda_j \|\chi_{E_j}\|_{\Lambda(0,1)} = \|f\|_{\Lambda(0,1)}.$$

Now if f is simple but no longer nonnegative, we use the same for the positive part of f and for the negative part of f .

Suppose that f is an arbitrary function in $\Lambda(0, 1)$ and let f_n be a sequence of simple integrable functions converging to f in $\Lambda(0, 1)$. Then

$$\|T(f_n) - T(f_m)\|_{Y(0,1)} \leq \|T(f_n - f_m)\|_{Y(0,1)} \lesssim \|f_n - f_m\|_{\Lambda(0,1)},$$

and Tf_n is Cauchy, hence convergent in $Y(0, 1)$. Since limits are unique in $Y(0, 1)$, it follows that $\lim Tf_n = Tf$ and

$$\|Tf\|_{Y(0,1)} = \lim \|Tf_n\|_{Y(0,1)} \lesssim \lim \|f_n\|_{\Lambda(0,1)} = \|f\|_{\Lambda(0,1)}$$

as we wished to show. □

Next proposition provides the optimal r.i. range space for the operator H_α^β and a given r.i. domain space. The proof can be obtained by simple modification of the proof of [11, Theorem 4.5], where $\beta = 1$ and $\alpha = 1/n$ and therefore is omitted.

Proposition 3.5. *Let $X(0, 1)$ be an r.i. space, $0 < \alpha < 1$, $\beta > 0$ and $\alpha + 1/\beta \geq 1$. Then*

$$Y'(0, 1) := \left\{ f \in \mathcal{M}(0, 1), \|f\|_{Y'} := \|(H_\alpha^\beta)' f^*\|_{X'(0,1)} < \infty \right\}$$

is an r.i. space, such that the associate space $Y(0, 1)$ is the smallest space among r.i. spaces rendering $H_\alpha^\beta: X(0, 1) \rightarrow Y(0, 1)$ true.

The construction of the optimal r.i. domain for H_α^β and a given r.i. range space is similar to that in [16, Theorem 3.3], as well as its proof, needing only trivial modifications. The fact that $\mu_f = \mu_h$ is denoted by $f \sim h$.

Proposition 3.6. *Let $Y(0, 1)$ be an r.i. space such that $Y(0, 1) \hookrightarrow L^{\frac{1}{\beta(1-\alpha)}, 1}(0, 1)$. Then*

$$X(0, 1) := \left\{ f \in \mathcal{M}(0, 1), \|f\|_X := \sup_{h \sim f} \|H_\alpha^\beta h\|_Y < \infty \right\}$$

is the largest r.i. space satisfying $H_\alpha^\beta: X(0, 1) \rightarrow Y(0, 1)$.

Remark 3.7. Now, under the same assumptions on α and β as in Proposition 3.5 one can readily calculate the optimal endpoint estimates

$$H_\alpha^\beta: L^1(0, 1) \rightarrow L^{\frac{1}{\beta(1-\alpha)}, 1}(0, 1) \quad (3.6)$$

and

$$H_\alpha^\beta: L^{\frac{1}{\alpha}, 1}(0, 1) \rightarrow L^\infty(0, 1). \quad (3.7)$$

The relation (3.6) shows that the assumption in Proposition 3.6 cause no lose of generality.

Let us also discuss the assumption on fundamental function

$$\sup_{0 < t < 1} \varphi(t^{\frac{1}{\beta}}) t^{\alpha-1} = \infty$$

in Theorems A and B. If this condition is not satisfied, then

$$\varphi(t) \leq Ct^{\beta(1-\alpha)}, \quad t \in (0, 1),$$

for some $C > 0$, which is equivalent to

$$L^{\frac{1}{\beta(1-\alpha)}, \infty}(0, 1) \subseteq M_\varphi(0, 1),$$

hence, thanks to (3.6), also to

$$H_\alpha^\beta: L^1(0, 1) \rightarrow M_\varphi(0, 1).$$

Since $L^1(0, 1)$ is the largest r.i. space, we can see that this considered assumption cause no relevant restriction to target spaces.

Proof of Theorem C. We first prove the inequality “ \gtrsim ”. Let α and β be as in the theorem and let us set

$$\psi(t) = t \sup_{t < s < 1} \varphi(s^{\frac{1}{\beta}}) s^{\alpha-1}, \quad t \in (0, 1).$$

Then, by Lemma 3.2, $\psi(t)$ is quasiconcave function on $(0, 1)$ and $\psi(t) \geq t^\alpha \varphi(t^{1/\beta})$ for $t \in (0, 1)$. We claim that ψ is up to equivalence the smallest function with this property. Indeed, let $\eta(t)$ be a quasiconcave function on $[0, 1)$ and $\eta(t) \geq t^\alpha \varphi(t^{1/\beta})$ for $t \in (0, 1)$. Then

$$\begin{aligned} \varphi(s^{\frac{1}{\beta}}) s^{\alpha-1} &\leq \frac{\eta(s)}{s}, \quad s \in (0, 1), \\ \sup_{t < s < 1} \varphi(s^{\frac{1}{\beta}}) s^{\alpha-1} &\leq \sup_{t < s < 1} \frac{\eta(s)}{s}, \quad t \in (0, 1). \end{aligned}$$

The right hand side of the last inequality equals $\eta(t)/t$ by quasiconcavity of η . Then multiplying by t gives that $\psi(t) \leq \eta(t)$ for $t \in (0, 1)$.

Now by Proposition 3.6 we have

$$\begin{aligned} \varphi_X(t) &= \sup_{h \sim \chi_{(0,t)}} \left\| \int_{s^\beta}^1 y^{\alpha-1} h(y) dy \right\|_Y \geq \left\| \int_{s^\beta}^1 y^{\alpha-1} \chi_{(0,t)}(y) dy \right\|_Y \\ &= \left\| \chi_{(0,t^{1/\beta})}(s) \int_{s^\beta}^t y^{\alpha-1} dy \right\|_Y \simeq \left\| \chi_{(0,t^{1/\beta})}(s) (t^\alpha - s^{\alpha\beta}) \right\|_Y \\ &\geq \left\| \chi_{(0,t^{1/\beta}/2)}(s) (t^\alpha - t^\alpha 2^{-\alpha\beta}) \right\|_Y \simeq t^\alpha \left\| \chi_{(0,t^{1/\beta}/2)}(s) \right\|_Y \\ &\simeq t^\alpha \left\| \chi_{(0,t^{1/\beta})}(s) \right\|_Y = t^\alpha \varphi_Y(t^{\frac{1}{\beta}}). \end{aligned}$$

Hence $\varphi_X(t) \gtrsim t^\alpha \varphi(t^{1/\beta})$ and by the claim $\psi(t) \lesssim \varphi_X(t)$.

Let us focus on the inequality “ \lesssim ”. Let $t \in (0, 1/2)$ then

$$\begin{aligned}
\varphi_X(t) &= \sup_{h \sim \chi_{(0,t)}} \left\| \int_{s^\beta}^1 y^{\alpha-1} h(y) dy \right\|_Y \\
&= \sup_{h \sim \chi_{(0,t)}} \sup_{0 < s < 1} \left(\int_{r^\beta}^1 y^{\alpha-1} h(y) dy \right)^{**} (s) \varphi(s) \\
&= \sup_{h \sim \chi_{(0,t)}} \sup_{0 < s < 1} \frac{\varphi(s)}{s} \int_0^s \int_{r^\beta}^1 y^{\alpha-1} h(y) dy dr \\
&= \sup_{h \sim \chi_{(0,t)}} \sup_{0 < s < 1} \frac{\varphi(s)}{s} \int_0^1 y^{\alpha-1} h(y) \int_0^{\min\{y^{\frac{1}{\beta}}, s\}} dr dy \\
&= \sup_{0 < s < 1} \sup_{h \sim \chi_{(0,t)}} \frac{\varphi(s)}{s} \left(\int_0^{s^\beta} y^{\alpha+\frac{1}{\beta}-1} h(y) dy + s \int_{s^\beta}^1 y^{\alpha-1} h(y) dy \right) \\
&= \sup_{0 < s < 1} \sup_{\substack{0 < z < s^\beta \\ s^\beta < z+t < 1}} \frac{\varphi(s)}{s} \left(\int_z^{s^\beta} y^{\alpha+\frac{1}{\beta}-1} dy + s \int_{s^\beta}^{z+t} y^{\alpha-1} dy \right).
\end{aligned}$$

