Conditional and partial copulas and measures of associations

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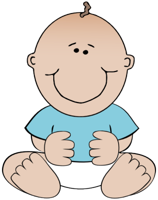
Outline

Life expectancies of males and females

CIA World Factbook contains various demographic and economic characteristics of countries.

We are interested in the relationship of the **life expectancies** of males (Y_1) and females (Y_2) across countries.



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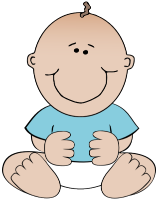

We are interested in the relationship of the **life expectancies** of males (Y_1) and females (Y_2) across countries.

Question: How does this relationship change when GDP per capita (X) is taken into consideration?

Measures of dependence (concordance)

Pearson correlation coefficient – $\rho(Y_1, Y_2) = \frac{\operatorname{cov}(Y_1, Y_2)}{\sqrt{\operatorname{var}(Y_1)}\sqrt{\operatorname{var}(Y_2)}}$.

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Kendall's tau –

$$\tau(Y_1, Y_2) = 2\,\mathbb{P}\big[(Y_1 - Y_1')(Y_2 - Y_2') > 0\big] - 1,$$

where (Y_1', Y_2') is an independent copy of (Y_1, Y_2).

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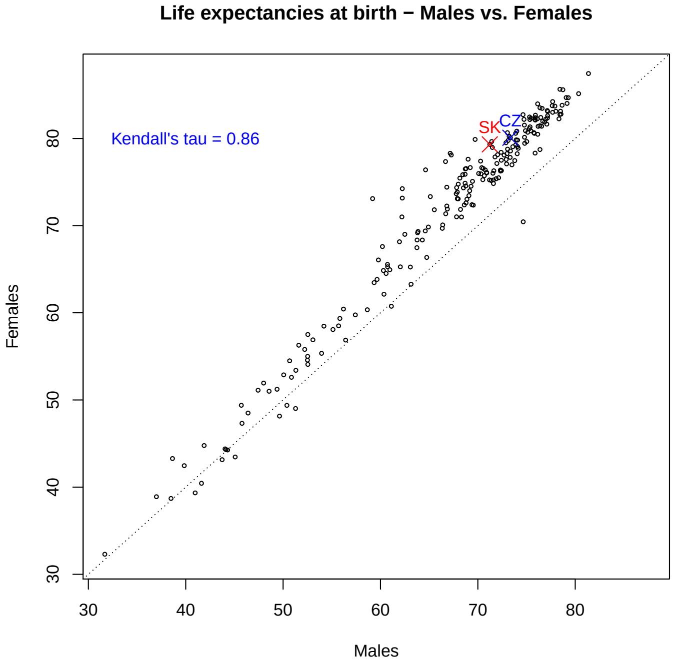
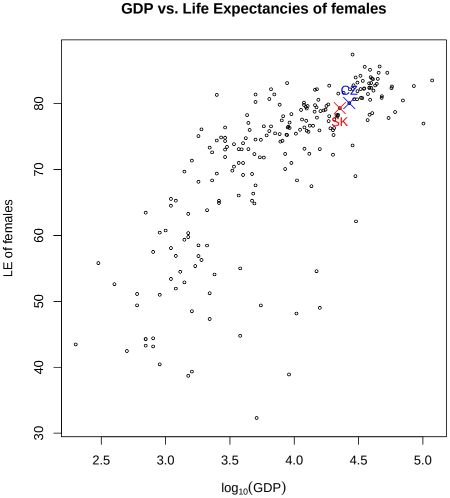
Other measures - Spearman's rho,
Blomqvist beta

