# Modelling with Jump Processes and Optimal Control


## JAKUB PETRÁSEK

petrasek@karlin.mff.cuni.cz

Department of Probability and Mathematical Statistics, Charles University, Prague



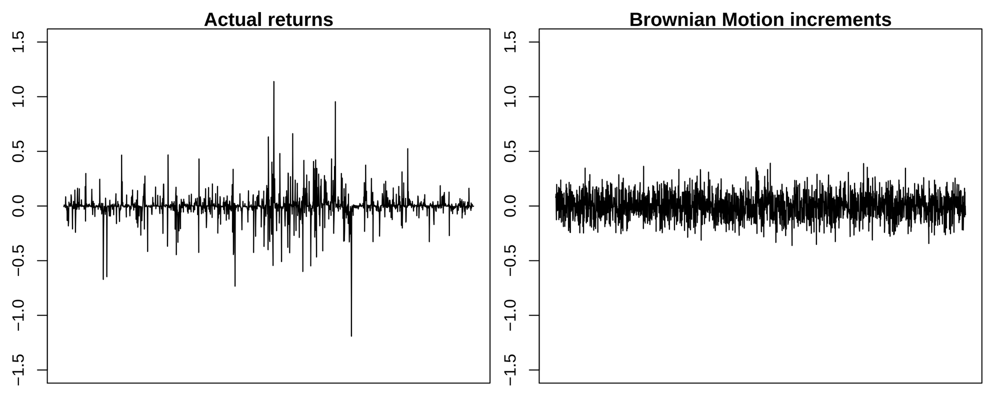
## SUMMARY

**The poster aims at optimal consumption and portfolio choice under jump-diffusion models. Firstly, we argue that continuous models are not satisfactory for modelling high-frequency data and as an alternative we present jump-diffusion models. We propose a practical approach for modelling such a data and calibrate two jump-diffusion models. The main part of the poster deals with optimal control. Formulas for optimal consumption and portofolio choice are presented. An alternative expression for the optimals is proposed and the influence of skewness and kurtosis on optimals is discussed.**


## MOTIVATION

### Empirical facts of financial time series

Long ago statisticians noticed that returns respectively relative returns of financial prices are not necessarily normally distributed. In 1963 a french mathematician Benoìt B. Mandelbrot noticed that the increments of cotton prices showed heavy tails. The common property of processes having heavy-tailed increments is càdlàg trajectory, i.e. path is no longer continuous.




In the left picture 6 seconds shots of future price returns (in ticks), in the right one Brownian motion with the same mean and volatility.

### Possible solutions

1. One can repair classical diffusion models by adding stochastic volatility term, however, this term becomes unrealistically large as the process jumps.
2. Much more plausible way is to allow jumps in the model. Note that it is obviously a generic property for any jump model to exhibit sudden large movements and heavy tails.

### Volatility clustering

We assume that clusters are essentially caused by variable activity at markets. We change the time so that the series is constantly active with respect to the new transformed time. If the activity is high, we slower the time and vice versa.

### Lévy Process

**Definition**

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a filtered probability space. A predictable process $L_t$ is called a *Lévy process* if it is continuous in probability and has stationary, independent increments.

**Decomposition**

Let $L_t$ be a Lévy process. Then $L_t$ has the decomposition

$$L_t = bt + \sigma W_t + \int_{|z|\leq 1} z\tilde{N}(t, \mathrm{d}z) + \int_{|z|>1} zN(t, \mathrm{d}z), \quad 0 \leq t < \infty.$$

where $b \in \mathbb{R}$, $\sigma \geq 0$, $(\tilde{N})$ $N$ is a (compensated) Poisson random measure with a Lévy measure $\nu$.

## MODELLING

### We proceed in three steps

1. Make the series stationary
   - we assume that the non-stationarity is caused by variable intensity of trading,
   - overcome by appropriate time change.
2. Select a model that satisfies the empirical facts (moments, variation, tail behavior),
3. Calibrate the parameters of the model.
   - MLE method used (analytical density assumed).

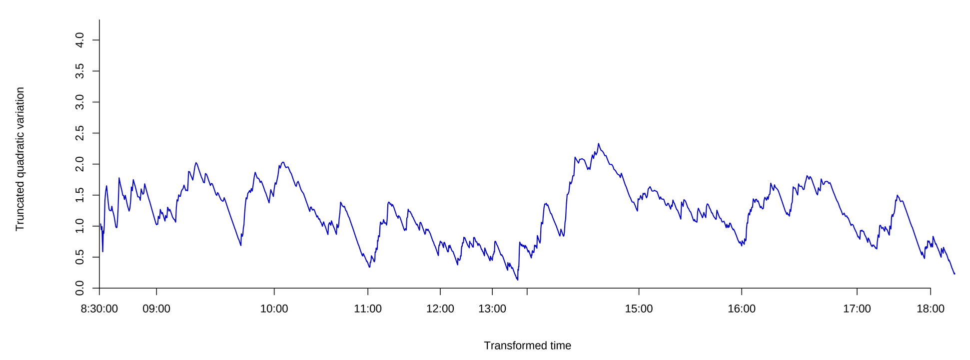
### How to measure volatility?