Denote

$$V(s, z, t) = \frac{\varphi(s)}{s} \left(\int_z^{s^\beta} y^{\alpha+\frac{1}{\beta}-1} dy + s \int_{s^\beta}^{z+t} y^{\alpha-1} dy \right).$$

We split the area over which the supremum is taken into three disjoint regions, namely

$$\varphi_X(t) \leq \sup_{\substack{0 < s < t^{\frac{1}{\beta}} \\ 0 < z < s^\beta}} V(s, z, t) + \sup_{\substack{t^{\frac{1}{\beta}} < s < (1-t)^{\frac{1}{\beta}} \\ s^\beta - t < z < s^\beta}} V(s, z, t) + \sup_{\substack{(1-t)^{\frac{1}{\beta}} < s < 1 \\ s^\beta - t < z < 1-t}} V(s, z, t).$$

Now

$$\begin{aligned}
\sup_{\substack{0 < s < t^{\frac{1}{\beta}} \\ 0 < z < s^\beta}} V(s, z, t) &\leq \sup_{0 < s < t^{\frac{1}{\beta}}} \frac{\varphi(s)}{s} \left(\int_0^{s^\beta} y^{\alpha+\frac{1}{\beta}-1} dy + s \int_{s^\beta}^{s^\beta+t} y^{\alpha-1} dy \right) \\
&\leq \sup_{0 < s < t^{\frac{1}{\beta}}} \frac{\varphi(s)}{s} \left(\int_0^{s^\beta} s^{\beta(\alpha-1)+1} dy + s \int_0^t y^{\alpha-1} dy \right) \\
&\lesssim \sup_{0 < s < t^{\frac{1}{\alpha}}} \frac{\varphi(s)}{s} (s s^{\alpha\beta} + s t^\alpha) \\
&\lesssim \varphi(t^{\frac{1}{\beta}}) t^\alpha \\
&\leq t \sup_{t < s < 1} \varphi(s^{\frac{1}{\beta}}) s^{\alpha-1},
\end{aligned}$$

$$\begin{aligned}
\sup_{\substack{t^{\frac{1}{\beta}} < s < (1-t)^{\frac{1}{\beta}} \\ s^\beta - t < z < s^\beta}} V(s, z, t) &\leq \sup_{\substack{t^{\frac{1}{\beta}} < s < (1-t)^{\frac{1}{\beta}} \\ s^\beta - t < z < s^\beta}} \frac{\varphi(s)}{s} \left(\int_{s^\beta-t}^{s^\beta} y^{\alpha+\frac{1}{\beta}-1} dy + s \int_{s^\beta}^{s^\beta+t} y^{\alpha-1} dy \right) \\
&\leq \sup_{t^{\frac{1}{\beta}} < s < (1-t)^{\frac{1}{\beta}}} \frac{\varphi(s)}{s} \left(t s^{\beta(\alpha-1)+1} + s t s^{\beta(\alpha-1)} \right) \\
&\lesssim t \sup_{t^{\frac{1}{\beta}} < s < 1} \varphi(s) s^{\beta(\alpha-1)} \\
&\lesssim t \sup_{t < s < 1} \varphi(s^{\frac{1}{\beta}}) s^{\alpha-1}
\end{aligned}$$

and

$$\begin{aligned}
\sup_{\substack{(1-t)^{\frac{1}{\beta}} < s < 1 \\ s^\beta - t < z < 1-t}} V(s, z, t) &\leq \sup_{(1-t)^{\frac{1}{\beta}} < s < 1} \frac{\varphi(s)}{s} \left(\int_{s^\beta - t}^{s^\beta} y^{\alpha + \frac{1}{\beta} - 1} dy + s \int_{s^\beta}^1 y^{\alpha - 1} dy \right) \\
&\leq \sup_{(1-t)^{\frac{1}{\beta}} < s < 1} \frac{\varphi(s)}{s} \left(t s^{\beta(\alpha - 1) + 1} + s(1 - s^\beta) s^{\beta(\alpha - 1)} \right) \\
&\leq \sup_{(1-t)^{\frac{1}{\beta}} < s < 1} \frac{\varphi(s)}{s} \left(t s^{\beta(\alpha - 1) + 1} + t s^{\beta(\alpha - 1) + 1} \right) \\
&\lesssim t \sup_{(1-t)^{\frac{1}{\beta}} < s < 1} \varphi(s) s^{\beta(\alpha - 1)} \\
&\lesssim t \sup_{t < s < 1} \varphi(s^{\frac{1}{\beta}}) s^{\alpha - 1}.
\end{aligned}$$

Finally

$$\varphi_X(t) \lesssim t \sup_{t < s < 1} \varphi(s) s^{\alpha - 1}, \quad t \in (0, 1/2).$$

□

Proof of Theorem B. Consider the Orlicz space $L^A(0, 1)$ and the Marcinkiewicz space $M(0, 1)$ from the assumption of the theorem. They satisfy the inequality

$$\left\| \int_{t^\beta}^1 g(s) s^{\alpha - 1} ds \right\|_{M(0,1)} \lesssim \|g\|_{L^A(0,1)}, \quad g \in \mathcal{M}^+.$$

By the L^1 duality, this is the same as

$$\left\| t^{\alpha - 1} \int_0^{t^{\frac{1}{\beta}}} f(s) ds \right\|_{L^{\tilde{A}}(0,1)} \lesssim \|f\|_{M'(0,1)}, \quad f \in \mathcal{M}^+.$$

where \tilde{A} is the complementary Young function to A . This is equivalent to

$$\left\| t^{\alpha - 1} \int_0^{t^{\frac{1}{\beta}}} f^*(s) ds \right\|_{L^{\tilde{A}}(0,1)} \lesssim \|f\|_{M'(0,1)}, \quad f \in M'(0, 1).$$

Indeed, one implication follows just by passing to only nonincreasing functions with the fact that $\|f\|_{M'(0,1)} = \|f^*\|_{M(0,1)}$, and the other holds thanks to the Hardy-Littlewood inequality applied to functions f and $\chi_{(0, t^{1/\beta})}$.

Using the fact that $M'(0, 1)$ is a Lorentz endpoint space and passing to the characteristic functions while keeping Proposition 3.4 in mind, this is equivalent to

$$\left\| t^{\alpha - 1} \int_0^{t^{\frac{1}{\beta}}} \chi_{(0,a)}(s) ds \right\|_{L^{\tilde{A}}(0,1)} \lesssim \varphi_{M'}(a), \quad a \in (0, 1). \quad (3.8)$$

Let us compute the left hand side. Clearly

$$\begin{aligned}
\left\| t^{\alpha - 1} \int_0^{t^{\frac{1}{\beta}}} \chi_{(0,a)}(s) ds \right\|_{L^{\tilde{A}}(0,1)} &= \|t^{\alpha - 1} \chi_{(0,a^\beta)}(t) \cdot t^{\frac{1}{\beta}} + t^{\alpha - 1} \chi_{(a^\beta, 1)}(t) \cdot a\|_{L^{\tilde{A}}(0,1)} \\
&\leq \|t^{\alpha + \frac{1}{\beta} - 1} \chi_{(0,a^\beta)}(t)\|_{L^{\tilde{A}}(0,1)} + a \|t^{\alpha - 1} \chi_{(a^\beta, 1)}(t)\|_{L^{\tilde{A}}(0,1)}.
\end{aligned}$$

We suppose that $a \in (0, 2^{-1/\beta})$, since we are interested only in values of a near zero. We show that the second summand dominates the first one. Indeed, for any r.i. norm we have

$$a \|t^{\alpha - 1} \chi_{(a^\beta, 1)}(t)\| \geq a \|t^{\alpha - 1} \chi_{(a^\beta, 2a^\beta)}(t)\| \geq a(2a^\beta)^{\alpha - 1} \|\chi_{(a^\beta, 2a^\beta)}(t)\| \simeq a^{\beta(\alpha - 1) + 1} \|\chi_{(0, a^\beta)}(t)\| \geq \|t^{\alpha + \frac{1}{\beta} - 1} \chi_{(0, a^\beta)}(t)\|$$

Therefore we can state that

$$\left\| t^{\alpha-1} \int_0^{t^{\frac{1}{\beta}}} \chi_{(0,a)}(s) ds \right\|_{L^{\tilde{A}}(0,1)} \simeq a \|t^{\alpha-1} \chi_{(a^\beta,1)}(t)\|_{L^{\tilde{A}}(0,1)}. \quad (3.9)$$

