SEKVENČNÍ TESTOVÁNÍ STABILITY VE FUNKCIONÁLNÍM MODELU CAPM

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Sequential testing

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Introduction

joint work with A.Aue, S.Hörmann, L.Horváth, J.Steinebach

- Sequential testing
- Functional data
- Dependent observations
- Particular linear model

Capital asset pricing (CAPM) model:

$$\mathbf{r}_k(t) = (\mathbf{1} - \boldsymbol{\beta}_k)\gamma + \boldsymbol{\beta}_k r_{M,k}(t) + \mathbf{e}_k(t), \; k = 1, 2, \dots, \; t \in (0,1)$$

$t \in (0,1)$ (usually one trading day)

$\mathbf{r}_k$ — vector of daily log-returns – vector of functions on an interval $(0,1)$-corresponding to d risky assets (logaritmická míra výnosu d akcií)

$r_{M,k}$ —log-return of observable market portfolio (logaritmická míra výnosu tržního portfolia akcií)

γ – return on a risk free assets, scalar (unknown parameter)(míra výnosu bezrizikových akcií)

$\boldsymbol{\beta}_k$ - d-dimensional unknown vector

original CAPM models considered by Sharpe (1964), Lintner (1965), Merton ((1973) etc. (not for functional data)

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Here: functional data, weak dependence, sequential setup

k-th observations:

$(\mathbf{r}_k(t_j), r_{M,k}(t_j))$, $0 < t_1 < \ldots t_J < 1$ -k-th observation ($(d+1)J$ dimensional random vectors)

$k = 1, 2, \ldots,$ J large

$\boldsymbol{\beta}_k, \gamma$-unknown parameters (finite dimensional)

Training data of size m with no change in parameters are assumed to be available

Sequential setup: portfolio manager has to decide on-line whether to hold to to sell assets in his portfolio

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Testing problem and test procedures

$H_0 : \boldsymbol{\beta}_1 = \ldots = \boldsymbol{\beta}_k = \ldots$

against a change in $\boldsymbol{\beta}$ after $m + k^*$ observations

m - size of training data

k^*– change point, structural break point (unknown)

to detect possible instability

Procedure based on a functionals of the difference of LSE of $\boldsymbol{\beta}$ based on the training data and based on " the new data"

LSE $\widehat{\boldsymbol{\beta}}_{\ell,\ell+k}$ of $\boldsymbol{\beta}$ based on observations $\ell+1,\dots,\ell+k$

$$\widehat{\boldsymbol{\beta}}_{\ell,\ell+k} = \Big(\sum_{i=\ell+1}^{\ell+k}\sum_{j=1}^{J}(r_{M,i}(s_j) - \bar{r}_{M,\ell,k}(s_j))^2\Big)^{-1}$$

$$\times \sum_{i=\ell+1}^{\ell+k}\sum_{j=1}^{J}(r_{M,i}(s_j) - \bar{r}_{M,\ell,k}(s_j)(\mathbf{r}_i(s_j) - \bar{\mathbf{r}}_m(s_j))$$

$$\bar{r}_{M,\ell,k}(s_j) = \frac{1}{k}\sum_{i=\ell+1}^{\ell+k} r_{M,i}(s_j), \quad , \quad \bar{\mathbf{r}}_{\ell,k}(s_j) = \frac{1}{k}\sum_{i=\ell+1}^{\ell+k}\mathbf{r}_i(s_j)$$

Natural test procedures based on

$$V_k = \Big(\widehat{\boldsymbol{\beta}}_{m,m+k} - \widehat{\boldsymbol{\beta}}_{0,m}\Big)^T \widehat{\mathbf{Q}}_m^{-1}\Big(\widehat{\boldsymbol{\beta}}_{m,m+k} - \widehat{\boldsymbol{\beta}}_{0,m}\Big)$$

$\widehat{\mathbf{Q}}_m$ – suitable estimator of the respective variance matrix

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Equivalently:

$$V_k = \mathbf{R}_k^T(\widehat{\mathbf{D}}_m)^{-1}\mathbf{R}_k$$

$$R_k = \frac{1}{J}\sum_{i=m+1}^{m+k}\sum_{j=1}^{J}(r_{M,i}(s_j) - \overline{r}_{M,m,k}(s_j)(\mathbf{r}_i(s_j) - \overline{\mathbf{r}}_m(s_j))$$

$$-\frac{U_{m,m+k}}{U_m}\sum_{i=m+1}^{m+k}\sum_{j=1}^{J}(r_{M,i}(s_j) - \overline{r}_{M,\ell,k}(s_j)(\mathbf{r}_i(s_j) - \overline{\mathbf{r}}_m(s_j))$$

$$U_m = \frac{1}{J}\sum_{i=1}^{m}\sum_{j=1}^{J}(r_{M,i}(s_j) - \overline{r}_{M,m}(s_j))^2$$

