Qualification Round — Category II

February 21, 2014, room T7

Problem II.1 Determine for which integers a the Diophantine equation

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{a}{xyz}$$

has infinitely many integer solutions (x, y, z) such that $gcd(a, xyz) = 1$. (10 points)

Problem II.2 Let P be a partially ordered set where every chain and every antichain are finite. Show that P is finite. (10 points)

Problem II.3 Let G be a finite commutative group. Show that $\sum_{a \in G} a \neq 0$ if and only if G has exactly one member of order 2. (10 points)

Problem II.4 Let $f_1 : (0, 1] \to \mathbb{R}$ and define $f_{n+1}(x) := x^{f_n(x)}$ for $x \in (0, 1]$ and $n = 1, 2, \ldots$. Denote $a_n := \lim_{x \to 0+} f_n(x)$ if it exists.

(a) Let m be such that a_m exists, $a_m \neq 0$. Prove that $|a_k - a_{k+1}| = 1$ for all $k \geq m + 2$.

(b) Does there exist f_1 such that $a_m = 0$ for all $m \in \mathbb{N}$?

(10 points)

A set M is *a chain* if every two members of M are comparable (for all x, $y \in M$ it holds that $x \leq y$ or $y \leq x$). A set M is *an antichain* if no two members of M are comparable (if for x, $y \in M$ we have $x \leq y$ or $y \leq x$, then neccessarily $x = y$).