Bisobolev mappings and non-isotropic elliptic equations

Carlo Sbordone

University of Naples
Federico II

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Joint work with

- S. Hencl (Prague)
- C. Capozzoli, L. D'Onofrio, F. Farroni
  N. Fusco, F. Giannetti, R. Giova, L. Greco, G. Moscariello,
  A. Passarelli di Napoli, T. Ricciardi, R. Schiattarella, C. Trombetti,
  G. Zecca (Napoli)
- M. Carozza (Benevento)
- P. Di Gironimo (Salerno)

Outline:

- Definitions and main properties of planar bisobolev mappings
- Sequences of bisobolev mappings
- Composition of bisobolev mappings

$\Omega \subset \mathbb{R}^2$ open; $f : \Omega \xrightarrow{\text{onto}} \Omega' \subset \mathbb{R}^2$ homeomorphism

Definition 1

f is bisobolev iff:

$$f,\ f^{-1} \in W^{1,1}_{\mathrm{loc}}$$

$$\left(f = (u, v) : \Omega \xrightarrow{\text{onto}} \Omega';\ \ f^{-1} = (x, y) : \Omega' \xrightarrow{\text{onto}} \Omega\right)$$

While one can be interested in the question how the Sobolev regularity of f reflects in the regularity of its inverse ([Hencl, Koskela (2006)], [Hencl, Koskela, Malý (2006)], [Csörney, Hencl, Malý (2010)]) here we start from mappings which have the same Sobolev regularity of the inverse.

Remark 1:

If $f \in W^{1,1}_{\mathrm{loc}}$ only, then automatically $f^{-1} \in \mathrm{BV}_{\mathrm{loc}}$ and

(1) $$|\nabla y|(\Omega') = \int_\Omega \left|\frac{\partial f}{\partial x}\right| \, dz$$

(2) $$|\nabla x|(\Omega') = \int_\Omega \left|\frac{\partial f}{\partial y}\right| \, dz$$

[Di Gironimo, D'Onofrio, S. ,Schiattarella, Ann. Fenn., (2011)]
[Hencl, Koskela, Onninen, Arch. Ration. Mech. Anal., (2007)], qualitatively

Remark 2:

If we further weaken and assume $f \in \mathrm{BV}_{\mathrm{loc}}$ only, then (1) and (2) extend to

$$|\nabla y|(\Omega') = \left|\frac{\partial f}{\partial x}\right|(\Omega) \tag{3}$$

$$|\nabla x|(\Omega') = \left|\frac{\partial f}{\partial y}\right|(\Omega) \tag{4}$$

[D'onofrio, Schiattarella, to appear]

The advantage of quantitative equalities is seen considering sequences.

If we go to bisobolev precise results hold true:

Theorem 1 (Hencl, Moscariello, Passarelli, S., (2009))

Let $f = (u, v) : \Omega \to \Omega'$ be a bisobolev homeomorphism. Then

$$C_u = \{z : |\nabla u(z)| = 0\} = \{z : |\nabla v(z)| = 0\} = C_v \qquad a.e.$$

The reason is that (u, v) is a solution of a non trivial linear system whose symmetric coefficient matrix $A(z)$ has $\det A(z) = 1$

$$\star\, \nabla v = A(z) \nabla u \tag{5}$$

1. easy case: $f^{-1} \in W^{1,2}$
2. general bisobolev case
3. ACL-homeomorphism

Theorem 2

A sufficient condition for $f \in W^{1,1}_{loc} \cap Hom$ to be a bisobolev is

$$J_f(z) > 0 \ a.e.$$

On the other hand bisobolev homeomorphisms may verify

$$J_f(z) = 0 \text{ on } |Z_f| > 0.$$

The point is that we are assuming only

$$f, f^{-1} \in W^{1,1}_{loc}.$$

In the category of $W^{1,p}$ bisobolev the case $1 \leq p < 2$ is critical because the (N)-property of Lusin

$$|E| = 0 \implies |f(E)| = 0$$

is missing (**Ponomarev 1971**) (while it holds true for $p = 2$)

In particular, the area formula (for $J_f \geq 0$) holds as an inequality

$$\int_B J_f \leq |f(B)|$$

CLASSICAL RESULTS:

Theorem A (Lehto- Virtanen Th.6.1. III)

If $f \in W^{1,2}_{loc} \cap Hom$, then f satisfies the (N)-property.

(More general results $Df \in L^{2)}_b$ or $K_f \in L^{1)}_b$ (Giannetti - Passarelli))

Theorem B (Gehring-Lehto, 1959)

If $f \in W^{1,1}_{loc} \cap Hom$, then f is differentiable a.e.

(false if $n > 2$; $|Df| \in L^p$, $p > n-1$ is sufficient)

CLASSICAL RESULTS:

Theorem C (Sard)

$f : \Omega \longrightarrow \Omega'$ *bi-Sobolev,* $Z_f = \{z : J_f(z) = 0\}$.
$\Longrightarrow \exists N_0$ *such that* $|N_0| = 0$ *and* $|f(Z_f \setminus N_0)| = 0$.

Theorem D (Hencl, Maly (2009) for a recent proof)

If f is a planar bisobolev homeomorphism, then

$$\text{either } J_f \geq 0 \text{ a.e., or } J_f \leq 0 \text{ a.e.}$$

True also for ACL-homeomorphism.

Proof of Theorem 1 in the particular case $f^{-1} \in W^{1,2}$

Claim:

$$|C_u| = |C_v| = |Z_f| = 0$$

In fact

$$C_u \subset Z_f = \{z : J_f(z) = 0\}$$

Theo C of Sard $\implies |f(Z_f)| = 0$

Theo A for $\mathbf{f^{-1}}$ $\implies f^{-1}$ verifies (N) condition
$\implies 0 = |Z_f| = \left|f^{-1}\left(f\left(Z_f\right)\right)\right|$
$\implies |C_u| = 0.$

Similarly $|C_v| = 0.$

bi-ACL homeomorphism

Definition 2

A real function $u = u(x, y)$ continuous in $\Omega \subset \mathbb{R}^2$ is said **absolutely continuous on lines** in Ω if for every rectangle

$$]a, b[\times]c, d[\subset\subset \Omega$$

u is absolutely continuous as a function of the real variable x on a.e. segment $I_y =]a, b[\times\{y\}$ and as a function of y on a.e. segment $\{y\}\times]c, d[$.

