INSENSITIVITY REGION FOR THE VARIANCE COMPONENTS IN MIXED LINEAR MODEL

HANA BOHÁČOVÁ

hana.bohacova@upce.cz
Institute of mathematics, University of Pardubice

SUMMARY

In linear regression models the estimator of variance components needs a suitable choice of a starting point for an iterative procedure for a determination of the estimate. The aim of this paper is to find a criterion for a decision whether a linear regression model enables to determine the estimate reasonably and whether it is possible to do so when using the given data.

MIXED LINEAR MODEL

Let us consider following regression model (according to [4], page 62):

$$\mathbf{Y} \sim N_n\left(\mathbf{X}\boldsymbol{\beta}, \boldsymbol{\Sigma}_{\boldsymbol{\theta}}\right),$$

where $\mathbf{X}$ is a $n \times k$ know design matrix, $r(\mathbf{X}) = k$,
$\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_k)'$ is a vector of unknown fixed effects parameters,

$$\boldsymbol{\Sigma}_{\boldsymbol{\theta}} = \sum_{i=1}^{r} \theta_i \mathbf{V}_i.$$

$\theta_1, \theta_2, \ldots, \theta_r > 0$ are unknown variance components
$\mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_r$ are known symmetrical positive definite matrices,
$\boldsymbol{\Sigma}_{\boldsymbol{\theta}}$ is positive definite.

MAXIMAL LIKELIHOOD ESTIMATORS IN MIXED LINEAR MODEL

The ML estimators of $\boldsymbol{\beta}$ and $\boldsymbol{\theta}$ are

$$\widehat{\boldsymbol{\beta}} = \left(\mathbf{X}'\boldsymbol{\Sigma}_{\boldsymbol{\theta}}{}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\boldsymbol{\Sigma}_{\boldsymbol{\theta}}{}^{-1}\mathbf{Y},$$

$$\widehat{\boldsymbol{\theta}} = \mathbf{S}^{-1}_{(\mathbf{M_X}\boldsymbol{\Sigma}_{\boldsymbol{\theta}}\mathbf{M_X})^+}\begin{pmatrix} \mathbf{Y}'\left(\mathbf{M_X}\boldsymbol{\Sigma}_{\boldsymbol{\theta}}\mathbf{M_X}\right)^+\mathbf{V}_1\left(\mathbf{M_X}\boldsymbol{\Sigma}_{\boldsymbol{\theta}}\mathbf{M_X}\right)^+\mathbf{Y} \\ \vdots \\ \mathbf{Y}'\left(\mathbf{M_X}\boldsymbol{\Sigma}_{\boldsymbol{\theta}}\mathbf{M_X}\right)^+\mathbf{V}_r\left(\mathbf{M_X}\boldsymbol{\Sigma}_{\boldsymbol{\theta}}\mathbf{M_X}\right)^+\mathbf{Y} \end{pmatrix}.$$

Both the above stated estimators are the functions of $\boldsymbol{\theta}$, we have to choose some starting value $\boldsymbol{\theta}_0$ and get the estimates iteratively. We will focus on the estimator of $\boldsymbol{\theta}$ in what follows.

INSENSITIVITY REGION FOR A LINEAR FUNCTION OF $\boldsymbol{\theta}$

The task now is to find a set $\mathcal{N}_{\mathbf{g}'\boldsymbol{\theta},\boldsymbol{\theta}_0}$ with following property (where $\mathbf{g}$ is a given r-dimensional vector which determines a desirable linear combination of the variance components)

$$\boldsymbol{\theta}_0 + \delta\boldsymbol{\theta} \in \mathcal{N}_{\mathbf{g}'\boldsymbol{\theta},\boldsymbol{\theta}_0} \Longrightarrow$$

$$\sqrt{\mathrm{Var}_{\boldsymbol{\theta}_0}\left[\mathbf{g}'\widehat{\boldsymbol{\theta}}(\boldsymbol{\theta}_0 + \delta\boldsymbol{\theta})\right]} \leq (1+\varepsilon)\sqrt{\mathrm{Var}_{\boldsymbol{\theta}_0}\left[\mathbf{g}'\widehat{\boldsymbol{\theta}}(\boldsymbol{\theta}_0)\right]}.$$

Such a set is called an insensitivity region for $\mathbf{g}'\boldsymbol{\theta}$. It can be expressed in this form

$$\mathcal{N}_{\mathbf{g}'\boldsymbol{\theta},\boldsymbol{\theta}_0} = \left\{\boldsymbol{\theta}_0 + \delta\boldsymbol{\theta} : \delta\boldsymbol{\theta}'\mathbf{W}_{\mathbf{g}}\delta\boldsymbol{\theta} \leq 4\varepsilon\mathbf{g}'\mathbf{S}^{-1}_{(\mathbf{M_X}\boldsymbol{\Sigma}_{\boldsymbol{\theta}}\mathbf{M_X})^+}\mathbf{g}\right\}.$$