At this moment, it is the time for using Lemma 3.1. We need the part (3.4) with (3.3) for $\xi = \alpha - 1 < 0$, $R = 1$ and $\varepsilon = 1 - 2^{-1/\beta}$. The assumption (3.1) can be rendered as satisfied without any loss of generality since the domain is of finite measure, hence the appropriate Young function can be redefined on $(0, 1)$ without any effect to the corresponding Orlicz space. Note also that we are using the complementary Young function \tilde{A} instead of A . Hence we conclude that (3.8) is equivalent to

$$\frac{a^{\beta(\alpha-1)+1}}{E_{\alpha-1}^{-1}(a^{-\beta})} \lesssim \varphi_{M'}(a), \quad a \in (0, 2^{-\frac{1}{\beta}}). \quad (3.10)$$

Now we substitute $t = a^{-\beta}$ and use the fact that $\varphi_{M'}(a) \varphi(a) = a$. We get

$$\varphi(t^{-\frac{1}{\beta}}) t^{1-\alpha} \lesssim E_{\alpha-1}^{-1}(t), \quad t \in (2^{\frac{1}{\beta}}, \infty). \quad (3.11)$$

Let us define

$$F(t) = \bar{\varphi}(t^{-\frac{1}{\beta}}) t^{1-\alpha}, \quad t \in (0, 1),$$

where the function $\bar{\varphi}(t)$ is taken from Lemma 3.2. We claim that $F(t)$ is the least nondecreasing majorant of $\varphi(t^{-1/\beta}) t^{1-\alpha}$. Indeed,

$$\bar{\varphi}(t) = t^{\beta(1-\alpha)} \sup_{t < s < 1} \varphi(s) s^{\beta(\alpha-1)}, \quad t \in (0, 1),$$

hence

$$\bar{\varphi}(t^{-\frac{1}{\beta}}) t^{1-\alpha} = \sup_{0 < s < t} \varphi(s^{-\frac{1}{\beta}}) s^{1-\alpha}, \quad t \in (0, 1),$$

and the claim follows.

Since the function $E_{\alpha-1}$ is strictly increasing as well as its inverse, we can enlarge the left hand side of the inequality (3.11) by $F(t)$. Hence we can equivalently continue by

$$F(t) \lesssim E_{\alpha-1}^{-1}(t), \quad t \in (2^{\frac{1}{\beta}}, \infty). \quad (3.12)$$

Now Lemma 3.3 comes to play with $u(t) = \bar{\varphi}(t^{1/\beta}) t^\alpha$. By Lemma 3.2 u is quasiconcave and strictly increasing on $(0, 1)$. Next,

$$\lim_{t \rightarrow 0^+} \frac{u(t)}{t} = \lim_{t \rightarrow 0^+} \sup_{t < s < 1} \varphi(s^{\frac{1}{\beta}}) s^{\alpha-1} = \infty$$

thanks to the assumption of the theorem. Also, u is right continuous at the origin since

$$u(t) = t \sup_{t < s < 1} \varphi(s^{\frac{1}{\beta}}) s^{\alpha-1} \leq \varphi(1) t^\alpha.$$

We obtain a Young function B such that $\tilde{B}^{-1}(t) \simeq F(t)$. Theorem C ensures that the space $L^B(0, 1)$ has the same fundamental function as the optimal r.i. domain $X(0, 1)$ in $H_\alpha^\beta: X(0, 1) \rightarrow M(0, 1)$. Using this and passing to inverse functions, (3.12) is equivalent to the existence of some constant $C > 0$ such that

$$E_{\alpha-1}(t) \leq \tilde{B}(Ct), \quad t \in (c, \infty),$$

where $c = E_{\alpha-1}^{-1}(2^{1/\beta}) > 0$. This is however equivalent to

$$E_{\alpha-1}(t) \lesssim \tilde{B}(Ct), \quad t \in (2, \infty),$$

which is nothing but

$$\int_0^t \frac{\tilde{A}(s)}{s^{1/(1-\alpha)+1}} ds \lesssim \frac{\tilde{B}(Ct)}{t^{1/(1-\alpha)}}, \quad t \in (2, \infty).$$

Finally observe that the quantities $\int_0^t \tilde{A}(s) s^{1/(\alpha-1)-1} ds$ and $\int_1^t \tilde{A}(s) s^{1/(\alpha-1)-1} ds$ are comparable since $t \in (2, \infty)$. One can now immediately observe that the resulting inequality does not depend on the behavior of the Young function \tilde{A} on the interval $(0, 1)$. \square

4. Proof of Theorem A

Before proving Theorem A we need several auxiliary results. The next theorem is the crucial ingredient in the proof of the main result and it reveals the constructive approach to the nonexistence of an optimal Orlicz domain space in appropriate situations.

Theorem 4.1. Let Young functions A and B satisfy for $0 < \alpha < 1$ and some $C > 0$ the inequality

$$\int_1^t \frac{A(s)}{s^{1/(1-\alpha)+1}} ds \lesssim \frac{B(Ct)}{t^{1/(1-\alpha)}}, \quad t \in (2, \infty). \quad (4.1)$$

Denote $G(t) = t^{1/(\alpha-1)} B(Ct)$. If

$$\limsup_{t \rightarrow \infty} G(t) = \infty \quad (4.2)$$

and

$$\limsup_{t \rightarrow \infty} \frac{1}{G(Kt)} \int_1^t \frac{G(s)}{s} ds = \infty \quad (4.3)$$

for every $K \geq 1$, then there exists a Young function A_1 satisfying $A_1 \asymp A$ and also

$$\int_1^t \frac{A_1(s)}{s^{1/(1-\alpha)+1}} ds \lesssim \frac{B(C_1 t)}{t^{1/(1-\alpha)}}, \quad t \in (2, \infty).$$

Proof. Let A and G be the functions from the assumptions. First, we establish an upper bound for A . Namely, for $t \in (1, \infty)$

$$G(2t) \gtrsim \int_1^{2t} \frac{A(s)}{s^{1/(1-\alpha)+1}} ds \geq \int_t^{2t} \frac{A(s)}{s^{1/(1-\alpha)+1}} ds \geq A(t) \int_t^{2t} s^{1/(\alpha-1)-1} ds \simeq A(t) t^{1/(\alpha-1)}. \quad (4.4)$$

Using this, we obtain the existence of $\beta > 0$ such that

$$\beta B(2Ct) > A(t), \quad t \in (1, \infty). \quad (4.5)$$

Now we fix this β and, for every $t \in (1, \infty)$, we define the set

$$G_t = \left\{ s \in (1, \infty), \frac{A(s)}{s} \geq \beta \frac{B(2Ct)}{t} \right\}.$$

Since $A(s)/s$ is a nondecreasing mapping from $(0, \infty)$ onto itself, the sets G_t are upper segments. In particular, G_t is nonempty for every $t \in (1, \infty)$. Let us define $\tau = \tau_t = \inf G_t$. Observe that, for $t \in (1, \infty)$ and $s \in (1, t)$,

$$\frac{A(s)}{s} \leq \frac{A(t)}{t} < \beta \frac{B(2Ct)}{t}$$

thanks to the estimate (4.5). Hence $\tau_t > t$ for every t . Moreover, since $A(t)/t$ is continuous, we have the equality

$$\frac{A(\tau)}{\tau} = \beta \frac{B(2Ct)}{t}, \quad t \in (1, \infty). \quad (4.6)$$

Let K be a real number such that $K \geq 1$. Then

$$\limsup_{t \rightarrow \infty} \frac{A(\tau_t)}{\tau_t} \cdot \frac{t}{A(2Kt)} = \infty. \quad (4.7)$$

Indeed, suppose that there exists $K \geq 1$ and some $L > 0$ such that there is for all $t \in (1, \infty)$ the estimate

$$\frac{A(\tau)}{\tau} \cdot \frac{t}{A(2Kt)} < L,$$

or equivalently

$$\frac{A(2Kt)}{t} > L^{-1} \frac{A(\tau)}{\tau}. \quad (4.8)$$

Now for $t > 2$ the following holds:

$$\begin{aligned} G(Kt) &\gtrsim \int_1^{Kt} \frac{A(s)}{s^{1/(1-\alpha)+1}} ds \geq \int_K^{Kt} \frac{A(s)}{s^{1/(1-\alpha)+1}} ds \\ &\simeq \int_{1/2}^{t/2} \frac{A(2Ks)}{s^{1/(1-\alpha)+1}} ds && \text{(by change of variables)} \\ &\gtrsim \int_{1/2}^{t/2} \frac{A(\tau_s)}{\tau_s} s^{1/(\alpha-1)} ds && \text{(by (4.8))} \\ &\simeq \int_{1/2}^{t/2} \frac{G(2s)}{s} ds && \text{(by (4.6))} \\ &\simeq \int_1^t \frac{G(s)}{s} ds. \end{aligned}$$

This contradicts (4.3) for this K .