It is well known that a continuous function $u : \Omega \to \mathbb{R}$ which is absolutely continuous on lines (ACL for short) in Ω, possesses finite partial derivates a.e. in Ω.

bi-ACL homeomorphism

In the following we will assume that $f = (u, v) : \Omega \subset \mathbb{R}^2 \to \Omega' \subset \mathbb{R}^2$ is a homeomorphism with u and v ACL together with the components of the inverse f^{-1} and call such a mapping a **bi-ACL homeomorphism**.

Remark 3:

bi-ACL homeomorphism are a wider class than bi-Sobolev homeomorphism. A mapping f is in $W^{1,1}$ iff it is in ACL and $|Df| \in L^1$.

Theorem 1' (Moscariello, Passarelli, S., (2009))

Let $f = (u, v) : \Omega \subset \mathbb{R}^2 \xrightarrow{onto} \Omega' \subset \mathbb{R}^2$ be a bi-ACL homeomorphism. Then the components of f have the same critical points, i.e.

$$C_u = \{z \in \Omega : \ |\nabla u| = 0\} = \{z \in \Omega : \ |\nabla v| = 0\} = C_v = Z_f \ \text{a.e.}$$

hence $Df(x)$ vanishes a.e. on Z_f.

Proposition (Schiattarella, (2009))

Let $f : \Omega \xrightarrow{onto} \Omega'$ be a bi-ACL homeomorphism. If f belongs to the Sobolev space $W^{1,1}_{loc}(\Omega; \mathbb{R}^2)$ then f is bisobolev and

$$\int_{\Omega'} |Df^{-1}(w)|dw = \int_{\Omega} |Df(z)|dz \tag{6}$$

Proof of Theorem 1'

Suppose, by contradiction, that there exists a Borel set $A \subset Z_f$; $|A| > 0$ such that

$$|\nabla u(z)| = 0 \qquad |\nabla v(z)| > 0 \text{ a.e. } z \in A.$$

1. **Fubini:** $0 \overset{Sard}{=} |f(A)| = \int \mathcal{H}^1 (f(A) \cap l(t)) \, d\mathcal{H}^1(t)$
   $l(t) =$ horizontal line segment

2. **Coarea:**

   $0 < \int_A |\nabla v(z)| \, dz = \int \mathcal{H}^1 \left(f^{-1}(f(A)) \cap l(t) \right) \, dH^1(t)$

3. **(N)-property of $\mathbf{f^{-1}}$ with respect to Hausdorff $\mathcal{H}^1$- measure:**

   $\mathcal{H}^1 (f(A) \cap l(t)) = 0 \quad \Longrightarrow \quad \mathcal{H}^1 \left(f^{-1}(f(A)) \cap l(t) \right) = 0$

In a paper [Moscariello, Passarelli, S. , Commun. Pure Appl. Anal. 9, (2010)] examples of bi-ACL homeomorphism which are not bisobolev.
Winding around one point: polar coordinates

$$f(r,\varphi) = \left[r, \varphi + \frac{1}{r}\right]$$

$$f^{-1}(s,\theta) = \left[s, \theta - \frac{1}{s}\right].$$

Remark 4:
The condition

$$J_f(z) = 0 \qquad \Longrightarrow \qquad |Df(z)|^2 = 0 \quad a.e. z \in \Omega$$

implies the (distortion) inequality

$$(\diamondsuit) \qquad |Df(z)|^2 \leq K(z) J_f(z)$$

to be satisfied a.e. in Ω for some measurable function $1 \leq K(z) < \infty$ a.e.

$$\left(\text{Recall Hadamard inequality} \quad J_f(z) \leq |Df(z)|^2\right)$$

The smallest function $K \geq 1$ for which $(\diamondsuit)$ holds is called the distortion function of f and is denoted by K_f.

Remark 5:
By symmetry, for bi-ACL homeomorphism f, also $f^{-1} : \Omega' \longrightarrow \Omega$ satisfies a distortion inequality

$$(\diamondsuit\,\diamondsuit) \qquad \left|Df^{-1}(w)\right|^2 \leq H(w)\, J_{f^{-1}}(w) \text{ a.e. in } \Omega'$$

Theorem 3 (Greco, S. , Trombetti (2007))

Let f be bi-ACL homeomorphism $f : \Omega \xrightarrow{onto} \Omega'$. Let K be the distortion of f and H the distortion of f^{-1}. Then

$$H(w) = K\left(f^{-1}(w)\right) \;\; a.e. \;\; w \in \Omega'.$$

(More general results $n \geq 2$ Fusco, Moscariello, S. Calc. Var. (2008))

The interplay between the Theory of mappings $f : \Omega \subset \mathbb{R}^2 \longrightarrow \Omega' \subset \mathbb{R}^2$ and planar PDE's goes back to Morrey (1938).
We will show , as a consequences of Theorem 1, that bisobolev maps represent a class of mappings which permits a far reaching generalization of Morrey's results.

The space of Sobolev mappings is the largest space in which one can begin to discuss what it means to be a solution (u, v) to the degenerate elliptic system

$$\star \nabla v = A(z) \nabla u \tag{7}$$

Here

$$\star \nabla v = \star \begin{pmatrix} v_x \\ \\ v_y \end{pmatrix} = \begin{pmatrix} v_y \\ \\ -v_x \end{pmatrix}$$

Theorem 4

To each bi-ACL homeomorphism $f = (u, v) : \Omega \overset{onto}{\longrightarrow} \Omega'$ *there corresponds a unique (non trivial) degenerate elliptic system such that u and v satisfy* (7), *where:* $A : \Omega \longrightarrow \mathbb{R}^{2\times 2}$ *is a measurable matrix valued function such that* ${}^tA(z) = A(z)$, $\det A(z) = 1$ *a.e. and* $\forall \xi \in \mathbb{R}^2$,

$$\frac{|\xi|^2}{K_f(z)} \leq \langle A(z)\xi, \xi\rangle \leq K_f(z)\, |\xi|^2 \text{ a.e. } z \in \Omega.$$

Proof of Theorem 4

Define $A(z) = (\, a_{i\,j}(z)\,)$ as follows:

$$a_{11}(z) = \frac{v_y^2(z) + u_y^2(z)}{J_f(z)},$$

$$a_{12}(z) = a_{21}(z) = -\frac{v_x(z)v_y(z) + u_x(z)u_y(z)}{J_f(z)}$$

$$a_{22}(z) = \frac{v_x^2(z) + u_x^2(z)}{J_f(z)}$$

for $z \in R_f = \{z \in \Omega : f \text{ is differentiable at } z \text{ and } J_f(z) > 0\}$, while

$$a_{ij}(z) = \delta_{ij} \qquad \text{if } z \in \Omega \setminus R_f .$$

By a direct computation if $z \in R_f$.

