APPROXIMATION OF STATIONARY DENSITY IN SOME AUTOREGRESSIVE MODELS

Pavel Ranocha
Department of Probability and Statistics, Charles University, Prague
ranocha@karlin.mff.cuni.cz

SUMMARY

We describe methods of calculating of stationary density in some autoregressive models of time series. Exact solution of this problem is known only in few special cases. We propose algorithms which approximate such solution and deal with their convergence properties. In detail we study several types of processes – AR(1) (including multivariate), AR(2) and AAR(1).

INTRODUCTION

Let $\{X_t\}_{t\geq 0}$ be a strictly stationary time series. We wish to evaluate its stationary distribution given the distribution of innovations. This is quite complicated problem if an analytic solution in closed form is required. If the desired distribution is absolutely continuous, it is usually a solution of some integral equation.

An explicit solution of such problem is known only in few cases. The most trivial is, of course, a linear AR(1) model driven by Gaussian noise. Among those non-trivial we can mention an absolute autoregression

$$X_t = a|X_{t-1}| + \varepsilon_t \tag{1}$$

with innovations ε_t having Gaussian, Cauchy, Laplace and discrete rectangular distribution (see [3] and [6]), and a threshold autoregression

$$X_t = \begin{cases} \alpha X_{t-1} + \varepsilon_t & \text{if } X_{t-1} \geq 0 \\ \beta X_{t-1} + \varepsilon_t & \text{if } X_{t-1} < 0 \end{cases}$$

where the noise process ε_t has Laplace distribution (see [5]).

LINEAR AUTOREGRESSION

If an analytic solution is not known, we seek for a numerical approximation. For general nonlinear models, several methods have been developed, see e.g. [6]. If the model of interest is linear, we recommend to use one of the following procedures.

Algorithm of Anděl & Hrach

Consider an AR(1) process $\{X_t\}$ defined by

$$X_t = \rho X_{t-1} + \varepsilon_t, \qquad t = 1, 2, \ldots, \tag{2}$$

$\rho \in (-1, 1)$, ε_t are iid r.v. with density f. It is well-known that the stationary density h of X_t satisfies an integral equation

$$h(x) = \int_{\mathbb{R}} f(x - \rho u) h(u)\,\mathrm{d}u.$$

Anděl and Hrach [1] suggested to define a sequence

$$h_n(x) = \int_{\mathbb{R}} f(x - \rho u) h_{n-1}(u)\,\mathrm{d}u, \qquad n = 1, 2, \ldots,$$

h_0 is an arbitrary density. They proved that under some quite mild condition on f, the sequence h_n converges to h pointwise. For some specific distributions (exponential, uniform) it was shown that the rate of convergence is geometrical (see [1], [2] and references therein). We showed that the same result holds (and moreover, that the convergence is uniform) for very wide class of distributions – it suffices to assume that f is piecewise absolutely continuous, its derivative is integrable and the distribution has its first moment finite. (Let us call such density *piecewise smooth*.)

This algorithm was generalized to multidimensional AR(1) process ([2]) and onedimensional AR(2) process ([1]).

Haiman's procedure

Consider again an AR(1) process (2). It is obvious that its stationary distribution is the same as that of $\varepsilon_1 + \rho\varepsilon_2 + \rho^2\varepsilon_3 + \cdots$. In [4], Haiman studied asymptotic behaviour of the sequence of partial sums

$$Y_n = \varepsilon_1 + \rho\varepsilon_2 + \cdots + \rho^n \varepsilon_{n+1}.$$

He showed that the density h_n of Y_n converges to the stationary density h uniformly and with geometrical speed under the following conditions: the support of f is compact, f is differentiable everywhere, its derivative is bounded and $\rho > 0$. These assumptions are extremely strong and there are not many distributions (of those widely used) which satisfy them.

However, we relaxed these conditions significantly and showed that Haiman's assertions remain valid if f is piecewise smooth. No assumption on the support of f is needed.

ABSOLUTE AUTOREGRESSION

For the absolute regression model (1), a special procedure for computation of its stationary density was developed. If the density f of ε_t is symmetrical around origin, then the density h of X_t equals

$$h(y) = 2\int_0^\infty g(x) f(y - ax)\,\mathrm{d}x,$$

where g is the density of ξ_t in AR(1) model $\xi_t = a\xi_{t-1} + \varepsilon_t$.

Density g is usually unknown, but we can approximate it using either algorithm of Anděl and Hrach or Haiman's procedure. We proved that if f is piecewise smooth, then in both cases we obtain a sequence of densities which converge to the desired stationary density uniformly and with geometrical speed.

FUTURE RESEARCH

We will continue studying extensions of the algorithm of Anděl and Hrach to autoregressive processes of higher order and higher dimension, in particular we focus on its convergence properties.

We also work on the generalization of Haiman's procedure to model AR(p), $p > 1$.

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