From estimate (4.7), we can take an increasing sequence $t_j \in (2, \infty)$, $j \geq 2$, such that

$$\lim_{j \rightarrow \infty} \frac{A(\tau_j)}{\tau_j} \cdot \frac{t_j}{A(jt_j)} = \infty, \quad (4.9)$$

where we define $\tau_j = \tau_{t_j}$. We can also choose this sequence to ensure $t_{j+1} > \tau_j$. We claim that without loss of generality we can assume that $2t_j < \tau_j$ for every index $j \geq 2$. Indeed, suppose that there exists a subsequence j_k in \mathbb{N} such that $\tau_{j_k} \leq 2t_{j_k}$. Then $A(\tau_{j_k}) \leq A(2t_{j_k})$ and

$$\frac{A(\tau_{j_k})}{\tau_{j_k}} \cdot \frac{t_{j_k}}{A(j_k t_{j_k})} \leq \frac{A(2t_{j_k})}{t_{j_k}} \cdot \frac{t_{j_k}}{A(\frac{j_k}{2} 2t_{j_k})} \leq \frac{A(2t_{j_k})}{A(2t_{j_k})} \cdot \frac{2}{j_k} = \frac{2}{j_k} \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

which is impossible due to (4.9).

At this moment, we can define a function A_1 by the formula

$$A_1(t) = \begin{cases} A(t_j) + \frac{A(\tau_j) - A(t_j)}{\tau_j - t_j} (t - t_j), & t \in (t_j, \tau_j), j \in \mathbb{N}, \\ A(t), & \text{otherwise.} \end{cases}$$

Obviously, $A_1 \geq A$ and A_1 is a Young function. Moreover, for $j \in \mathbb{N}$, $j \geq 2$,

$$\begin{aligned} \frac{A_1(2t_j)}{A(jt_j)} &= \frac{A(t_j) + \frac{A(\tau_j) - A(t_j)}{\tau_j - t_j} t_j}{A(jt_j)} \\ &\geq \frac{A(\tau_j) - A(t_j)}{A(jt_j)} \cdot \frac{t_j}{\tau_j} \\ &\geq \frac{A(\tau_j) - A(\frac{\tau_j}{2})}{A(jt_j)} \cdot \frac{t_j}{\tau_j} && \text{(since } 2t_j < \tau_j) \\ &\geq \frac{1}{2} \cdot \frac{A(\tau_j)}{\tau_j} \cdot \frac{t_j}{A(jt_j)} && \text{(since } A(\tau_j/2) \leq A(\tau_j)/2), \end{aligned}$$

and the latter tends to infinity as $j \rightarrow \infty$ by (4.9). Therefore

$$\limsup_{t \rightarrow \infty} \frac{A_1(t)}{A(\lambda t)} = \infty$$

for every $\lambda > 2$, which is precisely $A_1 \gg A$.

It remains to show that A_1 satisfies the condition (4.1) with A replaced by A_1 . Let $t \in (2, \infty)$ be fixed. We find $j \in \mathbb{N}$ such that $t \in [t_j, t_{j+1})$. Then we have

$$\begin{aligned} \int_1^t \frac{A_1(s)}{s^{1/(1-\alpha)+1}} ds &\leq \int_1^t \frac{A(s)}{s^{1/(1-\alpha)+1}} ds + \sum_{k=1}^j \int_{t_k}^{\tau_k} \left(A(t_k) + \frac{A(\tau_k) - A(t_k)}{\tau_k - t_k} (s - t_k) \right) s^{1/(\alpha-1)-1} ds \\ &\leq 2 \int_1^t \frac{A(s)}{s^{1/(1-\alpha)+1}} ds + \sum_{k=1}^j \frac{A(\tau_k) - A(t_k)}{\tau_k - t_k} \int_{t_k}^{\tau_k} \frac{(s - t_k)}{s^{1/(1-\alpha)+1}} ds. \end{aligned}$$

We can follow with estimates of the latter integral. Since $1/(\alpha - 1) < -1$, we have for $k \in \mathbb{N}$ such that $1 \leq k \leq j$,

$$\int_{t_k}^{\tau_k} \frac{(s - t_k)}{s^{1/(1-\alpha)+1}} ds \leq \int_{t_k}^{\tau_k} s^{1/(\alpha-1)} ds \leq \int_{t_k}^{\infty} s^{1/(\alpha-1)} ds \simeq t_k^{1/(\alpha-1)+1}.$$

This together with the fact that $2t_k < \tau_k$ gives

$$\int_1^t \frac{A_1(s)}{s^{1/(1-\alpha)+1}} ds \lesssim 2 \int_1^t \frac{A(s)}{s^{1/(1-\alpha)+1}} ds + 2 \sum_{k=1}^j \frac{A(\tau_k)}{\tau_k} t_k^{1/(\alpha-1)+1}.$$

Since (4.6) implies

$$\frac{A(\tau_k)}{\tau_k} t_k^{1/(\alpha-1)+1} = \beta 2^{1/(1-\alpha)} G(2t_k),$$

we have

$$\int_1^t \frac{A_1(s)}{s^{1/(1-\alpha)+1}} ds \lesssim G(t) + \sum_{k=1}^j G(2t_k).$$

Because the sequence t_j could be taken arbitrarily fast growing, we can assume without loss of generality that $G(2t_i) \geq \sum_{k=1}^{i-1} G(2t_k)$ thanks to the fact that G is unbounded by (4.2). Adding all the estimates together, we finally obtain that

$$\int_1^t \frac{A_1(s)}{s^{1/(1-\alpha)+1}} ds \lesssim G(t) + G(2t_j) \lesssim G(2t), \quad t \in (2, \infty),$$

which proves the theorem. \square

The following auxiliary fact is based on the idea of L'Hôpital's rule and the proof is very similar to the proof of the original result, hence we omit it.

Proposition 4.2. *Suppose that f and g are real functions having finite derivatives on some neighbourhood of infinity. If $g(x) \rightarrow \infty$ as $x \rightarrow \infty$, then*

$$\liminf_{x \rightarrow \infty} \frac{f'(x)}{g'(x)} \leq \liminf_{x \rightarrow \infty} \frac{f(x)}{g(x)}.$$

Theorem 4.3. *Let $G: (0, \infty) \rightarrow (0, \infty)$ be a continuous nondecreasing function satisfying Δ_2 condition. Then the following are equivalent.*

(i)

$$\limsup_{t \rightarrow \infty} \frac{1}{G(Kt)} \int_1^t \frac{G(s)}{s} ds = \infty \quad \text{for every } K \geq 1;$$

(ii)

$$\limsup_{t \rightarrow \infty} \frac{1}{G(t)} \int_1^t \frac{G(s)}{s} ds = \infty;$$

(iii)

$$\liminf_{t \rightarrow \infty} \frac{G(Kt)}{G(t)} = 1 \quad \text{for every } K \geq 1.$$

Proof. The equivalence (ii) \Leftrightarrow (i) is trivial, since the quantities $G(t)$ and $G(Kt)$ are comparable for every fixed $K \geq 1$ thanks to the fact that $G \in \Delta_2$.

Let us focus on the implication (iii) \Rightarrow (ii). Let $K \geq 1$ be fixed and suppose $t > 1$. Then

$$\int_1^{Kt} G(s) \frac{ds}{s} \geq \int_t^{Kt} G(s) \frac{ds}{s} \geq G(t) \int_t^{Kt} \frac{ds}{s} = G(t) \log K.$$

Dividing both sides by $G(Kt)$ we obtain

$$\log K \frac{G(t)}{G(Kt)} \leq \frac{1}{G(Kt)} \int_1^{Kt} \frac{G(s)}{s} ds.$$

Taking the limes superior as $t \rightarrow \infty$ on both sides of the inequality, we get

$$\log K = \log K \limsup_{t \rightarrow \infty} \frac{G(t)}{G(Kt)} \leq \limsup_{t \rightarrow \infty} \frac{1}{G(Kt)} \int_1^{Kt} \frac{G(s)}{s} ds =: L,$$

where L is independent of K . Since $\log K \leq L$ for arbitrary K , L has no other option but to equal infinity.