If $J_f(z) = 0$ then $|Df(z)| = 0$ by Theorem 1 therefore the system (7) is clearly satisfied.

Remark 6:

Conversely if there exists a non-trivial degenerate elliptic system (7) such that u, v represents a solution, then

$$C_u = C_v.$$

In fact if $\nabla v(z_0) = 0$ then

$$0 = A(z_0)\,\nabla u(z_0) = \begin{pmatrix} a_{11}(z_0)u_x(z_0) + a_{12}(z_0)u_y(z_0) \\ \\ a_{12}(z_0)u_x(z_0) + a_{22}(z_0)u_y(z_0) \end{pmatrix}$$

since $\det A(z_0) = 1$, we deduce $\nabla u(z_0) = 0$.

Remark 7:

Actually we have

$$A(z) = [G_f^*(z)]^{-1} \qquad \text{a.e. } z \in \Omega.$$

Moreover u and v have finite energy:

$$\begin{aligned}\int_\Omega \langle A(z)\nabla u(z), \nabla u(z)\rangle \, dz &= \int_\Omega \langle A(z)\nabla v(z), \nabla v(z)\rangle \, dz \\ &= \int_\Omega J_f(z) \, dz \leq |f(\Omega)|.\end{aligned}$$

Corollary

Let $f : \Omega \subset \mathbb{R}^2 \xrightarrow{onto} \Omega' \subset \mathbb{R}^2$ *be a homeomorphism. Then*

$$f = (u, v) \ \textit{bi-ACL homeomorphism}$$

$$\Updownarrow$$

$$\star \nabla v = A(z) \nabla u \ \textit{in} \ \Omega, \ \textit{where}$$

$$(*) \begin{cases} \frac{I}{K(z)} \leq A(z) = {}^t A(z) \leq K(z) I \\ \det A(z) = 1 \ a.e. \\ 1 \leq K(z) < \infty \ a.e. \ \textit{Borel} \end{cases}$$

Remark 8:
Hence condition (*) is inherited by the inverse $f^{-1} = (s, t)$ with $H(w) = K(f^{-1}(w))$ and $B(w) = A(f^{-1}(w))$.

The regularity of the deformation quotients $K_f(z)$ and $K_{f^{-1}}(w)$ is strictly related with the $W^{1,p}$ bi-Sobolev regularity of f.

Namely

$$K_f \text{ and } K_{f^{-1}} \text{ belong to } L^1$$

if and only if

$$f \text{ is } W^{1,2}\text{bisobolev.}$$

Moreover (Hencl, Moscariello, Passarelli, S. (2009))

$$\int_\Omega |Df(z)|^2 \, dz = \int_{\Omega'} K_{f^{-1}}(w) \, dw$$

and

$$\int_{\Omega'} \left|Df^{-1}(w)\right|^2 \, dw = \int_\Omega K_f(z) \, dz.$$

A different matter is the existence problem of homeomorphic solutions to the elliptic system

$$\star\, \nabla v = A(x) \nabla u$$

The assumption $K \in L^1$ or even stronger $K \in L^p$ for $p > 1$ is not sufficient.

in Ryazanov-Srebro-Yakubov (2001))

The assumption $K \in \mathrm{EXP}$ is the right one.
(Iwaniec, S. Ann. Inst. H. Poincaré, (2001)

The "continuity" of the operator

$$f \longrightarrow A_f = [G_f]^{-1}$$

has been studied by Capozzoli, Carozza (2008).

The "right" topology on coefficient matrices is the Γ- convergence.

Bisobolev mappings (in plane) are simply homeomorphisms $f : \Omega \to \Omega'$ ($\Omega, \Omega' \subset \mathbb{R}^2$ domains) such that

i) f belongs to Sobolev class $W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$

and

ii) $f = (u, v)$ satisfies the Beltrami system

$$D^t f(z) Df(z) = J_f(z) G_f(z) \qquad \text{a.e.} z \in \Omega$$

where the measurable symmetric matrix field G_f satisfies:

- $\det G_f(z) = 1$,
- $\forall \xi \in \mathbb{R}^2, \qquad \dfrac{|\xi|^2}{K_f(z)} \leq \langle G_f(z)\xi, \xi\rangle \leq K_f(z)|\xi|^2$
  and
- $K_f : \Omega \to [1, \infty[$ is Borel.

Theorem

$$\mathcal{A}_f = {G_f}^{-1} \qquad f = (u, v) \qquad \Longrightarrow \qquad \begin{cases} \operatorname{div} \mathcal{A}_f \nabla u = 0 \\ \operatorname{div} \mathcal{A}_f \nabla v = 0 \end{cases}$$

Theorem 0 (Fusco, Moscariello, S. , (2008))

$f_j : \Omega \xrightarrow{onto} \Omega'$ *bisobolev*

$$\left.\begin{array}{l} f_j \overset{W^{1,1}}{\rightharpoonup} f \in W^{1,1}_{\mathrm{loc}} \cap Hom \\ K_{f_j} \rightharpoonup K \quad \sigma(L^1, L^\infty) \end{array}\right) \Longrightarrow \left(\begin{array}{l} f \; bisobolev \\ K_f(z) \leq K(z) \\ \int_\Omega K_f \leq \liminf_j \int_\Omega K_{f_j} \end{array}\right.$$

Moreover:

$$\left.\begin{array}{l} f_j \xrightarrow{C^0} f \\ K^*_{f_j}(z) \leq K(z) \end{array}\right) \Longrightarrow \left(\begin{array}{l} f_j^{-1} \overset{W^{1,1}}{\rightharpoonup} f^{-1} \\ f_j^{-1} \xrightarrow{C^0} f^{-1} \end{array}\right.$$

SHARP: EXAMPLE (D'Onofrio, Schiattarella, to appear)

Theorem 1

Let be $f_j \in W^{1,2} \cap$ Hom such that $\{f_j\}$ converges weakly in $W^{1,1}$ and c-uniformly to a map $f \in W^{1,1}$. Then f admits a.e. a right inverse $h \in \mathrm{BV}$, that is $f(h(w)) = w$ a.e. and

$$\|h\|_{\mathrm{BV}} \leq C \int_\Omega |Df|$$

The Jacobians of $f_j = (u_j, v_j)$ bisobolev

$$J_f \in L^1(\Omega)$$

(weak continuity)