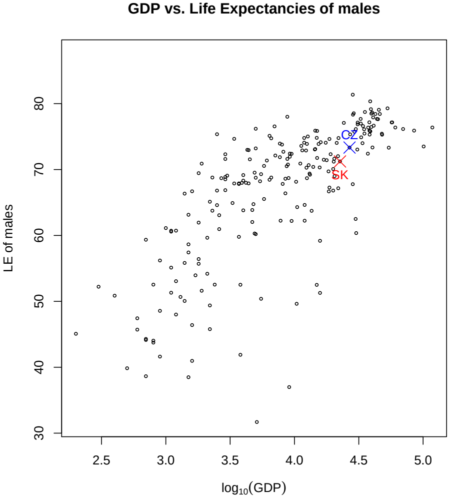
$$\beta = \mathbb{P}\left[\left(Y_1 - F_1^{-1}(0.5)\right)\left(Y_2 - F_2^{-1}(0.5)\right) > 0\right] - \mathbb{P}\left[\left(Y_1 - F_1^{-1}(0.5)\right)\left(Y_2 - F_2^{-1}(0.5)\right) < 0\right],$$

. . .


Life expectancies at birth − Males vs. Females
Kendall's tau = 0.86
SK
CZ
Females
Males
30
40
50
60
70
80

Life expectancies vs. GDP





Partial vs. conditional correlation coefficients

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Pearson partial correlation coefficient
Estimation: First compute 'the residuals'

$$\hat{\varepsilon}_{1i} = Y_{1i} - \hat{a}_1 - \hat{b}_1 X_i, \qquad \hat{\varepsilon}_{2i} = Y_{2i} - \hat{a}_2 - \hat{b}_2 X_i$$

where $(\hat{a}_1, \hat{b}_1)$ are the LS estimates of the parameters of the linear model $Y_1 = a_1 + b_1 X + \varepsilon_1$ and similarly for $(\hat{a}_2, \hat{b}_2)$.

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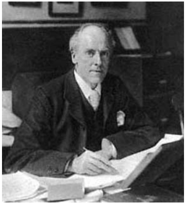
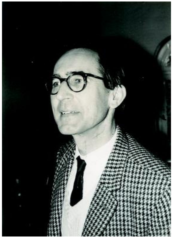
Then

$$\hat{\rho}\big((Y_1, Y_2) \,|\, X\big) := \hat{\rho}(\hat{\varepsilon}_1, \hat{\varepsilon}_2).$$

Population version of Pearson partial correlation coefficient may be also computed as



$$\rho((Y_1, Y_2)|X) = \frac{\rho(Y_1, Y_2) - \rho(Y_1, X)\,\rho(Y_2, X)}{\sqrt{1-\rho^2(Y_1, X)}\,\sqrt{1-\rho^2(Y_2, X)}}.$$



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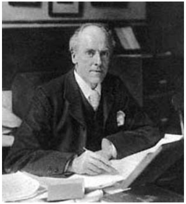

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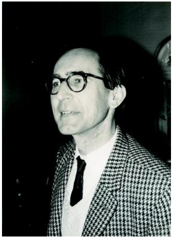
Kendall's partial tau



$$\tau((Y_1, Y_2)|X) = \frac{\tau(Y_1, Y_2) - \tau(Y_1, X)\,\tau(Y_2, X)}{\sqrt{1-\tau^2(Y_1, X)}\,\sqrt{1-\tau^2(Y_2, X)}}.$$

Kendall, M. G. (1942). Partial rank correlation. *Biometrika*, 32:277–283.

Conditional correlation coefficients

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Conditional correlation coefficient measures correlation of Y_1 and Y_2 when $X = x$.

If the conditional dependence structure of (Y_1, Y_2) given $X = x$ does not change with x, then the concepts of partial and conditional dependence coincide.

Estimating conditional measures of dependence

We observe independent triples $\{(Y_{1i}, Y_{2i}, X_i)\}_{i=1}^{n}$.