To prove (ii) \Rightarrow (iii), let $K \geq 1$ be fixed and let us define $f(t) = \int_1^t G(Ks) \frac{ds}{s}$ and $g(t) = \int_1^t G(s) \frac{ds}{s}$. Then both f and g are continuous and have derivatives, namely $f'(t) = G(Kt)/t$, $g'(t) = G(t)/t$. Since (ii) holds, it has to be $g(t) \rightarrow \infty$ as $t \rightarrow \infty$. Using Proposition 4.2, we get

$$\begin{aligned} 0 &\leq \liminf_{t \rightarrow \infty} \frac{G(Kt)}{G(t)} - 1 \\ &\leq \liminf_{t \rightarrow \infty} \frac{\int_1^t G(Ks) \frac{ds}{s}}{\int_1^t G(s) \frac{ds}{s}} - 1 \\ &\leq \liminf_{t \rightarrow \infty} \frac{\int_K^{Kt} G(s) \frac{ds}{s} - \int_1^t G(s) \frac{ds}{s}}{\int_1^t G(s) \frac{ds}{s}} \\ &\leq \liminf_{t \rightarrow \infty} \frac{G(t)}{\int_1^t G(s) \frac{ds}{s}} \frac{\int_t^{Kt} G(s) \frac{ds}{s}}{G(t)}. \end{aligned}$$

Since $\liminf_{t \rightarrow \infty} G(t) / \int_1^t G(s) \frac{ds}{s} = 0$, it suffices to show that $\frac{1}{G(t)} \int_t^{Kt} G(s) \frac{ds}{s}$ is bounded. To this end we use the fact that G is nondecreasing and, due to $G \in \Delta_2$, there is some $c > 0$ such that $G(Kt) \leq cG(t)$ for big t . For such a t we have

$$\frac{1}{G(t)} \int_t^{Kt} G(s) \frac{ds}{s} \leq \frac{G(Kt)}{G(t)} \int_t^{Kt} \frac{ds}{s} \leq c \log K.$$

□

Proof of Theorem A. The equivalence of (ii) and (v) follows directly from Theorem B.

The condition (v) holds if and only if (iv) holds thanks to the consequence of [15, Theorem A].

In order to show (i) \Rightarrow (v) assume that (v) is not satisfied, i.e.,

$$\limsup_{t \rightarrow \infty} \frac{1}{G(Kt)} \int_1^t \frac{G(s)}{s} ds = \infty$$

for some constant $K > 1$. Now for any Orlicz space $L^A(0, 1)$ satisfying $H_\alpha^\beta: L^A(0, 1) \rightarrow M(0, 1)$ there exists a constant C_A such that

$$\int_1^t \frac{\tilde{A}(s)}{s^{1/(1-\alpha)+1}} ds \lesssim \frac{\tilde{B}(C_A t)}{t^{1/(1-\alpha)}}, \quad t \in (2, \infty)$$

due to Theorem B. If the function G is unbounded, then Theorem 4.1 ensures the existence of a Young function A_1 such that the space $L^{A_1}(0, 1)$ is strictly larger than $L^A(0, 1)$ and still renders the inequality above true, with possibly different constants. In case when the function G is equivalent to a constant function (which coincides with the situation when $M(0, 1) = L^\infty(0, 1)$) the same is true by similar construction argument described in [9, Theorem 6.4]. Now again by Theorem B one has $H_\alpha^\beta: L^{A_1}(0, 1) \rightarrow M(0, 1)$ and no optimal Orlicz domain exists. This contradicts (i).

To prove (iii) \Rightarrow (i) we claim that $L^B(0, 1)$ is among the Orlicz spaces $L^A(0, 1)$ the largest space rendering

$$H_\alpha^\beta: L^A(0, 1) \rightarrow M(0, 1).$$

Indeed let $L^A(0, 1)$ be any of such spaces. By the optimality of $X(0, 1)$, we have $L^A(0, 1) \subseteq X(0, 1)$ and thus we have the inequality between appropriate fundamental functions

$$\varphi_X(t) \lesssim \varphi_{L^A}(t), \quad t \in (0, 1).$$

Since the space $L^B(0, 1)$ is defined in a way that its fundamental function coincides with φ_X , one gets that $\varphi_{L^B}(t) \lesssim \varphi_{L^A}(t)$ which implies $A(t) \leq B(Ct)$ for some $C > 0$, hence $L^A(0, 1) \subseteq L^B(0, 1)$ and $L^B(0, 1)$ is optimal.

The equivalence of (ii) and (iii) follows directly from the definition of the optimal r.i. space, and the equivalence of (v) and (vi) has already been proved in Theorem 4.3. □

Remark 4.4. Note that the proof of the implication (iii) \Rightarrow (i) does not depend on the target space, so it can be used to prove the optimality in positive cases for any r.i. target space Y .

5. Examples and applications

5.1 Sobolev embeddings on John domains

We begin by the easiest case of Sobolev embeddings, namely those acting on John domains. We will use the reduction theorem from [10]. Recall that a bounded open set Ω in \mathbb{R}^n is called a John domain if there exist a constant $c \in (0, 1)$ and a point $x_0 \in \Omega$ such that for every $x \in \Omega$ there exists a rectifiable curve $\varpi: [0, l] \rightarrow \Omega$, parameterized by arclength, such that $\varpi(0) = x$, $\varpi(l) = x_0$, and

$$\text{dist}(\varpi(r), \partial\Omega) \leq cr, \quad r \in [0, l].$$

We will use the reduction principle for John domains proved in [10, Theorem 6.1]. It can be read as follows.

Let $n \in \mathbb{N}$, $n \geq 2$, and let $m \in \mathbb{N}$. Assume that Ω is a John domain in \mathbb{R}^n . Let $\|\cdot\|_{X(0,1)}$ and $\|\cdot\|_{Y(0,1)}$ be rearrangement-invariant function norms. Then the following assertions are equivalent.

(i) The Hardy type inequality

$$\|H_{m/n}^1 f\|_{Y(0,1)} \leq C \|f\|_{X(0,1)}$$

holds for some constant C and for every nonnegative $f \in X(0, 1)$.

(ii) The Sobolev embedding

$$W^m X(\Omega) \hookrightarrow Y(\Omega)$$

holds.

Recall that

$$W^m X(\Omega) = \left\{ u \in \mathcal{M}(\Omega), u \text{ is } m\text{-times weakly differentiable in } \Omega \text{ and } |\nabla^k u| \in X(\Omega), k = 0, 1, \dots, m \right\}.$$

Here, $\nabla^m u$ denotes the vector of all m -th order weak derivatives of u and $\nabla^0 u = u$. The norm is then given by

$$\|u\|_{W^m X(\Omega)} := \sum_{k=0}^m \|\nabla^k u\|_{X(\Omega)}.$$

Now, one can select any r.i. space $X(0, 1)$ and seek to find an optimal range space. Let Ω be a John domain in \mathbb{R}^n , $m \in \mathbb{N}$ such that $m < n$ and consider the spaces $L^p \log^q L(\Omega)$ or $L^p \log^q \log L(\Omega)$, $p > 1$ and $q \in \mathbb{R}$ or $p = 1$ and $q \geq 0$. By [10, Theorem 6.12 and Example 6.14] (see also [6, Example 1 and 2]), we have

$$W^m L^p \log^q L(\Omega) \hookrightarrow \begin{cases} L^{\frac{np}{n-mp}} \log^{\frac{nq}{n-mp}} L(\Omega), & 1 \leq p < \frac{n}{m}, \\ \exp L^{\frac{n}{n-m(1+q)}}(\Omega), & p = \frac{n}{m}, q < \frac{n}{m} - 1, \\ \exp \exp L^{\frac{n}{n-m}}(\Omega), & p = \frac{n}{m}, q = \frac{n}{m} - 1, \\ L^\infty(\Omega), & p > \frac{n}{m} \text{ or } p = \frac{n}{m}, q > \frac{n}{m} - 1, \end{cases}$$

and

$$W^m L^p \log^q \log L(\Omega) \hookrightarrow \begin{cases} L^{\frac{np}{n-mp}} \log^{\frac{nq}{n-mp}} \log L(\Omega), & 1 \leq p < \frac{n}{m}, \\ \exp(L^{\frac{n}{n-m}} \log^{\frac{mq}{n-m}} L)(\Omega), & p = \frac{n}{m}, \\ L^\infty(\Omega), & p > \frac{n}{m}, \end{cases}$$

and all the targets are optimal among all Orlicz spaces.