Theorem 2 (Dal Maso, S. (1994))

$$f_j, f \in W^{1,2}(\Omega);\ \ f_j \overset{W^{1,1}}{\rightharpoonup} f \qquad \Longrightarrow \qquad J_{f_j} \rightharpoonup J_f \ \ \textit{weakly in } L^1_{\mathrm{loc}}(\Omega).$$

Theorem 3 (Iwaniec, Martin)

$$P(t) = t^2 \log^{-1}(e+t), \quad f_j, f \in W^{1,P}(\Omega)$$

$$f_j \overset{W^{1,P}}{\rightharpoonup} f \qquad \Longrightarrow \qquad J_{f_j} \rightharpoonup J_f \ \textit{weakly in } L^1_{\mathrm{loc}}(\Omega).$$

EXAMPLE:

$z = (x, y) \in (0,1)^2$

$$h_j(x) = \int_0^x \varphi_j(s)\, ds \qquad k_j(y) = \int_0^y \psi_j(s)\, ds$$

h, k strictly increasing;

$$\frac{1}{\varphi_j} \rightharpoonup \frac{1}{\varphi_-}; \quad \varphi_j \rightharpoonup \varphi_+; \quad \frac{1}{\psi_j} \rightharpoonup \frac{1}{\psi_-}; \quad \psi_j \rightharpoonup \psi_+ \qquad \sigma(L^1, L^\infty).$$

$$f_j(x, y) \stackrel{\text{def}}{=} (h_j(x), k_j(y))$$

$$Df_j(z) = \begin{pmatrix} \varphi_j(x) & 0 \\ 0 & \psi_j(y) \end{pmatrix} \xrightharpoonup{L^1} \begin{pmatrix} \varphi_+(x) & 0 \\ 0 & \psi_+(y) \end{pmatrix} = Df(z)$$

$$f(x,y) = \left(\int_0^x \varphi_+ , \int_0^y \psi_+ \right)$$

$$J_{f_j}(z) = \varphi_j(x)\psi_j(y) \rightharpoonup J_f = \varphi_+(x)\psi_+(y)$$

Notice:

$$\varphi_-(s) \leq \varphi_+(s)$$

$$\psi_-(t) \leq \psi_+(t)$$

The Beltrami matrix of $f_j = (u_j, v_j)$

$$\mathcal{A}_{f_j} = \begin{cases} \left[\dfrac{D^t f_j(z) Df_j(z)}{J_{f_j}(z)}\right]^{-1} & \text{if } J_{f_j}(z) > 0 \\ I, & \text{if} J_{f_j}(z) = 0 \end{cases}$$

satisfies

$$^t\mathcal{A}_{f_j} = \mathcal{A}_{f_j}$$

$$\det \mathcal{A}_{f_j} = 1 \qquad \text{a.e.}$$

$$\frac{|\xi|^2}{K_j(z)} \leq \langle \mathcal{A}_{f_j}\xi, \xi\rangle \leq K_j(z)|\xi|^2$$

$$\begin{cases} \operatorname{div}\left(\mathcal{A}_{f_j}\nabla u_j\right) = 0 \\ \operatorname{div}\left(\mathcal{A}_{f_j}\nabla v_j\right) = 0 \end{cases}$$

So u_j, v_j are distributional solutions with finite energy:

$$\int_\Omega \langle \mathcal{A}_{f_j} \nabla u_j, \nabla u_j \rangle = \int_\Omega J_{f_j} = \int_\Omega \langle \mathcal{A}_{f_j} \nabla v_j, \nabla v_j \rangle$$

Actually, adopting the notation

$$f_j = \left(f_j^{(1)}, f_j^{(2)} \right)$$

we have

$$\langle \mathcal{A}_{f_j}(z) \nabla f_j^{(r)}, \nabla f_j^{(s)} \rangle = J_{f_j}(z) \delta_{rs}$$

For the example

$$f_j(z) = \left(\int_0^x \varphi_j, \int_0^y \psi_j \right)$$

under the assumptions

$$\varphi_j \rightharpoonup \varphi_+ \qquad \frac{1}{\varphi_j} \rightharpoonup \frac{1}{\varphi_-}$$

$$\psi_j \rightharpoonup \psi_+ \qquad \frac{1}{\psi_j} \rightharpoonup \frac{1}{\psi_-}$$

in $\sigma(L^1, L^\infty)$ we have

$$\mathcal{A}_{f_j}(z) = \begin{pmatrix} \frac{\psi_j(y)}{\varphi_j(x)} & 0 \\ 0 & \frac{\varphi_j(x)}{\psi_j(y)} \end{pmatrix} \rightharpoonup \begin{pmatrix} \frac{\psi_+(y)}{\varphi_-(x)} & 0 \\ 0 & \frac{\varphi_+(x)}{\psi_-(y)} \end{pmatrix} = \mathcal{A}_f^+(z)$$

(weakly in L^1_{loc})

$$f(x,y) = \left(\int_0^x \varphi_+, \int_0^y \psi_+ \right)$$

$$\det \mathcal{A}_f^+(z) = \frac{\psi_+(y)}{\varphi_-(x)} \frac{\varphi_+(x)}{\psi_-(y)} > 1$$

Definition of G-convergence

Let $A_j(z)$ and $A(z)$, $j \in \mathbb{N}$ be symmetric matrices satisfying uniform ellipticity

$$\frac{|\xi|^2}{K} \leq \langle A_j(z)\xi, \xi\rangle \leq K|\xi|^2 \tag{8}$$

$$\frac{|\xi|^2}{K} \leq \langle A(z)\xi, \xi\rangle \leq K|\xi|^2 \tag{9}$$

a.e. $z \in \Omega \subset \mathbb{R}^2$, bounded simply connected domain, $\forall \xi \in \mathbb{R}^2$.

We say that

$$A_j \xrightarrow{G} A \tag{10}$$

iff for any $\xi \in \mathbb{R}^2$, the unique solutions $u_j \in W_0^{1,2}(\Omega) + \langle \xi, z \rangle$ of the Dirichlet problems

$$\begin{cases} \operatorname{div}(A_j(z)\nabla u_j) = 0 & \text{in } \Omega \\ u_j(z) = \langle \xi, z \rangle & \text{on } \partial\Omega \end{cases} \tag{11}$$

converge weakly in $W^{1,2}$ to the (unique) solution $u \in W_0^{1,2}(\Omega) + \langle \xi, z \rangle$ of the Dirichlet problem

$$\begin{cases} \operatorname{div}(A(z)\nabla u) = 0 & \text{in } \Omega \\ u(z) = \langle \xi, z \rangle & \text{on } \partial\Omega \end{cases} \tag{12}$$

Theorem 4 (S.Spagnolo (1967))

Any sequence A_j satisfying (8) contains a subsequence A_{j_r} G-converging to a symmetric matrix A satisfying (9).