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Refinement: The pair (Y_{1i}, Y_{2i}) is given a weight

$$w_{ni}(x, h_n) = \frac{K\left(\frac{X_i - x}{h_n}\right)}{\sum_{j=1}^n K\left(\frac{X_j - x}{h_n}\right)}, \qquad i = 1, \ldots, n,$$

where K is a given kernel, e.g. the Epanechnikov kernel

$$K(x) = \tfrac{3}{4}\,(1 - x^2)\,\mathbb{I}\{|x| \leq 1\},$$

and h_n is the width of a 'smoothing window'.

Conditional Kendall's tau

(Standard) **Kendall's tau**

$$\hat{\tau}(Y_1, Y_2) = \frac{4}{n(n-1)} \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \mathbb{I}\{Y_{1i} < Y_{1j}, Y_{2i} < Y_{2j}\} - 1.$$

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Conditional Kendall's tau: $\hat{\tau}(x) := \hat{\tau}((Y_1, Y_2)|X = x)$

$$\hat{\tau}(x) = \frac{4}{A_n(x,h)} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ni}(x, h_n)\, w_{nj}(x, h_n)\, \mathbb{I}\{Y_{1i} < Y_{1j}, Y_{2i} < Y_{2j}\} - 1,$$

where $A_n(x,h) = 1 - \sum_{i=1}^{n} w_{ni}^2(x, h_n)$.

Which quantity are we estimating and where do copulas come in?

What is a copula?

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According to *Sklar's theorem* (see e.g. Nelsen (2006)) there exists a bivariate function C such that

$$P(Y_1 \leq y_1, Y_2 \leq y_2) = H(y_1, y_2) = C(F_1(y_1), F_2(y_2)).$$

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The function C is called a **copula** and it completely describes the dependence of (Y_1, Y_2). It is in itself a joint cumulative distribution function on $[0, 1]^2$ with uniform marginals.

Conditional copula

Suppose that in fact we observe a three-dimensional vector (Y_1, Y_2, X) and we are interested in the *conditional dependence structure* of (Y_1, Y_2) for a given value of $X = x$.

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be the *conditional joint distribution function* with F_{1x} and F_{2x} being the corresponding marginals of H_x.

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$$H_x(y_1, y_2) = C_x(F_{1x}(y_1), F_{2x}(y_2)).$$

The function C_x is called the **conditional copula function** (Patton, 2006).

Patton, J. A. (2006). Modeling asymmetric exchange rate dependence. *International Economic Review*, 47(2):527–556.

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We suggest that the **partial copula function** is called the copula corresponding to (U_1, U_2), i.e.

$$\bar{\mathrm{C}}(u_1, u_2) \quad = \quad \mathbb{P}(U_1 \leq u_1, U_2 \leq u_2)$$

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$$\begin{aligned}\bar{\mathrm{C}}(u_1, u_2) &= \mathbb{P}(U_1 \le u_1, U_2 \le u_2) \\ &= \int \mathbb{P}(U_1 \le u_1,\, U_2 \le u_2 \,|\, X = x) f_X(x)\, dx \\ &= \int C_x(u_1, u_2) f_X(x)\, dx.\end{aligned}$$

That is why some researchers call $\bar{\mathrm{C}}$ an *average copula*.

Simplified pair-copula construction

Recall that

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Often, it is assumed that C_x does not change with x, which implies

$$\bar{\mathrm{C}} = C_x, \qquad \forall x \in \mathrm{supp}(X).$$

Acar, E., Genest, C., and Nešlehová, J. (2012). Beyond simplified pair-copula constructions. *J. Multivariate Anal.* Available online.

Hobæk Haff, I., Aas, K., and Frigessi, A. (2010). On the simplified pair-copula construction–simply useful or too simplistic? *J. Multivariate Anal.*, 101(5):1296–1310.