Following notation is used:

$$\left\{\mathbf{C}_{\mathbf{A},\mathbf{B}}\right\} = \mathrm{Tr}\left(\mathbf{V}_i\mathbf{A}\mathbf{V}_j\mathbf{B}\right),$$

$$\left\{S_{\mathbf{A}}\right\}_{i,j} = \mathrm{Tr}\left(\mathbf{A}\mathbf{V}_i\mathbf{A}\mathbf{V}_j\right),$$

$$\mathbf{a}_k' = \mathbf{g}'\mathbf{S}^{-1}_{\left(\mathbf{M_X}\boldsymbol{\Sigma}_{\boldsymbol{\theta}_0}\mathbf{M_X}\right)^+}$$

$$\times\mathbf{C}_{\left(\mathbf{M_X}\boldsymbol{\Sigma}_{\boldsymbol{\theta}_0}\mathbf{M_X}\right)^+\mathbf{V}_k\left(\mathbf{M_X}\boldsymbol{\Sigma}_{\boldsymbol{\theta}_0}\mathbf{M_X}\right)^+,\left(\mathbf{M_X}\boldsymbol{\Sigma}_{\boldsymbol{\theta}_0}\mathbf{M_X}\right)^+}\mathbf{S}^{-1}_{\left(\mathbf{M_X}\boldsymbol{\Sigma}_{\boldsymbol{\theta}_0}\mathbf{M_X}\right)^+},$$

$$\mathbf{b_g}' = \mathbf{g}'\mathbf{S}^{-1}_{\left(\mathbf{M_X}\boldsymbol{\Sigma}_{\boldsymbol{\theta}_0}\mathbf{M_X}\right)^+},$$

$$\mathbf{V_g} = \sum_{i=1}^{r}\mathbf{b_g}_i\mathbf{V}_i.$$

Matrix $\mathbf{W_g}$ which establishes the quadratic form in the insensitivity region is as follows

$$\mathbf{W_g} = \mathrm{Var}_{\boldsymbol{\theta}_0}\frac{\partial\mathbf{g}'\widehat{\boldsymbol{\theta}}(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}} =$$

$$= 8\begin{pmatrix}\mathbf{a}_1' \\ \vdots \\ \mathbf{a}_r'\end{pmatrix}\mathbf{S}_{(\mathbf{M_X}\boldsymbol{\Sigma}_{\boldsymbol{\theta}_0}\mathbf{M_X})^+}\left(\mathbf{a}_1, \ldots, \mathbf{a}_r\right)$$

$$+8\mathbf{S}_{(\mathbf{M_X}\boldsymbol{\Sigma}_{\boldsymbol{\theta}_0}\mathbf{M_X})^+\mathbf{V_g}(\mathbf{M_X}\boldsymbol{\Sigma}_{\boldsymbol{\theta}_0}\mathbf{M_X})^+}$$

$$-8\begin{pmatrix}\mathbf{a}_1' \\ \vdots \\ \mathbf{a}_r'\end{pmatrix}\mathbf{C}_{(\mathbf{M_X}\boldsymbol{\Sigma}_{\boldsymbol{\theta}_0}\mathbf{M_X})^+,(\mathbf{M_X}\boldsymbol{\Sigma}_{\boldsymbol{\theta}_0}\mathbf{M_X})^+\mathbf{V_g}(\mathbf{M_X}\boldsymbol{\Sigma}_{\boldsymbol{\theta}_0}\mathbf{M_X})^+}$$

$$-8\mathbf{C}'_{(\mathbf{M_X}\boldsymbol{\Sigma}_{\boldsymbol{\theta}_0}\mathbf{M_X})^+,(\mathbf{M_X}\boldsymbol{\Sigma}_{\boldsymbol{\theta}_0}\mathbf{M_X})^+\mathbf{V_g}(\mathbf{M_X}\boldsymbol{\Sigma}_{\boldsymbol{\theta}_0}\mathbf{M_X})^+}\left(\mathbf{a}_1, \ldots, \mathbf{a}_r\right).$$

PROPERTIES OF MATRIX $\mathbf{W_G}$

According to the numerical studies it seems $\mathbf{W}_g$ is a singular matrix. With respect to its quite complicated form it is a problem to prove this in general. The next question is (providing $\mathbf{W}_g$ really is not regular) what is its column space and whether it is orthogonal to the considered starting value $\boldsymbol{\theta}_0$ (as it is in case of the insensitivity region for the linear function of $\boldsymbol{\beta}$).

FUTURE PLANS

The presented work should be understood as the first step in analyzing the insensitivity regions for the linear function of the variance components vector. Many questions still remain without satisfactory answers. Among all the structure and properties of matrix $\mathbf{W_g}$ and properties of the insensitivity regions on their own should be studied in detail.

References.

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