Definition of Beltrami matrices

For $f : \Omega \to \Omega'$ bisobolev, its <u>Beltrami matrix</u> is

$$\mathcal{A}_f(z) = \begin{cases} \left[\frac{D^t f(z) Df(z)}{J_f(z)}\right]^{-1} & \text{if } J_f(z) > 0 \\ I & \text{otherwise} \end{cases} \tag{13}$$

$$\det \mathcal{A}_f = 1 \qquad {}^t\mathcal{A}_f = \mathcal{A}_f$$

Definition of Γ-convergence

Let $A_j(z)$ and $A(z)$, $j \in \mathbb{N}$ be symmetric matrices satisfying a.e. $z \in \Omega$;

$$0 \leq \langle A_j(z)\xi, \xi\rangle \leq K_j(z)|\xi|^2 \qquad \forall \xi \in \mathbb{R}^2$$

$$0 \leq \langle A(z)\xi, \xi\rangle \leq K(z)|\xi|^2 \qquad \forall \xi \in \mathbb{R}^2$$

for K_j, $K \in L^1(\Omega)$.
We say that

$$A_j \xrightarrow{\Gamma} A$$

iff the two conditions (i), (ii) are satisfied:

(i) $u_j, u \in W^{1,\infty}(\Omega)$ and $u_j \xrightarrow{L^1(\Omega)} u \;\Rightarrow$

$$\int_\Omega \langle A\nabla u, \nabla u\rangle \leq \lim_j{}' \int_\Omega \langle A_j \nabla u_j, \nabla u_j\rangle$$

(ii) $\forall u \in W^{1,\infty}(\Omega)\; \exists w_j \in W^{1,\infty}(\Omega) \,:\, w_j \xrightarrow{L^1(\Omega)} u$ and

$$\int_\Omega \langle A\nabla u, \nabla u\rangle = \lim_j \int_\Omega \langle A_j \nabla w_j, \nabla w_j\rangle$$

Theorem 5 (Marcellini, S. , J.Math. Pures Appl., (1977))

If $K_j \rightharpoonup K_0$ in $\sigma(L^1, L^\infty)$ then there exists a subsequence A_{j_r} such that

$$A_{j_r} \xrightarrow{\Gamma} A$$

for A satisfying $0 \leq \langle A(z)\xi, \xi\rangle \leq K_0(z)|\xi|^2$

Remark 1:
If $1 \leq K_j(z) \leq K_0$ are equibounded, then

$$A_j \xrightarrow{G} A \quad \Leftrightarrow \quad A_j \xrightarrow{\Gamma} A$$

On the other hand, in general, no relation with convergence of solutions to Dirichlet problems.

Remark 2: If $\|K_j\|_{L^1} \leq C_0 \ \forall j \in \mathbb{N}$ the $\Gamma-$compactness result of Marcellini, S. fails.

$$F_j(u) = \int_\Omega K_j(z)|\nabla u|^2 dz$$

$\Omega = [0,1]^2 \ni z = (x,y)$

$$K_j(z) = \begin{cases} j & \text{in } \ \Omega_j = \Omega \cap |x| < \frac{1}{j} \\ \\ 1 & \text{in } \ \Omega \setminus \Omega_j \end{cases}$$

$$F_j(u) \xrightarrow{\Gamma} \int_\Omega |\nabla u|^2 + |u(0,1) - u(0,-1)|^2$$

for $u \in C^1(\overline{\Omega}) : x \to u(x,1)$ and $x \to u(x,-1)$ are constant near zero and $u(0,y) = my + q$.

Definition of Γ_Q –convergence

For sequences $A_j(z)$ satisfying

$$\frac{|\xi|^2}{K_j(z)} \leq \langle A_j(z)\xi, \xi\rangle \leq K_j(z)|\xi|^2$$

$$\fint_\Omega e^{\frac{K_j(z)}{\lambda_0}}\, dz \leq C_0 \qquad \text{for } 0 < \lambda_0 < 1/2 \text{ and } C_0 \geq 1$$

let $P(t) = t^2 \log^{-1}(e+t),\ Q(t) = t\log(e+t)$.

We say that

$$A_j \xrightarrow{\Gamma_Q} A$$

iff

(i_Q) $u_j, u \in W^{1,Q}(\Omega),\ u_j \xrightarrow{L^Q(\Omega)} u \Rightarrow \int_\Omega \langle A\nabla u, \nabla u\rangle \leq \liminf_j{}' \int_\Omega \langle A_j \nabla u_j, \nabla u_j\rangle$

(ii_Q) $\forall u \in W^{1,Q}(\Omega)\ \exists w_j \in W^{1,Q}(\Omega)\ :\ w_j \xrightarrow{L^Q(\Omega)} u$ and

$$\int_\Omega \langle A\nabla u, \nabla u\rangle = \lim_j \int_\Omega \langle A_j \nabla w_j, \nabla w_j\rangle$$

Theorem 6 (Capozzoli, Carozza, Ric. Mat. (2009))

Let $f_j, f \in W^{1,1}(\Omega;\mathbb{R}^2) \cap \mathrm{Hom}$ *satisfy*

(*i*) $|Df_j(z)|^2 \le K_j(z) J_{f_j}(z) \qquad a.e.\ z \in \Omega$

(*ii*) $\exists\, 0 < \lambda < 1/2,\ \exists C_0 \ge 1:$

$$\fint_\Omega e^{\frac{K_j(z)}{\lambda}}\, dz \le C_0 \qquad \forall j \in \mathbb{N}$$

(*iii*) $f_j \rightharpoonup f$ *weakly in* $W^{1,1}_{\mathrm{loc}}$

Then f *is bisobolev and we have*

$$\fint_\Omega e^{\frac{K_f(z)}{\lambda}}\, dz \le C_0$$

and

$$\mathcal{A}_{f_j} \xrightarrow{\Gamma} \mathcal{A}_f$$

Under previous EXP assumptions we have, for $v \in W^{1,Q}(\Omega) \subset W^{1,P}(\Omega)$

$$c \int_\Omega |\nabla v|^2 \log^{-1}\left(e + \frac{|\nabla v|}{|\nabla v|_\Omega}\right) \leq \int_\Omega \langle A_j \nabla v, \nabla v \rangle$$

$$\leq C \int_\Omega |\nabla v|^2 \log\left(e + \frac{|\nabla v|}{|\nabla v|_\Omega}\right)$$

We need the regularity results Astala-Gill-Rohde-Saksman (2009)

Theorem 7

$f:\Omega\to\Omega'$ bisobolev, if $\exists\ 0<\lambda_0<1/2$ such that $\int_\Omega e^{\frac{K_f(z)}{\lambda_0}}dz\leq C_0$, then:

$$\int_B |Df|^2\log\left(e+\frac{|Df|}{|Df|_\Omega}\right)\leq\int_{2B}J_f$$

for all concentric disks $B\subset\subset 2B\subset\Omega$.