Different versions of Kendall's tau

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Conditional Kendall's tau of (Y_1, Y_2) given $X = x$

$$\tau(x) = 4 \iint_{[0,1]^2} C_x(u_1, u_2)\, dC_x(u_1, u_2) - 1,$$

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Partial Kendall's tau of $(U_1, U_2) = (F_{1X}(Y_1), F_{2X}(Y_2))$

$$\bar{\tau} = 4 \iint_{[0,1]^2} \bar{\mathrm{C}}(u_1, u_2)\, d\bar{\mathrm{C}}(u_1, u_2) - 1,$$

where $\bar{\mathrm{C}}$ is the copula associated with (U_1, U_2).

Conditional copula estimation – overview

- **Parametric approach** – the conditional copula C_x belongs to a given parametric family whose parameters depend on the covariate through a known parametric function – Patton (2006), time-series literature, ...

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- **Nonparametric approach** – OGV.

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The **conditional copula** C_x may be expressed as

$$C_x(u_1, u_2) = H_x\left(F_{1x}^{-1}(u_1), F_{2x}^{-1}(u_2)\right).$$

Empirical conditional copula

A straightforward estimator of the conditional copula is

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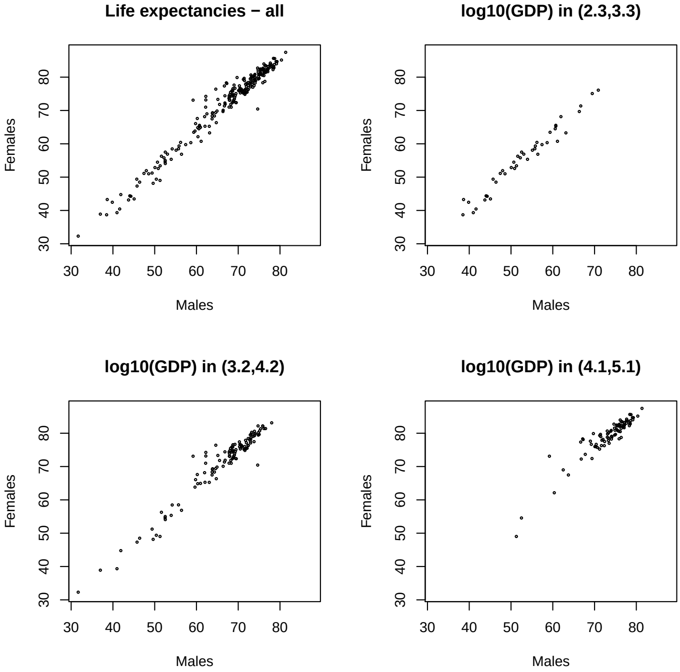
Life expectancies − all
log10(GDP) in (2.3,3.3)
log10(GDP) in (3.2,4.2)
log10(GDP) in (4.1,5.1)
Females
Males
30
40
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Conclusion: Before calculation of conditional measures of dependence, first remove the effect of the covariate on the marginal distributions.

A general transformation of the marginals

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- the margins of (U_{1i}, U_{2i}) are uniform;
- the conditional copula is the same for original (Y_{1i}, Y_{2i}) and transformed (U_{1i}, U_{2i}).

A general transformation of the marginals - practice

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To be fully 'nonparametric' we use nonparametric estimators

$$
\begin{aligned}
\widehat{F}_{1x_i g_1}(y_1) &= \sum_{j=1}^{n} w_{nj}(x_i, g_{1n}) \, \mathbb{I}\{Y_{1i} \leq y_1\}, \\
\widehat{F}_{2x_i g_2}(y_2) &= \sum_{j=1}^{n} w_{nj}(x_i, g_{2n}) \, \mathbb{I}\{Y_{2i} \leq y_2\},
\end{aligned}
$$

with g_{n1}, g_{n2} being sequences of bandwidths going to zero.

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Under appropriate regularity assumptions

$$\sup_{u_1,u_2} \left| \sqrt{n\,h_n} \left(\widetilde{C}_x(u_1,u_2) - \widetilde{C}_x^{(or)}(u_1,u_2) \right) \right| = o_P(1)$$

where $\widetilde{C}_x$ is the estimator of the conditional copula based on $(\widetilde{U}_{1i}, \widetilde{U}_{2i})$ and $\widetilde{C}_x^{(or)}$ is an ‘oracle estimator’ based on unobserved (U_{1i}, U_{2i}).