Let $L$ be a Lévy process, $\Delta_t^n = \{t_0, \ldots, t_n\}$ a partition of $[0, t]$. Then

$$\sum_{\Delta_t^n} (L_{t_i} - L_{t_{i-1}})^2 \xrightarrow{P} \sigma^2 t + \sum_{s\in[0,t]} [\Delta(L_s)]^2, \quad \|\Delta_t^n\| \to 0.$$

Consistent estimator of $\sigma^2$ is truncated Quadratic Variation [Mancini, 2001]

$$\sum_{\Delta_t^n} (L_{t_i} - L_{t_{i-1}})^2 \mathrm{I}_{\left[|L_{t_i} - L_{t_{i-1}}| < \alpha(\Delta_{t_i})^{\bar{\omega}}\right]}, \text{ with } 0 < \bar{\omega} < 1/2.$$




Transformed time by truncated quadratic variation.

### Proposed Models

Merton Jump-Diffusion [Merton, 1976]

$$L_t = \alpha t + \sigma W_t + \sum_{i=1}^{N_t} Y_i, \quad t \geq 0,$$

and Normal Inverse Gaussian [Barndorff-Nielsen, 1998]

Let $L_t = \mu t + B(T_t)$, where

$$T_t = \inf\{s > 0; \beta W_s + \alpha s = \delta t\},$$

and $B_t$ is a Brownian motion with drift $\theta$ and volatility $\sigma$.

The objective is to minimize usual likelihood function $L(\Theta|(\mathcal{Z}, \bar{\mathcal{T}}))$, where $(\mathcal{Z}, \bar{\mathcal{T}})$ are observed data (increment of size $z$ over time period $\Delta\bar{t}$). Let us remark, that estimated models are far superior to classical continuous model. They also show satisfactory fit across the whole spectrum of transformed time intervals $\Delta\bar{t}$.

**?** Having the estimated model we can quantify the additional risk due to jumps. How much should we change our capital invested in stocks if we allow jumps in the model?

## OPTIMAL CONTROL

### Model set-up

An investor invests in two assets

- A riskfree bond that pays interest rate $r$,
- A risky asset with dynamics

$$\mathrm{d}S_t = S(t^-)\left(\alpha \mathrm{d}t + \sigma \mathrm{d}W_t + \int_{-1}^{\infty} z\tilde{N}(\mathrm{d}t, \mathrm{d}z)\right).$$

An investor controls the number of stocks $\Delta_t$ in his portfolio and consumption $C_t \geq 0$ for $t \geq 0$. We further denote $\theta_t = \frac{\Delta_t S_{t^-}}{X_{t^-}}$ the proportion of capital invested in risky asset at time $t$ and $c_t = \frac{C_t}{X_{t^-}}$ the consumption proportion. Then it is easy to see that the portfolio follows stochastic differential equation

$$\mathrm{d}X_t = X_{t^-}\theta_t\left(\alpha \mathrm{d}t + \sigma \mathrm{d}W_t + \int_{-1}^{\infty} z\tilde{N}(\mathrm{d}t, \mathrm{d}z)\right) + rX_t\mathrm{d}t - c_t X_t \mathrm{d}t.$$

with $X(0) = x$, $\theta_t \in \mathcal{F}_{t^-}$, $c_t \in \mathcal{F}_t$.

$$U(x) = \frac{x^{1-p}}{1-p}, \text{ where}$$

$0 < p < 1$ for aggressive investor, $p > 1$ for conservative one.

The objective of an investor is to maximize the disctounted utility of consumption, we call it the value function $v$ and define as

$$\sup_{(\Delta_t, C_t)\in\mathcal{A}(x)} \int_0^{\infty} e^{-\beta t}\mathrm{E}\, U(C_t)\mathrm{d}t,$$

where $\mathcal{A}(x)$ is the set of admissible strategies, $\beta$ is a discount factor.

The following theorem gives the formulas for optimals. The theorem was presented in [Framstad et al., 1998] for aggressive investor and for processes with finite activity.

### Theorem (Optimal Consumption and Portfolio Choice)

Assume the model set-up and the objective

$$\begin{aligned}\theta_p^* &= \operatorname*{argmax}_{\theta_p} h(\theta_p) \quad \text{for} \quad 0 < p < 1,\\ &= \operatorname*{argmin}_{\theta_p} h(\theta_p) \quad \text{for} \quad p > 1,\end{aligned}$$

where

$$h(\theta_p) = \left\{(\alpha - r)\theta_p(1-p) - \frac{1}{2}\sigma^2\theta_p^2 p(1-p) + \int_{-1}^{\infty}\left((1+\theta_p z)^{1-p} - 1 - \theta_p z(1-p)\right)\nu(\mathrm{d}z)\right\}.$$

If $\beta - r(1-p) - h(\theta_p^*) > 0$ (finiteness of the value function) then $\theta_p^*$ is the optimal proportion, $c^* = (K(1-p))^{-1/p}$ is the optimal consumption and $v(z) = Kz^{1-p}$ is the value function, where $K = \frac{1}{1-p}\left(\frac{\beta - r(1-p) - h(\theta_p^*)}{p}\right)^{-p}$.