$\widehat{\mathbf{D}}_m$– a suitable standardization matrix

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Stopping rule

$$\tau_m(T) = \min\left\{k \le mT,\ V_k > cw(k/m)\right\}$$

$$\tau_m(T) = \infty \quad \textit{if} \quad V_k \le cw(k/m) \quad k \le mT$$

c– suitably chosen constant

$w(t)$– positive weight function, boundary function

T -typically large, max number of possible observations is $m(T+1)$

We wish: to test with asymptotic level α and consistency, i.e.,

$$\lim_{m\to\infty} P_{H_0}(\tau_m(T) < \infty) = \alpha,$$

$$\lim_{m\to\infty} P_{H_1}(\tau_m(T) < \infty) = 1.$$

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Theoretical results

Assumptions:

Ass. 1: For any $i \in \mathbb{Z}$, $r_{M,i}(s) = h(\boldsymbol{\xi}_i(s), \boldsymbol{\xi}_{i-1}(s), \ldots)$, $s \in [0,1]$, $h(\cdot)$ is a measurable real valued functional , $\{\boldsymbol{\xi}_i\}_i$ is sequence of i.i.d. random functions.

Ass. 2: For any $i \in \mathbb{Z}$, $\boldsymbol{\varepsilon}_i = \mathbf{g}(\boldsymbol{\zeta}_i, \boldsymbol{\zeta}_{i-1}, \ldots)$, where $\mathbf{g}(\cdot)$ is a measurable d-dimensional functional, $\{\boldsymbol{\zeta}_i\}_i$ is sequence of i.i.d. random functions $E\boldsymbol{\varepsilon}_i = 0$

Ass.3: $\{\boldsymbol{\xi}_i\}_i$ and $\{\boldsymbol{\zeta}_i\}_i$ are independent.

Ass. 4:

$$\|\sup_{s\in[0,1]} |r_{M,i}(s)|\|_4 < \infty, \qquad \max_{1\le j\le d} \|\sup_{s\in[0,1]} |\varepsilon_{i,\ell}(s)|\|_4 < \infty,$$

where $||V||_q = (E|V|^q)^{1/q}$, additionally some properties on a kind of continuity.

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Ass.5: $\lim_{h\to 0}(||\omega(r_{M,i},h)||_4 + ||\omega(\varepsilon_{i,\ell},h)||_4) = 0$

$1 \le \ell \le d$, $i \in \mathbb{Z}$– smoothness

$\omega(x;h) = \sup_{0\le t\le 1-h} \sup_{0\le s\le h} |x(t+h) - x(t)|$

Ass. 6: For any $i \in \mathbb{Z}$

$$\sum_{L=1}^{\infty} ||r_{iM} - r_{iM}^{(L)}||_4 < \infty$$

where

$$r_{iM}^{(L)} = h(\boldsymbol{\xi}_i, \boldsymbol{\xi}_{i-1}, \dots \boldsymbol{\xi}_{i-L+1}, \boldsymbol{\xi}_{i-L}^{(L)}, \boldsymbol{\xi}_{i-L-1}^{(L)}, \dots),$$

with

$$\boldsymbol{\xi}_{i-L}^{(L)}, \boldsymbol{\xi}_{i-L-1}^{(L)}, \dots$$

being i.i.d. with the same distribution as $\boldsymbol{\xi}_i$ and independent of $\{\boldsymbol{\xi}_i\}_i$

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Ass. 7: For any $i \in \mathbb{Z}$, $j = 1, \dots, d$

$$\sum_{L=1}^{\infty} ||\varepsilon_i - \varepsilon_i^{(L)}||_4 < \infty$$

where

$$\varepsilon_i^{(L)} = \mathbf{g}(\zeta_i, \zeta_{i-1}, \dots \zeta_{i-L+1}, \zeta_{i-L}^{(L)}, \zeta_{i-L-1}^{(L)}, \dots)$$

with

$$\zeta_{i-L}^{(L)}, \zeta_{i-L-1}^{(L)}, \dots$$

being i.i.d.with the same distribution as ζ_i and independent of $\{\zeta_i\}_i$.

Ass. 8. : $J = J_m \to \infty$, as $m \to \infty$

Ass. 9. : weight function w is positive continuous on $[0, 1]$.

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Then it can be proved:

$$\frac{1}{J}\sum_{j=1}^{J}(r_{M,i}(s_j) - \overline{r}_{M,\ell,k}(s_j)\varepsilon_i(s_j)$$

has approximately distribution as

$$\mathbf{z}_i = \int_0^1 (r_{M,i}(s) - Er_{M,i}(s))\varepsilon_i(s)ds$$

with dependence structure

$$\mathbf{D} = E(\mathbf{z}_0\mathbf{z}_0^T) + \sum_{i=1}^{\infty} E(\mathbf{z}_0\mathbf{z}_i^T) + \mathbf{z}_i\mathbf{z}_0^T)$$

It can be shown also that under no change

$$\max_{1 \le k \le mT} \{V_k / w(k/m)\}$$

behaves approximately as

$$\sup_{0 < t \le T} \frac{\sum_{j=1}^{d} W_j^2(t)}{w(t)}$$

$\{W_j(t),\ t \in (0, \infty)\}$ are independent Gaussian processes with zero mean and $var(W_j(t_1), W_j(t_2)) = \min(t_1, t_2) + t_1 t_2$.