Partial copula estimation

Recall that

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Then the 'oracle' estimator of $\bar{\mathrm{C}}$ would be an *empirical copula* estimator given by

$$C_n^{(or)}(u_1, u_2) = G_n\left(G_{1n}^{-1}(u_1), G_{2n}^{-1}(u_2)\right), \tag{1}$$

where

$$G_n(u_1, u_2) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{I}\{U_{1i} \le u_1, U_{2i} \le u_2\}, \tag{2}$$

and G_{1n} and G_{2n} are its corresponding marginals.

Partial copula estimation

Let $(\widetilde{U}_{1i}, \widetilde{U}_{2i})$ stand for (Y_{1i}, Y_{2i}) adjusted for the effect of X_i, e.g.

$$(\widetilde{U}_{1i}, \widetilde{U}_{2i}) = \left(\widehat{F}_{1x_ig_1}(Y_{1i}), \widehat{F}_{2x_ig_2}(Y_{2i})\right).$$

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Then we aim at the the following result:

$$\sup_{u_1,u_2} \left| \sqrt{n} \left(\widetilde{C}_n(u_1, u_2) - C_n^{(or)}(u_1, u_2) \right) \right| = o_P(1).$$

Life expectancies at birth

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$$\hat{\hat{\tau}} = \frac{4}{n(n-1)} \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbb{I}\{\widetilde{U}_{1i} < \widetilde{U}_{1j}, \widetilde{U}_{2i} < \widetilde{U}_{2j}\} - 1.$$

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$$\hat{\tau}(x) = \frac{4}{A_n(x,h)} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ni}(x,h_n)\, w_{nj}(x,h_n)\, \mathbb{I}\{\widetilde{U}_{1i} < \widetilde{U}_{1j}, \widetilde{U}_{2i} < \widetilde{U}_{2j}\} - 1,$$

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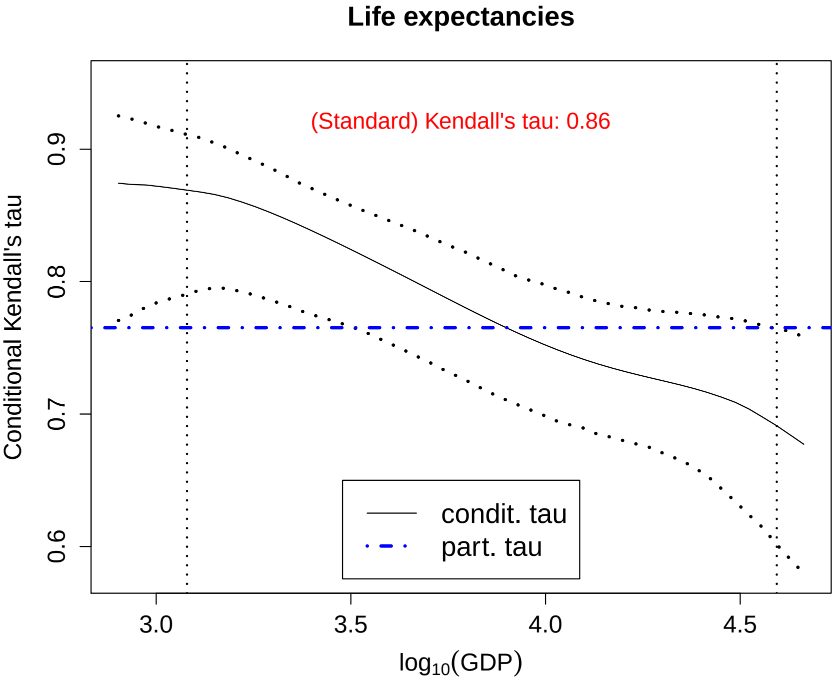
Life expectancies
(Standard) Kendall's tau: 0.86
Conditional Kendall's tau
0.6
0.7
0.8
0.9
3.0
3.5
4.0
4.5
log10(GDP)
condit. tau
part. tau

Uranium dataset

The dataset consists of the observed log-concentrations of seven chemicals in 655 water samples collected near Grand Junction, Colorado. The data can be found e.g. as a data set called `uranium` in the R-package `copula`.