To see explicitely the influence of skewness and kurtosis on the optimal proportion, the previous theorem can be rewritten.

### Proposition(Optimal portfolio as a function of cumulants)

Let

$$\int_{-1}^{\infty}\sum_{k=2}^{\infty}\left|\binom{1-p}{k}\theta_p^k z^k\right|\mathrm{d}z < \infty.$$

Then for $p > 0$

$$\begin{aligned}\theta_p^* &= \operatorname*{argmax}_{\theta_p}\left\{(\alpha - r)\theta_p(1-p) + \binom{1-p}{2}\sigma_J^2\theta_p^2 + \sum_{k=3}^{\infty}\binom{1-p}{k}\theta_p^k\kappa_k\right\}, p < 1\\ &= \operatorname*{argmin}_{\theta_p}\left\{(\alpha - r)\theta_p(1-p) + \binom{1-p}{2}\sigma_J^2\theta_p^2 + \sum_{k=3}^{\infty}\binom{1-p}{k}\theta_p^k\kappa_k\right\}, p > 1,\end{aligned}$$

where

$$\sigma_J^2 = \sigma^2 + \int_{-1}^{\infty} z^2\nu(\mathrm{d}z),$$

$\sigma^2$ is the volatility of the diffusion part, and $\kappa_k$ is the k-th cumulant.

We conclude that for both conservative and aggressive investors as skewness grows optimal proportion $\theta_p^*$ ascends. While as kurtosis grows $\theta_p^*$ descends. To sum up, an agent should not only be aware of heavy-tails but also of skewness. This fact becomes important when investing into stocks, whose skewness is observed to be negative. We may ask if it is possible to construct a jump process (heavy-tailed distribution) with positive skewness so that the optimal investment proportion is greater than the Merton proportion $\theta_p^* > \theta_p^{*M}$. Note that it is not obvious, since greater skewness also implies greater kurtosis?

### Example

Consider the following game:

- win $\frac{N}{2N-2}$ with probability $\frac{2}{N+1}$,
- lose $\frac{1}{2N-2}$ with probability $\frac{N-1}{N+1}$.

It holds that

$$\mathrm{E}\, Game = \mathrm{var}\, Game = \frac{1}{2N-2},$$

i.e. $\theta_p^{*M} = \frac{1}{p}$.

| $\theta_p^*$ | $N=3$ | $N=10$ | $N=100$ | $N=1000$ | $\theta_p^{*M}$ |
|---|---|---|---|---|---|
| $p=4$ | 0.293 | 0.355 | 0.376 | 0.378 | 0.250 |
| $p=10$ | 0.113 | 0.135 | 0.143 | 0.144 | 0.100 |
| $p=40$ | 0.028 | 0.033 | 0.035 | 0.035 | 0.025 |
| $p=70$ | 0.016 | 0.019 | 0.020 | 0.020 | 0.014 |
| skewness | 0.000 | 1.650 | 6.893 | 22.305 | |

## CONCLUSION

Financial series exhibit heavy tailed distribution, and these can be naturally modelled by jump processes. Jumps have a nontrivial effect on the undertaking risk. Using alternative expression for optimal proportion we conclude that

- positive skewness motivates the investor to invest more,
- large kurtosis forces the investor to behave more conservatively.

It is not necessarily true that heavy-tails escalate the risk, see the example.

In this work we have presented the theoretical influence of skewness and kurtosis on the optimal proportion, however, numerical study still remains to be performed.

**Acknowledgement.** Author would like to thank to Mgr. Karel Janeček Ph.D., MBA for his generous help and continuous encouragement and to the company RSJ Invest a.s. for the supply of data. I would also like to thank to Faculty of Mathematics and Physics, Charles University, for financial support for the presence at conference ROBUST 2010.

## References


O. E. Barndorff-Nielsen. Processes of Normal Inverse Gaussian Type. *Finance and Stochastic*, 1998.

R. Cont and P. Tankov. *Financial modelling with jump processes.* Chapman & Hall/CRC Financial Mathematics Series., 2004.

N. C. Framstad, B. Øksendal, and A. Sulem. Optimal consumption and portfolio in a jump diffusion market. *In A. Shyriaev et al (eds): Workshop on Mathematical Finance*, 1998.

C. Mancini. Disentangling the jumps of the diffusion in a geometric jumping Brownian motion. *Giornale dellÍnstituto Italiano degli Attuari.*, 2001.

R. C. Merton. Option pricing when underlying stock returns are discontinuous. *Journal Financial Economics*, 1976.

B. Øksendal and A. Sulem. *Applied stochastic control of jump diffusions. 2nd ed.* Universitext. Berlin: Springer., 2007.

S.T. Rachev, editor. *Handbook of Heavy Tailed Distributions in Finance.* Elsevier Science B.V., 2003.