Theorem 8 (Moscariello (1994))

$f:\Omega\to\Omega'$ bisobolev, $|Df|\in L^2\log^{-1}L(\Omega)$; then $J_f\in L\log\log_{\mathrm{loc}}(\Omega)$ and

$$\int_B J_f\log\log(e+J_f)dz\leq c\left[\int_{2B}|Df|^2\log^{-1}(e+|Df|^2)+1\right]$$

Proof of Theorem 6

STEP 1: By Theorem 5 we know that there exists a subsequence

$$A_{f_{j_r}} \xrightarrow{\Gamma} A \qquad \text{on } W^{1,\infty}$$

where $A = {}^tA$ satisfies

$$0 \leq \langle A(z)\xi, \xi \rangle \leq K(z)|\xi|^2.$$

If we prove

$$A(z) = \left[\frac{D^t f(z) Df(z)}{J_f(z)} \right]^{-1} \quad \text{a.e.}$$

this will imply better bound

$$\frac{|\xi|^2}{K(z)} \leq \langle A(z)\xi, \xi \rangle$$

and that the entire sequence will Γ-converge to A:

$$A_j \xrightarrow{\Gamma} A.$$

Since $\forall\, \Omega_1 \subset \Omega$

$$\begin{aligned}\int_{\Omega_1} \langle A_j \nabla u, \nabla u \rangle &\leq \int_{\Omega_1} K_j(z) |\nabla u|^2 \\ &\leq 2 \|K_j\|_{\mathrm{EXP}} \|\nabla u\|^2_{L^Q(\Omega_1)} \\ &\leq c \|\nabla u\|^2_{L^Q(\Omega_1)}\end{aligned}$$

the functionals

$$u \to \left(\int_\Omega \langle A_j \nabla u, \nabla u \rangle \right)^{1/2}$$

are equilipschitz in $W^{1,Q}_{\mathrm{loc}}$; a legitimate reason for passing from Γ-convergence to the stronger

$$A_j \xrightarrow{\Gamma_Q} A \qquad \text{on } W^{1,Q}$$

(by an abstract result on Γ- convergence)

STEP 2: Let us show that, for any $\Omega_1 \subset\subset \Omega$, if $f_{j_r} = (u_r, v_r)$, $f = (u, v)$, then

$$\int_{\Omega_1} \langle A(z)\nabla u, \nabla u\rangle = \lim_r \int_{\Omega_1} \langle A_{j_r}(z)\nabla u_r, \nabla u_r\rangle$$

$$\int_{\Omega_1} \langle A(z)\nabla v, \nabla v\rangle = \lim_r \int_{\Omega_1} \langle A_{j_r}(z)\nabla v_r, \nabla v_r\rangle$$

By Theorem 7 $u_r \overset{L^Q(\Omega_1)}{\longrightarrow} u$. Let w_r be a sequence in $W^{1,Q}(\Omega_1)$ such that $w_r \overset{L^Q(\Omega)}{\longrightarrow} u$ and

$$\int_{\Omega_1} \langle A(z)\nabla u, \nabla u\rangle = \lim_r \int_{\Omega_1} \langle A_{j_r}(z)\nabla v_r, \nabla v_r\rangle.$$

Fix $S \subset\subset \Omega_1$ and $\varphi \in C_0^1(\Omega_1)$ cut-off ($\varphi = 1$ on S), $t \in (0,1)$ yields

$$\int_{\Omega_1} \langle A_{j_r}(z)\nabla w_r, \nabla w_r\rangle \geq \int_{\Omega_1} \langle A_{j_r}(z)\nabla u_r, \nabla u_r\rangle(\varphi(z) - 1) - \frac{1-t}{t} c\|\nabla\varphi\|_\infty^2 \|w_r - u_r\|_{L^Q(\Omega_1)}^2$$

as $r \to \infty$, $t \to 0$

$$\begin{aligned}\int_{\Omega_1} \langle A(z)\nabla u, \nabla u\rangle &\geq \lim''_r \int_{\Omega_1} \langle A_{j_r}(z)\nabla u_r, \nabla u_r\rangle \\ &\geq \lim'_r \int_S \langle A_{j_r}(z)\nabla u_r, \nabla u_r\rangle \\ &\geq \int_S \langle A(z)\nabla u, \nabla u\rangle\end{aligned}$$

STEP 3:

Technical: use $K_f \in \mathrm{EXP}(\Omega)$,

$$|Df| \in L^2 \log^{-1} L(\Omega) \quad \Longrightarrow \quad J_f \in L \log\log L_{\mathrm{loc}}(\Omega)$$

Corollary

$\forall \varphi = \sum_{i=1}^{n} \mu_i \chi_{B_i}$

$$\left| \Omega_1 \setminus \left(\bigcup_{i=1}^{n} B_i \right) \right| = 0 \qquad \mu_i \geq 0$$

$$\lim_{r}{}' \int_{\Omega_1} \langle A_r \nabla u_{j_r}, \nabla u_{j_r} \rangle \varphi \geq \int_{\Omega_1} \langle A \nabla u, \nabla u \rangle \varphi$$