Let us investigate the optimal Orlicz domains.

Example 5.1. a) Case $Y(\Omega) = L^{\frac{np}{n-mp}} \log^{\frac{nq}{n-mp}} L(\Omega)$, $1 \leq p < \frac{n}{m}$. The space $Y(\Omega)$ is not a Marcinkiewicz space, but instead of $Y(\Omega)$ we can take the endpoint space $M_\varphi(\Omega)$ with the same fundamental function as the space $Y(\Omega)$, namely

$$\varphi(t) = t^{\frac{n-mp}{np}} \log^{\frac{q}{p}}\left(\frac{2}{t}\right), \quad t \in (0, 1).$$

Now, thanks to reduction principle the problem of Sobolev embedding is equivalent to the boundedness of the operator $H_{m/n}^1$.

By Theorem A, the optimal Orlicz domain space exists if and only if $H_{m/n}^1: L^B(0, 1) \rightarrow M_\varphi(0, 1)$ where, after some calculations, $B(t) = t^p \log^q t$ for large t . This is however the same as $W^m L^p \log^q L(\Omega) \hookrightarrow M_\varphi(\Omega)$ which is satisfied since

$$W^m L^p \log^q L(\Omega) \hookrightarrow Y(\Omega) \subseteq M_\varphi(\Omega).$$

Hence both domain and range spaces in

$$W^m L^p \log^q L(\Omega) \hookrightarrow L^{\frac{np}{n-mp}} \log^{\frac{nq}{n-mp}} L(\Omega)$$

are optimal among Orlicz spaces.

b) Case $Y(\Omega) = \exp L^{\frac{n}{n-m(1+q)}}(\Omega)$, $p = \frac{n}{m}$, $q < \frac{n}{m} - 1$. The Orlicz space $Y(\Omega)$ coincides with the Marcinkiewicz endpoint space $M_\varphi(\Omega)$ (cf. (2.1)) where

$$\varphi(t) = \log^{\frac{m}{n}(1+q)-1}\left(\frac{2}{t}\right), \quad t \in (0, 1).$$

Again by reduction principle and Theorem A we compute the Young function B and test the boundedness of $H_{m/n}^1$ on the space $L^B(0, 1)$ or check the condition using the function G . We get

$$G(t) = \log^{\frac{n-m(1+q)}{n-m}}(t), \quad t \in (1, \infty).$$

Since

$$\liminf_{t \rightarrow \infty} \frac{\log^{\frac{n-m(1+q)}{n-m}}(Ct)}{\log^{\frac{n-m(1+q)}{n-m}}(t)} = 1, \quad \text{for every } C \geq 1$$

and G satisfies the Δ_2 condition we conclude that the space $L^p \log^q L(\Omega)$ is not the largest Orlicz space rendering

$$W^m L^p \log^q L(\Omega) \hookrightarrow \exp L^{\frac{n}{n-m(1+q)}}(\Omega)$$

and no such Orlicz space exists. Just to compare, the space $L^B(0,1)$ from Theorem A is $L^p \log^{1+q-p} L(0,1)$ which is too large.

These two examples give us the outline how to use our results to investigate the optimal Orlicz domains. Other cases can be done in an analogous way and we just present the results. Observe that the optimal Orlicz domains exist in subcritical cases, i.e. when $1 \leq p < \frac{n}{m}$, otherwise every Orlicz domain space can be improved.

$Y(\Omega)$		$L^B(\Omega)$	$G(t)$
$L^{\frac{np}{n-mp}} \log^{\frac{nq}{n-mq}} L(\Omega)$	$1 \leq p < \frac{n}{m}$	$L^p \log^q L(\Omega)$	$t^{\frac{n}{m-n} + \frac{p}{p-1}} \log^{\frac{q}{1-p}}(t)$
$L^{\frac{np}{n-mp}} \log^{\frac{nq}{n-mq}} \log L(\Omega)$	$1 \leq p < \frac{n}{m}$	$L^p \log^q \log L(\Omega)$	$t^{\frac{n}{m-n} + \frac{p}{p-1}} \log^{\frac{q}{1-p}} \log(t)$
$\exp L^{\frac{n}{n-m(1+q)}}(\Omega)$	$p = \frac{n}{m}, q < \frac{n}{m} - 1$	$L^p \log^{1+q-p} L(\Omega)$	$\log^{\frac{n-m(1+q)}{n-m}}(t)$
$\exp \exp L^{\frac{n}{n-m}}(\Omega)$	$p = \frac{n}{m}, q = \frac{n}{m} - 1$	$L^p \log^{-q} \log L(\Omega)$	$\log \log(t)$
$\exp(L^{\frac{n}{n-m}} \log^{\frac{mq}{n-m}} L)(\Omega)$	$p = \frac{n}{m}$	$L^p \log^{1-p} L \log^q \log L(\Omega)$	$\log(t) \log^{\frac{mq}{m-n}} \log(t)$
$L^\infty(\Omega)$		$L^{\frac{n}{m}}(\Omega)$	1

Application of Theorem A for the operator $H_{m/n}^1$ and John domain

5.2 Sobolev embeddings on Maz'ya classes

Our next applications are in Sobolev embeddings on wider family of subsets so called Maz'ya classes.

Let Ω be a domain in \mathbb{R}^n , $n \geq 2$, with a normalised Lebesgue measure, i.e. $|\Omega| = 1$. Define the perimeter of a measurable set E in Ω

$$P(E, \Omega) = \mathcal{H}^{n-1}(\Omega \cap \partial^M E)$$

where $\partial^M E$ denotes the essential boundary of E . The isoperimetric function $I_\Omega: [0, 1] \rightarrow [0, \infty]$ of Ω is then given by

$$I_\Omega(s) = \inf \{ P(E, \Omega), E \subseteq \Omega, s \leq |E| \leq \frac{1}{2} \}, \quad s \in [0, \frac{1}{2}]$$

and $I_\Omega(s) = I_\Omega(1-s)$ if $s \in [\frac{1}{2}, 1]$.

Given $\alpha \in [\frac{1}{n}, 1]$, we denote by \mathcal{J}_α the *Maz'ya class* of all Euclidean domains Ω in \mathbb{R}^n such that

$$I_\Omega(s) \geq C s^\alpha \quad \text{for } s \in [0, \frac{1}{2}]$$

for some positive constant C .

The reduction theorem in the class \mathcal{J}_α [10, Theorem 6.4] takes the following form.

Let $n \in \mathbb{N}$, $n \geq 2$, $m \in \mathbb{N}$, and $\alpha \in [\frac{1}{n}, 1)$. Let $\|\cdot\|_{X(0,1)}$ and $\|\cdot\|_{Y(0,1)}$ be rearrangement-invariant function norms. Assume that there exists a constant C such that

$$\|H_{m(1-\alpha)}^1 f\|_{Y(0,1)} \leq C \|f\|_{X(0,1)} \quad (5.1)$$

for every nonnegative $f \in X(0,1)$. Then the Sobolev embedding

$$W^m X(\Omega) \hookrightarrow Y(\Omega) \quad (5.2)$$

holds for every $\Omega \in \mathcal{J}_\alpha$.

Conversely, if the Sobolev embedding (5.2) holds for every $\Omega \in \mathcal{J}_\alpha$, then the inequality (5.1) holds.

Notice the main difference between this statement and reduction principle for John domains. In the case of John domains the equivalence of Sobolev embedding and boundedness of Hardy type operator holds for every single domain Ω , while in the Maz'ya classes Ω has to range among all domains in \mathcal{J}_α .

