Řešení 4. soutěžní série

1. Existuje: Položíme $a_n = 2^{100} \cdot \left(\frac{2n+1}{2}\right)^n$. Tato posloupnost je rostoucí a pro $n \leq 100$ celočíselná. Čísla 2, $(2k+1)$ a $(2(k-1)+1)$ jsou nesoudělná a proto $gcd(a_k, a_{k-1}) = gcd\left(\frac{2^{100}}{2^k}, \frac{2^{100}}{2^{k-1}}\right) = 2^{100-k}$. Posloupnost největších dělitelů je tedy klesající, čehož jsme chtěli dosáhnout.

2. We want to have a telescoping series, i.e. to write it in the form

$$\sum \operatorname{arctg} a - \operatorname{arctg} b.$$

Well known (or easily derivable) formula says

$$\operatorname{arctg} a - \operatorname{arctg} b = \operatorname{arctg} \frac{a-b}{1+ab}.$$

One can see that $a := n+1$ and $b := n$ is the right choice and we have

$$\sum \operatorname{arctg} \frac{1}{n^2+n+1} = \sum \operatorname{arctg}(n+1) - \operatorname{arctg} n = \lim_{N\to\infty} \operatorname{arctg}(N+1) - \operatorname{arctg} 1 = \frac{\pi}{4}.$$

3. First observe, that the condition on identity matrix implies that the permutation permutes separately the diagonal and the rest of each matrix. Further, by considering appropriate permutation matrices (which are regular), the condition on regularity implies that if two entries lie in the same row or column, the same holds after permuting.
Let the entry at (i,i) be mapped to $(\pi(i), \pi(i))$; we claim that (i,j) is mapped to $(\pi(i), \pi(j))$ or $(\pi(j), \pi(i))$ for each i, j. This follows from the observation above: the image of (i,j) has to share the row/column with $(\pi(i), \pi(i))$ and the column/row with $(\pi(j), \pi(j))$.
Finally, observe that the choice between $(\pi(i), \pi(j))$ and $(\pi(j), \pi(i))$ is uniform for all entries: If not, there are two entries in the same row or column, say (i,j), (i,k), $i \neq j \neq k \neq i$, where the choice differs. This contradicts the observation above, since the pairs $(\pi(i), \pi(j))$, $(\pi(k), \pi(i))$ differ in both coordinates.
We deduce that the action of the permutation is of the form $A \mapsto PAP^{\mathrm{T}}$ or $A \mapsto PA^{\mathrm{T}}P^{\mathrm{T}}$, where P is a permutation matrix. This clearly implies the property from the statement.

4. Let A be the set in question. If f attains local maximum $a \in A$ in x_a, then there is $\delta_a > 0$ such that f does not attain greater values in $(x_a - \delta_a, x_a + \delta_a)$. It suffices to show that for every $n \in \mathrm{N}$, the set of those $a \in A$ such that $\delta_a > 1/n$ is countable (for $A = \bigcup_n A_n$). This follows from the fact that if $a, b \in A_n$ are distinct, then $|x_a - x_b| \geq 1/n$, hence A_n is the image of a discrete set.