We already know that $K_f \in \mathrm{EXP}(\Omega)$ and hence $J_f > 0$ a.e. in Ω.
Moreover, $\forall S \subset \Omega$

$$\int_S |Df_j|^2 \log^{-1}(\quad) \leq 2\lambda_0 \left[\int_S \left(J_{f_j} + e^{K_{f_j}/\lambda_0} - 1 \right) \right]$$

and (Moscariello) hence

$$\| J_{f_j} \log\log(\quad) \|_{L^1_{\mathrm{loc}}} \leq C_1$$

Since

$$J_{f_j}(z) = \langle A_{f_j}(z) \nabla u_j, \nabla u_j \rangle$$

we deduce $\exists F \in L^1(\Omega_1)$:

$$\int_{\Omega_1} \langle A_r(z) \nabla u_{j_r}, \nabla u_{j_r} \rangle \varphi \longrightarrow \int_{\Omega_1} F\varphi \qquad\qquad \forall \varphi \in C^0(\overline{\Omega_1})$$

and by Corollary

$$\int_{\Omega_1} \langle A \nabla u, \nabla u \rangle \varphi \leq \int_{\Omega_1} F \varphi$$

It is easy to arrive finally to

$$F(z) = \langle A(z) \nabla u, \nabla u \rangle \qquad \text{a.e. in } \Omega_1$$

and $\forall \in C^0(\overline{\Omega_1})$

$$(*) \qquad \int_{\Omega_1} \langle A_{f_j}(z) \nabla u_j, \nabla u_j \rangle \varphi \longrightarrow \int_{\Omega_1} \langle A(z) \nabla u, \nabla u \rangle \varphi$$

and similarly for v_j and v.

Now recall that (by definition of A_{f_j})

$$\langle A_{f_j}(z)\nabla f_j^{(r)}, \nabla f_j^{(s)}\rangle = J_{f_j}(z)\delta_{rs}$$

and use the weak continuity of Jacobian for $f_j \rightharpoonup f$ in $W^{1,P}$ (generalization of Reshetnjak Theorem), (Iwaniec, Martin Theorem 8.4.2) we get for the limit $f = \left(f^{(1)}, f^{(2)}\right) = (u, v)$

$$\langle A(z)\nabla f^{(r)}, \nabla f^{(s)}\rangle = J_f(z)\delta_{rs}$$

and consequently

$$A(z) = A_f(z)$$

since $J_f(z) > 0$ a.e.

1) Classical results on change of variables:

Let $f : \Omega \subset \mathbb{R}^2 \to \Omega' \subset \mathbb{R}^2$ be a homeomorphism

Theorem 1

f biLipschitz, $p \geq 1$,

$$w \in W^{1,p}_{\mathrm{loc}}(\Omega') \qquad \Longrightarrow \qquad w \circ f \in W^{1,p}_{\mathrm{loc}}(\Omega)$$

Theorem 2 (Reimann)

f K-quasiconformal,

$$w \in W^{1,2}_{\mathrm{loc}}(\Omega') \qquad \Longrightarrow \qquad w \circ f \in W^{1,2}_{\mathrm{loc}}(\Omega)$$

$$w \in \mathrm{BMO}(\Omega') \qquad \Longrightarrow \qquad w \circ f \in \mathrm{BMO}(\Omega)$$

Actually the composition operator $T_f(w) = w \circ f$ maps $W^{1,2}$ into $W^{1,2}$ continuously

Theorem 3 (Farroni-Giova, Studia Math. (2011))

f K-quasiconformal,

$$w \in \mathrm{EXP}(\mathbb{D}) \qquad \Longrightarrow \qquad w \circ f \in \mathrm{EXP}(\mathbb{D})$$

and

$$\frac{1}{1 + K \log K} \leq \frac{\|w \circ f\|_{\mathrm{EXP}(\mathbb{D})}}{\|w\|_{\mathrm{EXP}(\mathbb{D})}} \leq 1 + K \log K$$

SHARP using Astala's Theorem

$$w \in \mathrm{EXP}(\Omega) \Longleftrightarrow \exists \lambda > 0\colon \int_\Omega e^{\frac{w(x)}{\lambda}}\, dx < \infty$$

Theorem 4 (Lehto-Virtanen, Ziemer)

f $W^{1,2}$-bisobolev,

$$w \in W^{1,2}(\Omega') \qquad \Longrightarrow \qquad w \circ f \in W^{1,1}(\Omega).$$

2) Composing bisobolev maps:

Theorem 5 (Hencl, Koskela (2008))

$f : \Omega \to \Omega'$ *bisobolev*, $g : \Omega' \to \Omega''$ *bisobolev*,

$$\left.\begin{array}{c} f^{-1} \in W^{1,2} \\ g \in W^{1,2} \\ g^{-1} \text{ satisfies the (N)-condition of Lusin} \end{array}\right. \quad \Longrightarrow \quad g \circ f \ \text{bisobolev}$$

Remark 1:
The assumption g^{-1} satisfies (N) is not necessary (see Schiattarella (2010))

Theorem 6 (Greco, S., Schiattarella, Proc. Royal Soc. Edinburgh (2011))

$f : \Omega \to \Omega'$ *bisobolev,* $g : \Omega' \to \Omega''$ *biSobolev,*

$$\left.\begin{array}{l} |Df^{-1}| \in L^2 \log^{\alpha} L \\[2ex] |Dg| \in L^2 \log^{-\alpha} L \end{array}\right. \quad \Longrightarrow \quad g \circ f \ \textit{biSobolev}$$

and

$$K_{g \circ f}(z) \leq K_g(f(z)) K_f(z) \qquad a.e.\ z \in \Omega.$$

We have always that f^{-1} satisfies the (N)- condition

Proposition (Greco, S., Schiattarella)

$$\left.\begin{array}{l} f:\Omega\to\Omega', g:\Omega'\to\Omega'' \ \textit{measurable} \\ f^{-1} \ \textit{satisfies the (N)- condition} \end{array}\right. \implies g\circ f \ \textit{measurable}$$

To compare with classical facts:

$$f,\, g \text{ Borel} \implies g\circ f \text{ Borel}$$

$$f \text{ measurable}, g \text{ Borel} \implies g\circ f \text{ measurable}$$

Proof of Proposition

$$g \circ f : \Omega \longrightarrow \Omega''$$

We have to prove that:

$$(g \circ f)^{-1}(E'') = f^{-1}\left(g^{-1}(E'')\right) \text{ is measurable for all } E'' \subset \Omega'' \text{ open.}$$

$$g \text{ is a measurable mapping} \quad \Longrightarrow \quad g^{-1}(E'') \text{ is measurable.}$$

Then $\exists B'$ Borel $\supset g^{-1}(E'')$ such that $|B' \setminus g^{-1}(E'')| = 0$.

f^{-1} verifies the (N)- condition $\quad \Longrightarrow \quad |f^{-1}\left(B' \setminus g^{-1}(E'')\right)| = 0$
and the set

$$f^{-1}\left(g^{-1}(E'')\right) = f^{-1}(B'') \setminus f^{-1}\left(B' \setminus g^{-1}(E'')\right)$$

is measurable.