Let us mention similar examples for Orlicz spaces. Let m be an integer and $\alpha \in [\frac{1}{n}, 1)$ such that $m(1-\alpha) < 1$ and assume $p > 1$ and $q \in \mathbb{R}$ or $p = 1$ and $q \geq 0$. By [10, Theorem 6.12 and Example 6.14], we have

$$W^m L^p \log^q L(\Omega) \hookrightarrow \begin{cases} L^{\frac{p}{1-mp(1-\alpha)}} \log^{\frac{q}{1-mp(1-\alpha)}} L(\Omega), & 1 \leq p < \frac{1}{m(1-\alpha)}, \\ \exp L^{\frac{1}{1-(1+q)m(1-\alpha)}}(\Omega), & p = \frac{1}{m(1-\alpha)}, q < p-1, \\ \exp \exp L^{\frac{1}{1-m(1-\alpha)}}(\Omega), & p = \frac{1}{m(1-\alpha)}, q = p-1, \\ L^\infty(\Omega), & p > \frac{1}{m(1-\alpha)} \text{ or } p = \frac{1}{m(1-\alpha)}, q > p-1, \end{cases}$$

and

$$W^m L^p \log^q \log L(\Omega) \hookrightarrow \begin{cases} L^{\frac{p}{1-mp(1-\alpha)}} \log^{\frac{q}{1-mp(1-\alpha)}} \log L(\Omega), & 1 \leq p < \frac{1}{m(1-\alpha)}, \\ \exp\left(L^{\frac{1}{1-m(1-\alpha)}} \log^{\frac{mq(1-\alpha)}{1-m(1-\alpha)}} L\right)(\Omega), & p = \frac{1}{m(1-\alpha)}, \\ L^\infty(\Omega), & p > \frac{1}{m(1-\alpha)}. \end{cases}$$

Moreover, the target spaces are optimal among all Orlicz spaces, as Ω ranges in \mathcal{J}_α .

Now one can apply Theorem A for the operator $H_{m(1-\alpha)}^1$ in a analogous way as in Example 5.1 to investigate the optimal Orlicz domains.

As computation below shows, in the case $1 \leq p \leq 1/m(1-\alpha)$ the optimality is attained as Ω ranges through \mathcal{J}_α . In the remaining examples there exists some Ω in \mathcal{J}_α such that any Orlicz domain space in appropriate Sobolev embedding can be improved.

$Y(\Omega)$		$L^B(\Omega)$	$G(t)$
$L^{\frac{p}{1-mp(1-\alpha)}} \log^{\frac{q}{1-mp(1-\alpha)}} L(\Omega)$	$1 \leq p < \frac{1}{m(1-\alpha)}$	$L^p \log^q L(\Omega)$	$t^{\frac{1}{m(1-\alpha)-1} + \frac{p}{p-1}} \log^{\frac{q}{1-p}}(t)$
$L^{\frac{1p}{1-mp(1-\alpha)}} \log^{\frac{1q}{1-mp(1-\alpha)}} \log L(\Omega)$	$1 \leq p < \frac{1}{m(1-\alpha)}$	$L^p \log^q \log L(\Omega)$	$t^{\frac{1}{m(1-\alpha)-1} + \frac{p}{p-1}} \log^{\frac{q}{1-p}} \log(t)$
$\exp L^{\frac{1}{1-(1+q)m(1-\alpha)}}(\Omega)$	$p = \frac{1}{m(1-\alpha)}, q < p-1$	$L^p \log^{1+q-p} L(\Omega)$	$\log^{\frac{1-(1+q)m(1-\alpha)}{1-m(1-\alpha)}}(t)$
$\exp \exp L^{\frac{1}{1-m(1-\alpha)}}(\Omega)$	$p = \frac{1}{m(1-\alpha)}, q = p-1$	$L^p \log^{-q} \log L(\Omega)$	$\log \log(t)$
$\exp\left(L^{\frac{1}{1-m(1-\alpha)}} \log^{\frac{mq(1-\alpha)}{1-m(1-\alpha)}} L\right)(\Omega)$	$p = \frac{1}{m(1-\alpha)}$	$L^p \log^{1-p} L \log^q \log L(\Omega)$	$\log(t) \log^{\frac{mq(1-\alpha)}{m(1-\alpha)-1}} \log(t)$
$L^\infty(\Omega)$		$L^{m(1-\alpha)}(\Omega)$	1

Application of Theorem A for the operator $H_{m(1-\alpha)}^1$ and Maz'ya class

5.3 Sobolev trace embeddings

Our last application concerns the Sobolev trace embeddings.

An open set Ω in \mathbb{R}^n is said to have the cone property if there exists a finite cone Λ such that each point in Ω is the vertex of a finite cone contained in Ω and congruent to Λ .

Given an integer d such that $1 \leq d \leq n$ we denote by Ω_d the nonempty intersection of Ω with a d -dimensional affine subspace of \mathbb{R}^n .

The reduction principle for trace embeddings [7, Theorem 1.3] now has the following form.

Let Ω be a bounded open set with cone property in \mathbb{R}^n , $n \geq 2$. Assume that $m \in \mathbb{N}$ and $d \in \mathbb{N}$ are such that $1 \leq d \leq n$ and $d \geq n-m$. Let $\|\cdot\|_{X(0,1)}$ and $\|\cdot\|_{Y(0,1)}$ be rearrangement-invariant function norms. Then the following facts are equivalent.

(i) The inequality

$$\|H_{m/n}^{n/d} f\|_{Y(0,1)} \leq C \|f\|_{X(0,1)}$$

holds for some constant C and for every nonnegative $f \in X(0,1)$.

(ii) The Sobolev trace embedding

$$\text{Tr}: W^m X(\Omega) \rightarrow Y(\Omega_d)$$

holds.

Let Ω be a domain in \mathbb{R}^n with cone property, $m \in \mathbb{N}$, $m < n$, and consider again the spaces $L^p \log^q L(\Omega)$ or $L^p \log^q \log L(\Omega)$, $p > 1$ and $q \in \mathbb{R}$ or $p = 1$ and $q \geq 0$. By [7, Theorem 5.2, Example 5.3 and Example 5.4] we have

$$\text{Tr}: W^m L^p \log^q L(\Omega) \rightarrow \begin{cases} L^{\frac{pd}{n-mp}} \log^{\frac{qd}{n-mp}} L(\Omega_d), & 1 \leq p < \frac{n}{m}, \\ \exp L^{\frac{n}{n-m(1+q)}}(\Omega_d), & p = \frac{n}{m}, q < \frac{n}{m} - 1, \\ \exp \exp L^{\frac{n}{n-m}}(\Omega_d), & p = \frac{n}{m}, q = \frac{n}{m} - 1, \\ L^\infty(\Omega_d), & p > \frac{n}{m} \text{ or } p = \frac{n}{m}, q > \frac{n}{m} - 1, \end{cases}$$

and

$$\text{Tr}: W^m L^p \log^q \log L(\Omega) \rightarrow \begin{cases} L^{\frac{pd}{n-mp}} \log^{\frac{qd}{n-mp}} \log L(\Omega_d), & 1 \leq p < \frac{n}{m}, \\ \exp\left(L^{\frac{n}{n-m}} \log^{\frac{mq}{n-m}} L\right)(\Omega_d), & p = \frac{n}{m}, \\ L^\infty(\Omega_d), & p > \frac{n}{m}, \end{cases}$$

and the range spaces being optimal in the class of Orlicz spaces.

Now, using Theorem A for the operator $H_{m/n}^{n/d}$, one can investigate the optimal Orlicz domains. The situation is almost the same as in case of Sobolev embedding and hence we just present the results. Naturally, the optimality is attained only in the subcritical cases.

$Y(\Omega)$		$L^B(\Omega)$	$G(t)$
$L^{\frac{pd}{n-mp}} \log^{\frac{qd}{n-mp}} L(\Omega)$	$1 \leq p < \frac{n}{m}$	$L^p \log^q L(\Omega)$	$t^{\frac{n}{m-n} + \frac{p}{p-1}} \log^{\frac{q}{1-p}}(t)$
$L^{\frac{pd}{n-mp}} \log^{\frac{qd}{n-mp}} \log L(\Omega)$	$1 \leq p < \frac{n}{m}$	$L^p \log^q \log L(\Omega)$	$t^{\frac{n}{m-n} + \frac{p}{p-1}} \log^{\frac{q}{1-p}} \log(t)$
$\exp L^{\frac{n}{n-m(1+q)}}(\Omega)$	$p = \frac{n}{m}, q < \frac{n}{m} - 1$	$L^p \log^{1+q-p} L(\Omega)$	$\log^{\frac{n-m(1+q)}{n-m}}(t)$
$\exp \exp L^{\frac{n}{n-m}}(\Omega)$	$p = \frac{n}{m}, q = \frac{n}{m} - 1$	$L^p \log^{-q} \log L(\Omega)$	$\log \log(t)$
$\exp(L^{\frac{n}{n-m}} \log^{\frac{mq}{n-m}} L)(\Omega)$	$p = \frac{n}{m}$	$L^p \log^{1-p} L \log^q \log L(\Omega)$	$\log(t) \log^{\frac{mq}{m-n}} \log(t)$
$L^\infty(\Omega)$		$L^{\frac{n}{m}}(\Omega)$	1

Application of Theorem A for the operator $H_{m/n}^{n/d}$ and domain with cone property

5.4 Extension to other r.i. target spaces

As we have seen in Example 5.1 a) in the case when the optimality is attained one can extend the positive result to other r.i. target spaces. Let us now look closer on this phenomenon.