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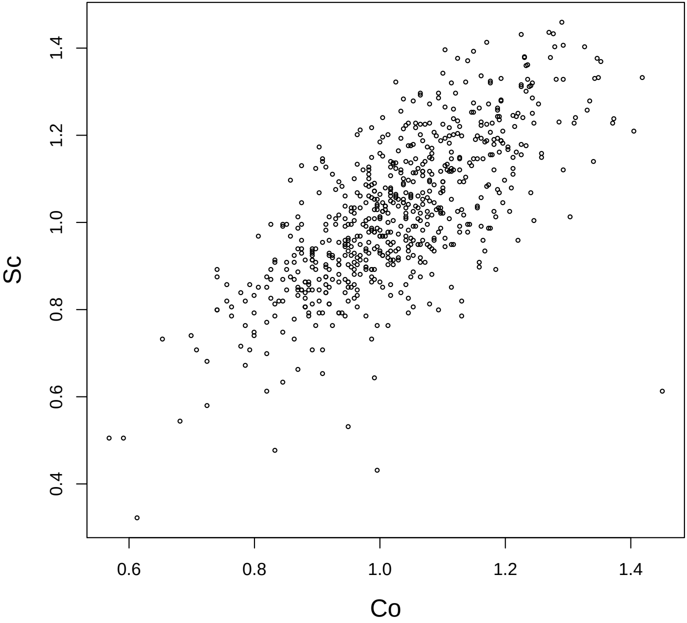
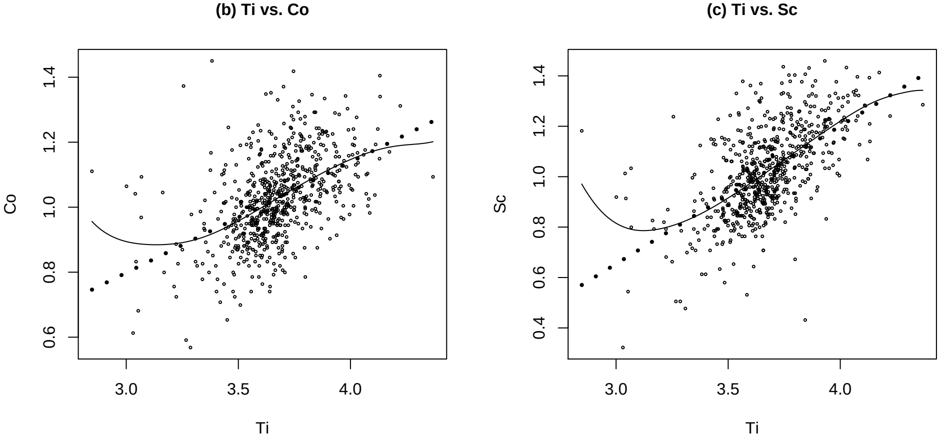
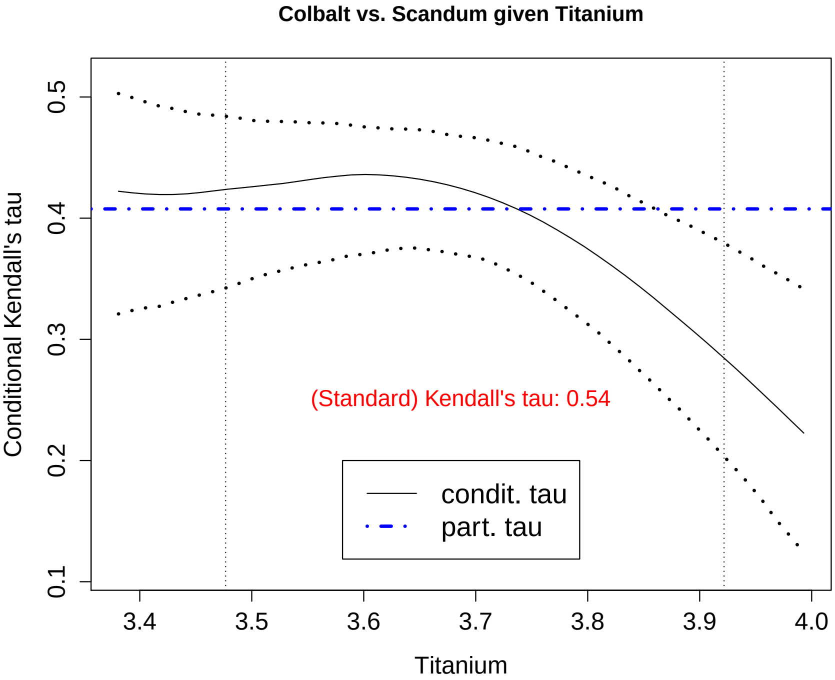
Let concentrate on cobalt (Co), scandium (Sc) and titanium (Ti).