Distortion of the composition

Theorem 7 (Greco, S., Schiattarella)

$f : \Omega \to \Omega'$, $g : \Omega' \to \Omega''$ bisobolev

Let

$$\left.\begin{array}{ll} (i) & K_g \in \mathrm{EXP}_{\mathrm{loc}}(\Omega') \\ & K_f \in L^2_{\mathrm{loc}}(\Omega) \end{array}\right) \Longrightarrow \left(\begin{array}{ll} & g \circ f \text{ bisobolev} \\ (j) & K_{g \circ f} \in L^1_{\mathrm{loc}}(\Omega) \end{array}\right.$$

SHARP:

(i) cannot be dropped

(j) is optimal

$$\exists\, f(x) = \frac{x}{|x|}\rho_1(|x|), \qquad g(x) = \frac{x}{|x|}\rho_2(|x|)$$

The inequality

$$\frac{|D(g\circ f)(z)|^2}{J_{g\circ f}(z)} \leq \frac{|Dg(f(z))|^2}{J_g(f(z))}\frac{|Df(z)|^2}{J_f(z)} \text{ for a.e. } z\in\Omega$$

does not require $J_g(w)>0$ a.e. $w\in\Omega$ but it is sufficient that, for a.e. $z\in\Omega$

$$J_g(f(z))=0 \quad\Longrightarrow\quad |Dg(f(z))|=0$$

and this is true, thanks to

$$g \text{ bisobolev and } f^{-1} \text{ verifies (N) condition}$$

$$\Longrightarrow \qquad |\{z\in\Omega : |Dg(f(z))|=0\}\setminus\{z\in\Omega : J_g(f(z))=0\}|$$

$$=\left|\{f^{-1}(w\in\Omega' : |Dg(w)|=0\}\setminus\{w\in\Omega' : J_g(w)=0\})\right|=0.$$

Recall that $K \in \mathrm{EXP}(\Omega)$ if there exists $\lambda > 0$ such that

$$\int_\Omega e^{\frac{K(z)}{\lambda}} dz < \infty.$$

Then, for many reason it comes out that the space

$$W^{1,P}_{loc}(\Omega), \qquad P(t) = t^2 \log^{-1}(e + t)$$

is a good starting point for the existence and regularity theory. We have the following:

Theorem 8

Let ${}^tA(z) = A(z)$ *be a measurable matrix field such that*

$$\begin{cases} \det A(z) = 1 & a.e.\ z \in \mathbb{D} \\ \frac{|\xi|^2}{K(z)} \leqslant \langle A(z)\xi, \xi\rangle \leqslant K(z)|\xi|^2 \end{cases}$$

for $K \in \mathrm{EXP}(\mathbb{D})$; $K(z) = 1$ *for* $|z| > 1$.
Then, there exists a unique bisobolev map $f = (u, v) \in W^{1,P}(\mathbb{R}^2)$ *solving the system*

$$\left[\frac{{}^tDf\ Df}{J_f}\right]^{-1} = A(z)$$

such that $J_f > 0$ *a.e.*

Remark 2:

$$\frac{|Df|^2}{\log(e + |Df|)} \leqslant 2\lambda(J_f + e^{\frac{K}{\lambda}} - 1).$$

Remark 3:
The condition

$$\det A(z) = 1$$

in some regularity problems can be removed because of the following algebraic lemma (Capone, Iwaniec-S.) (Malý)

Lemma

Given a vector $\underline{E} = (E_1, E_2)$ *and a symmetric matrix* $B \in \mathbb{R}^{2\times 2}$ *such that*

$$\frac{|\underline{X}|^2}{K} \leqslant \langle B\underline{X}, \underline{X}\rangle \leqslant K|\underline{X}|^2$$

for a $K \geqslant 1$; $\forall \underline{X} \in \mathbb{R}^2$.
Then, there exists a symmetric matrix $\mathcal{A} \in \mathbb{R}^{2\times 2}$ *such that*

$$\det \mathcal{A} = 1 \qquad \mathcal{A}\underline{E} = B\underline{E}$$

$$\frac{|\underline{X}|^2}{K} \leqslant \langle \mathcal{A}\underline{X}, \underline{X}\rangle \leqslant K|\underline{X}|^2$$

W_A-solutions to $\mathrm{div}(A\nabla) = 0$

Definition:

$K \in \mathrm{EXP}(\Omega)$. The function u

$$u \in W_A(\Omega) = \left\{ \varphi \in W^{1,1}(\Omega) : \int_\Omega \langle A(x)\nabla\varphi, \nabla\varphi\rangle < \infty \right\}$$

is a W_A-solution if

$$\int_\Omega \langle A\nabla u, \nabla\varphi\rangle = 0 \qquad \forall\varphi \in W_A(\Omega), \quad \text{supp } \varphi \subset\subset \Omega.$$

Theorem 9 (Ricciardi,Zecca Potential Anal., to appear)

Every W_A-solution u to

$$\mathrm{div} A\nabla u = 0$$

satisfies

$$|u(z) - u(z')|^2 \leqslant \frac{4\pi}{\lambda\left[\log\log\left(\frac{\mathcal{K}}{2\pi r^2}\right) - \log\log\left(\frac{\mathcal{K}e^3}{2\pi \bar{r}^2}\right)\right]} \int_{B_{\bar{r}}} \langle A\nabla u, \nabla u\rangle$$

$z, z' \in B(z_0, r)$, $r < \bar{r}e^{-3}$ *where* $B_{\bar{r}} = B(z_0, \bar{r}) \subset\subset \Omega$.

Theorem 10 (Ricciardi, Zecca)

If moreover $\det A = 1$ *a.e., then*

$$|u(z) - u(z')|^2 \leqslant \frac{C}{\left(\log\log \frac{\mathcal{K}}{2\pi r^2}\right) \log^\lambda \left(\frac{\mathcal{K}}{2\pi r^2}\right)} \int_{B_{\bar r}} \langle A \nabla u, \nabla u \rangle \, dz$$

$z, z' \in B(z_0, r)$, $r < \bar r e^{-6}$, $C = C(\lambda, \bar r, \mathcal{K})$.

In the spirit of mappings $f = (u, v)$, the W_A-condition means

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

in a stranger sense then in the distributional one.

To compare with Koskela-Onninen, Onninen-Zhong, Campbell-Hencl.