Let α and β be fixed and let $L^A(0, 1)$ be an optimal Orlicz space rendering the relation

$$H_\alpha^\beta: L^A(0, 1) \rightarrow M(0, 1)$$

true, where $M(0, 1)$ is a given Marcinkiewicz endpoint space. We know from Theorem A that not every Orlicz space is an optimal domain space; such spaces are exactly those for which the supremum operator S_α is bounded on their associate space.

However, we can go the opposite direction. Suppose that $L^A(0, 1)$ is a given Orlicz space such that the operator H_α^β is bounded on $L^{\tilde{A}}(0, 1)$. Now thanks to the result of [17], the operator S_α is bounded on some r.i. space $X'(0, 1)$ if and only if the $X(0, 1)$ is optimal r.i. domain space for some r.i. target space. By Proposition 3.5, the norm of the best r.i. target space, say $Y_{L^A}(0, 1)$, is given by

$$\|f\|_{(Y_{L^A})'(0,1)} = \left\| t^{\alpha-1} \int_0^{t^{\frac{1}{\beta}}} f^*(s) ds \right\|_{L^{\tilde{A}}(0,1)}.$$

The fundamental function of Y_{L^A} , say φ , then satisfies (cf. (3.9))

$$\begin{aligned} \varphi(t) &\simeq t^{\beta(1-\alpha)} E_{\alpha-1}^{-1}(t^{-\beta}) \\ &\simeq t^{\beta(1-\alpha)} \tilde{A}^{-1}(Kt^{-\beta}). \end{aligned}$$

Moreover, to the given Orlicz space $L^A(0, 1)$, we are able to compute the appropriate Marcinkiewicz space $M(0, 1)$. If we take a look at the proof of Theorem B again, we observe that in the case of optimality, the inequality (3.10) becomes actually equivalence, therefore the fundamental function of $M(0, 1)$ is equivalent to φ .

Consequently, we obtain that the space $L^A(0, 1)$ is the optimal Orlicz domain for every r.i. space $Y(0, 1)$ satisfying

$$Y_{L^A}(0, 1) \subseteq Y(0, 1) \subseteq M(0, 1).$$

Example 5.2. Let Ω be a bounded Lipschitz domain in \mathbb{R}^n , $n \geq 2$, and $1 < p < n$. One can easily observe that $S_{1/n}$ is bounded on $L^{p'}(0, 1)$, where $p' = p/(p-1)$. Then the optimal r.i. range space for the operator $H_{1/n}^1$ is the Lorentz space $L^{p^*, p}(0, 1)$, where $p^* = np/(n-p)$. Its fundamental function is equivalent to the power function t^{1/p^*} and therefore, for every fixed $q \in [p, \infty]$, the Lebesgue space $L^p(\Omega)$ is the largest Orlicz space which renders the embedding

$$W^1 L^p(\Omega) \hookrightarrow L^{p^*, q}(\Omega)$$

true.

Similarly, for a given integer $1 < m < n$, $1 < p < n/m$ and $q \in [p, \infty]$ we obtain that $L^p(\Omega)$ is the largest Orlicz space in

$$W^m L^p(\Omega) \hookrightarrow L^{\frac{np}{n-mp}, q}(\Omega).$$

Acknowledgement: I would like to express my thanks to Luboš Pick for careful reading of this paper and many valuable comments and suggestions.

References

- [1] C. Bennett and R. Sharpley, *Interpolation of operators*, Pure and Applied Mathematics, Vol. **129**, Academic Press Inc., Boston, MA, 1988.
- [2] A. P. Calderón, *Spaces between L^1 and L^∞ and the theorem of Marcinkiewicz*, Studia Math. **26** (1966), 273–299.
- [3] A. Cianchi, *Orlicz-Sobolev boundary trace embeddings*, Math. Zeit. **266** (2010), 431–449.
- [4] A. Cianchi, *Higher-order Sobolev and Poincaré inequalities in Orlicz spaces*, Forum Math. **18** (2006), 745–767.
- [5] A. Cianchi, *Optimal Orlicz-Sobolev embeddings*, Rev. Mat. Iberoam. **20** (2004), 427–474.
- [6] A. Cianchi, *A sharp embedding theorem for Orlicz-Sobolev spaces*, Indiana Univ. Math. J. **45** (1996), 39–65.
- [7] A. Cianchi and L. Pick, *Optimal Sobolev trace embeddings*, To appear in Trans. Amer. Math. Soc.
- [8] A. Cianchi and L. Pick, *Optimal Gaussian Sobolev embeddings*, J. Funct. Anal. **256** (2009), 11, 3588–3642.
- [9] A. Cianchi and L. Pick, *Sobolev embeddings into BMO, VMO, and L_∞* , Ark. Mat. **36** (1998), 2, 317–340.
- [10] A. Cianchi, L. Pick and L. Slavíková, *Higher-order Sobolev embeddings and isoperimetric inequalities*, Adv. Math. **273** (2015), 568–650.
- [11] D. E. Edmunds, R. Kerman and L. Pick, *Optimal Sobolev imbeddings involving rearrangement-invariant quasi-norms*, J. Funct. Anal. **170** (2000), 2, 307–355.
- [12] L. Gross, *Logarithmic Sobolev inequalities*, Amer. J. Math. **97** (1975), 1061–1083.
- [13] J. A. Hempel, G. R. Morris and N. S. Trudinger, *On the sharpness of a limiting case of the Sobolev imbedding theorem.*, Bull. Austral. Math. Soc. **3** (1970), 369–373.
- [14] R. Hunt, *On $L(p, q)$ spaces*, Enseignement Math. **12** (1966), 249–276.
- [15] R. Kerman, C. Phipps, L. Pick, *Marcinkiewicz interpolation theorems for Orlicz and Lorentz gamma spaces*, Publ. Mat. **58** (2014), 1, 3–30.
- [16] R. Kerman, L. Pick, *Optimal Sobolev Imbeddings*, Forum Math. **18** (2006), 4, 535–570.
- [17] R. Kerman, L. Pick, *Optimal Sobolev Imbeddings Spaces*, Studia Math. **192** (2009), 3, 195–217.
- [18] G. G. Lorentz, *On the theory of spaces Λ* , Pacific J. Math. **1** (1951), 411–429.
- [19] V. Maz’ya, *Sobolev Spaces*, Springer, Berlin, 1985.
- [20] R. O’Neil, *Convolution operators and $L(p, q)$ spaces*, Duke Math. J. **30** (1963), 129–142.
- [21] J. Peetre, *Espaces d’interpolation et théorème de Soboleff*, Ann. Inst. Fourier **16** (1966), 279–317.
- [22] L. Pick, *Optimal Sobolev Embeddings*, Rudolph Lipschitz Vorlesungsreihe, Vol. **43**, Rheinische Friedrich-Wilhelms-Universität, Bonn.
- [23] L. Pick, A. Kufner, O. John, S. Fučík, *Function spaces. Vol. 1*, De Gruyter Series in Nonlinear Analysis and Applications, Vol. **14**, Walter de Gruyter & Co., Berlin, 2013.
- [24] S. I. Pokhozhaev, *On eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$* , Dokl. Akad. Nauk SSR **165** (1965), 36–39.
- [25] N. S. Trudinger, *On imbeddings into Orlicz spaces and some applications*, J. Math. Mech. **17** (1967), 473–483.
- [26] V. I. Yudovich, *Some estimates connected with integral operators and with solutions of elliptic equations*, Soviet Math. Doklady **2** (1961), 746–749.

DEPARTMENT OF MATHEMATICAL ANALYSIS, FACULTY OF MATHEMATICS AND PHYSICS, CHARLES UNIVERSITY,
SOKOLOVSKÁ 83, 186 75 PRAHA 8, CZECH REPUBLIC

E-mail adress: musil@karlin.mff.cuni.cz