(a) Co vs. Sc




(b) Ti vs. Co
Co
Ti
0.6
0.8
1.0
1.2
1.4
3.0
3.5
4.0
(c) Ti vs. Sc
Sc
Ti
0.4
0.6
0.8
1.0
1.2
1.4
3.0
3.5
4.0


Colbalt vs. Scandum given Titanium
Conditional Kendall's tau
0.1
0.2
0.3
0.4
0.5
3.4
3.5
3.6
3.7
3.8
3.9
4.0
Titanium
(Standard) Kendall's tau: 0.54
condit. tau
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- Conditional/partial measures of correlation give us a deeper insight into the dependence structure of (Y_1, Y_2) when a covariate X is present.

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- We suggested a bootstrap method for the estimator of the conditional copula.
- We work at this moment on asymptotic properties of the nonparametric estimator of a partial copula.

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So far we consider X to be one-dimensional covariate, i.e. $X \in \mathbb{R}$.

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Multivariate covariate – $\mathbf{X} \in \mathbb{R}^d$

$$w_{ni}(\mathbf{x}, \mathbb{H}_n) = \frac{K_{\mathbb{H}_n}(\mathbf{X}_i - \mathbf{x})}{\sum_{j=1}^{n} K_{\mathbb{H}_n}(\mathbf{X}_j - \mathbf{x})}, \quad i = 1, \ldots, n,$$

where K is a d-variate kernel, $\mathbb{H}_n$ is a bandwidth matrix with the determinant $|\mathbb{H}_n|$ and $K_{\mathbb{H}_n}(\mathbf{y}) = K(|\mathbb{H}_n|^{-1/2}\mathbf{y})$.

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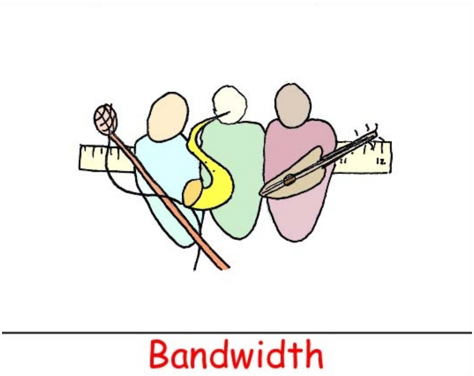
Functional covariate – $\mathcal{X} \in \mathcal{E}$

$$w_{ni}(\chi, h_n) = \frac{K\left(\frac{\|\mathcal{X}_i - \chi\|}{h_n}\right)}{\sum_{j=1}^{n} K\left(\frac{\|\mathcal{X}_j - \chi\|}{h_n}\right)}, \quad i = 1, \ldots, n,$$

where $\|\cdot\|$ stands for a norm on $\mathcal{E}$ and K is a given (univariate) kernel.