• Constant c can be obtained from simulation of the limit distribution or via bootstrap (block bootstrap), $c = c_\alpha$:

$$P(\sup_{0 < t \le T} \frac{\sum_{j=1}^{d} W_j^2(t)}{w(t)} > c_\alpha) = \alpha$$

- For properly chosen c the level of the test is asymptotically α, test is consistent.

- Matrix

$$\mathbf{D} = E(\mathbf{z}_0\mathbf{z}_0^T) + \sum_{i=1}^{\infty} E(\mathbf{z}_0\mathbf{z}_i^T) + \mathbf{z}_i\mathbf{z}_0^T)$$

is estimated by Bartlett type estimators based on training sample only.

$$\widehat{\mathbf{D}}_m = \widehat{\mathbf{S}}_m(0) + \sum_{k=1}^{m} h(k/q(m))(\widehat{\mathbf{S}}_m(k) + \widehat{\mathbf{S}}_m(-k))$$

$$\widehat{\mathbf{S}}_m(k) = \frac{1}{m}\sum_{i=1}^{m-k} \widehat{\mathbf{z}}_i\widehat{\mathbf{z}}_i^T$$

$h(.)$ — Bartlett kernel

$$\widehat{\mathbf{z}}_i = \frac{1}{J}\sum_{j=1}^{J}(r_{M,i}(s_j) - \widehat{r}_{M,m}(s_j))\widehat{\varepsilon}_i(s_j)$$

$$\widehat{\varepsilon}_i(s_j) = (r_i(s_j) - \widehat{r}_{i,m}(s_j)) - \widehat{\boldsymbol{\beta}}_m(r_{M,i}(s_j) - \widehat{r}_{M,m}(s_j))$$

Andrews (1991)

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Data

Monitoring portfolios beta in several stocks in the S&P 100 index 2001 and 2002.

Stocks in the S&P 100 index and the S&P 100 index market itself

5 stocks (Boeing, Exxon Mobile, AT& T Bank of America, Mircrosoft)

(a) 120 training days starting on January 29, 2001– July 19, 2001 (stable period)

(b) 80 training days starting on October 9, 2001 - February 2002

2 boundary functions — one for earlier changes (2), one for later changes (1) (test statistics is compared after each new observation with the value of boundary function at that point, large values indicate a change)

$$w_1(t) = (1+t)^2$$

$$w_2(t) = t\sqrt{3(1+t^2)+t} + 0.1$$

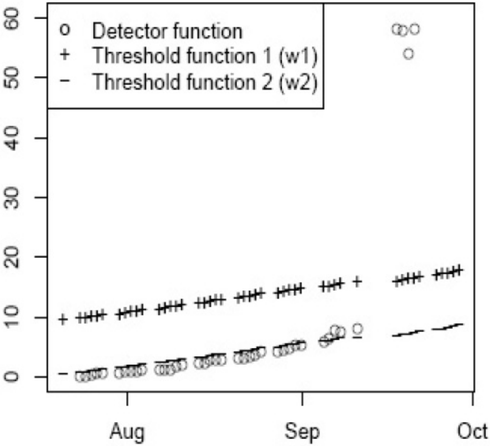

Figure 1: Detector V_k ($\circ\circ\circ$) and threshold functions w_1 ($+++$) and w_2 ($---$) for the monitoring procedure commencing on July 20, 2001. The significance level is set to $\alpha = 0.05$.

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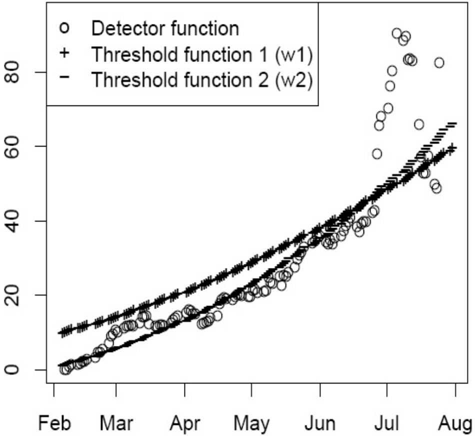

Figure 2: Detector V_k ($\circ\circ\circ$) and threshold functions w_1 ($+++$) and w_2 ($---$) for the monitoring procedure commencing on February 5, 2002. The significance level is set to $\alpha = 0.05$.

THANK YOU!!!!!