Řešení 5. soutěžní série

1. Denote the points $x_1, \ldots, x_{n+2}$ and WLOG $x_{n+2} = 0$. Take a nontrivial linear combination of $x_1, \ldots, x_{n+1}$ which is equal to zero: $\sum \lambda_i x_i = 0$. Set $\lambda_{n+2} := -\sum \lambda_i$. We take $I = \{x_i : \lambda_i > 0,\ i = 1, \ldots, n{+}2\}$ and $J = \{x_i : \lambda_i < 0,\ i = 1, \ldots, n{+}2\}$ and $\lambda = \sum_{i \in I} \lambda_i$. Then

$$\sum_{i \in I} \frac{\lambda_i}{\lambda} x_i = \sum_{i \in J} \frac{-\lambda_i}{\lambda} x_i$$

is the point in the intersection of the corresponding convex hulls.

2. Consider the equation

$$f(x) + \frac{f'(x)}{a(x)} = \frac{1}{a(x)}(f(x)a(x) + f'(x)) = g(x), \qquad g(x) \to 0.$$

The method of intagration factor yields

$$\left(e^{A(x)} f(x)\right)' = e^{A(x)} g(x) a(x),$$

where $A(x) = \int_0^x a(t)dt \to +\infty$ as $x \to +\infty$. Since $\lim e^{A(x)} = +\infty$, by L'Hospital Theorem we have

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \frac{e^{A(x)} f(x)}{e^{A(x)}} = \lim_{x \to +\infty} \frac{e^{A(x)} g(x) a(x)}{e^{A(x)} a(x)} = \lim_{x \to +\infty} g(x) = 0.$$

3. By the triangle inequality, the sequence is bounded. Let us prove that $\limsup = \liminf$. Fix n_0 and and let $k_0 = \varrho(x, f^{n_0}(x))/n_0$. It is enough to prove that for any $\varepsilon > 0$, for sufficiently large n we have $\varrho(x, f^n(x)) < (k_0 + \varepsilon)n$, which is clear since $\varrho(x, f^{jn_0}(x)) < jn_0 k_0$, $j = 1, 2, \ldots$.
For the second part, estimate $\varrho(y, f^n(y))$ as

$$\varrho(y, x) + \varrho(x, f^n(x)) + \varrho(f^n(y), f^n(x)),$$

the first and the last term being constant, thus the limits are the same after dividing by n.

4. Označme levou stranu $f(n)$. Pak ji můžeme interpretovat jako počet cest po mřížce doprava nahoru, které napřed jdou do bodu $[n-2k, k]$ a pak pokračují k kroků (takže jejich celková cesta je n). Všechny ostatní cesty délky n jsou takové, co neprochází žádným bodem tvaru $[n-2k, k]$, tedy nutně jdou do nějakého bodu $[n-2k-1, k]$ a pak nahoru do $[n-2k-1, k+1]$. Odstraněním tohoto kroku nahoru získáváme cestu délky $(n-1)$, která prochází některým z bodů $[(n-1)-2k, k]$. Z toho plyne rekurentní vztah $f(n) = 2^n - f(n-1)$. Indukcí pak snadno ověříme rovnost ze zadání.

Another solution: The RHS is a solution to a linear recurrence equation given by $a_{n+2} - a_{n+1} - 2a_n$, so if the LHS satisfies this recurrence as well (and the initial values agree, which is straightforward), the proposition is proved. But this is immediate using the identity

$$\binom{r}{k} = \binom{r-1}{k} + \binom{r-1}{k-1}.$$