Variational Wave Equations: The Methods of Young Measures and Mollifiers

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ABSTRACT

We study an elegant nonlinear wave equation which arises beautifully in a simplified liquid crystal model through the variational principle. The equation appears in several other disciplines as well. A distinctive feature of the equation is that the wave speed depends on the wave amplitude, in addition to its nonconservative appearance that is a natural consequence of the variational principle. We present recent results on the existence and uniqueness of weak solutions for the initial value problems for the equation. Our emphasis is to show the reader the various interesting phenomena that the equation possesses and the modern methods of Young measures, mollifiers, and related techniques.

We first present simple asymptotic equations for weakly nonlinear and unidirectional waves of the equation. The asymptotic equations to the nonlinear variational wave equation is what the Burgers equation to hyperbolic systems of conservation laws. We show the existence of two types of weak solutions to the initial value problem for the (first-order) asymptotic equation for data in the natural space of square integrable functions, and we establish the uniqueness of weak solutions for both the dissipative and conservative types.

For the full nonlinear variational wave equation, we point out that smooth initial data may evolve through the equation into singularities in finite time, a sequence of weak solutions may develop concentrations in the energy norm, while oscillations may persist. We then focus on the special case when the wave speed function is monotone. We present an invariant region in the phase space in which we establish that smooth data evolve smoothly for all time and any rough data in $H^1(\mathbb{R}) \times L^2(\mathbb{R})$ yield weak solutions to the Cauchy problem of the equation. For initial data outside the invariant region, we establish the global existence of weak solutions with initial Riemann invariants in $L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$. More precisely, the Cauchy problem for the nonlinear wave equation we are referring to is

$$\begin{cases} u_{tt} - c(u)[c(u)u_x]_x &= 0, \qquad\qquad t > 0, x \in \mathbb{R} \\ (u, \partial_t u)|_{t=0} &= (u_0, u_1)(x). \end{cases}$$