Further research

- The bandwidth selection problem. Plug-in bandwidths? Cross-validation procedure?



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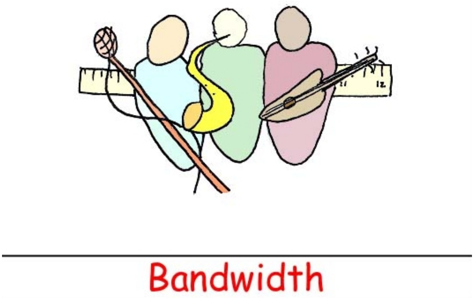

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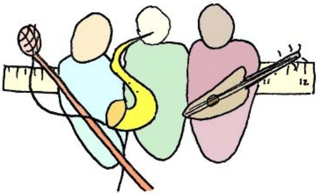
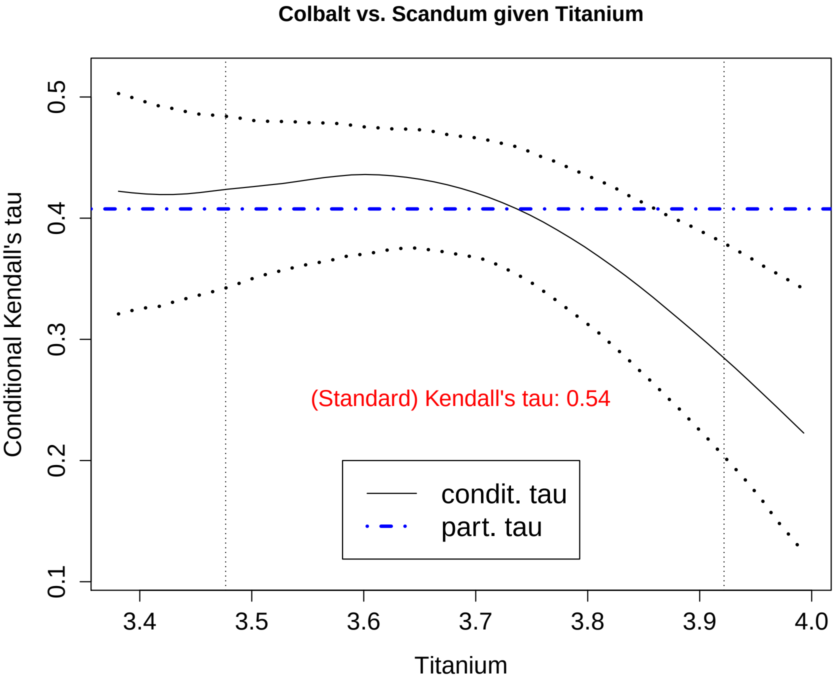


Bandwidth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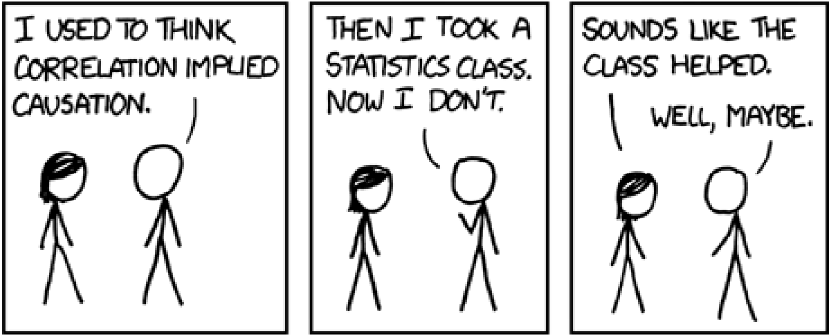

Děkuji za pozornost!