Convergence of the average cost in the case of jump diffusions.

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Contents.

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8. Compensated Poisson measure: $\tilde{N}(t, A) = N(t, A) - t\nu(A)$ ($\mathcal{F}$-martingal).

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1. For $A \subset \mathbb{R}$, $0 \notin \bar{A}$ and $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}^n$ borel measurable ($n \in \mathbb{N}$):

$$\int_{[0,t]} \int_A f(s,x)\tilde{N}(ds,dx)$$

$$= \sum_{0 \le s \le t} f(s, \Delta L_s)\mathbb{I}_{\Delta L_s \in A} - \int_{[0,t]} \int_A f(s,x)d\nu(x)ds$$

and $\int_0^t \int_{\mathbb{R}} f(s,x)\tilde{N}(ds,dx)$ we obtain by approximation.

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   2. ν has the density
      $$\left(c_+ \mathbf{I}_{(0,\infty)}(x) + c_- \mathbf{I}_{(-\infty,0)}\right) |x|^{-\alpha-1}$$
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   where $c_+, c_- \geq 0$.
   3. $$q = \mathbb{E}|L_1|^2 = \int_{\mathbb{R}} |x|^2 d\nu(x)$$

Itôo formula

1. Let $X_t = X_0 + \int_0^t F(s)ds + \int_0^t G(s)dL_s$, $F \in \mathbb{L}^{1,loc}(\Omega \times \mathbb{R}_+, \mathbb{R}^n)$, $G \in \mathbb{L}^{2,loc}(\Omega \times \mathbb{R}_+, \mathbb{R}^n)$, F, G progresive (denote $F \in \mathbb{L}_{\mathcal{F}}^{1,loc}(R_+, \mathbb{R}^n)$, $G \in \mathbb{L}_{\mathcal{F}}^{2,loc}(R_+, \mathbb{R}^n)$),

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2. $f \in \mathbb{C}^2(\mathbb{R}^n)$.
3. Then

$$f(X_t) = f(X_0) + \int_{[0,t]} f_x(X_s)F(s)ds$$

$$+ \int_{[0,t]} \int_{\mathbb{R}} (f(X_{s\_} + G(s)x) - f(X_{s\_}))\tilde{N}(ds, dx)$$

$$+ \int_{[0,t]} \int_{\mathbb{R}} (f(X_{s\_} + G(s)x) - f(X_{s\_}) - G(s)xf_x(X_{s\_}))d\nu(x)ds.$$

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2. It is possible to verify that this equation has a unique solution $X \in \mathbb{L}^{2,loc}_{\mathcal{F}}(\mathbb{R}_+)$.

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3. It is possible to prove the existence of the unique solution $X \in \mathbb{L}^2_{\mathcal{F}}[0, T]$ for all $T > 0$ and by splicing we obtain the solution on $\mathbb{L}^{2,loc}_{\mathcal{F}}(\mathbb{R}_+)$

Cost functional

1. $$J(U,T) = \int_0^T (\langle QX_s, X_s\rangle + \langle RU_s, U_s\rangle) ds, \ \ T > 0, \qquad (4)$$

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   1. Note that P is the limit of the solutions on $[0,T]$ of

      $$\dot{P}_t + P_t A + A^T P_t + Q - P_t BR^{-1}B^T P_t = 0 \tag{6}$$

      for $t\to\infty$.

Main result

Suppose that

1. $$\frac{\langle PX_t, X_t \rangle}{t} \to 0, \quad t \to \infty, \quad a.s., \tag{7}$$

2. let there exists $c_1 > 0$ such that

   1. $$\lim_{t\to\infty} \sup \frac{\int_0^t \langle PX_{s_-}, X_{s_-} \rangle ds}{t} \leq c_1 \;\; a.s., \tag{8}$$

   2. $$\lim_{t\to\infty} \sup \frac{\int_0^t \langle PX_{s_-}, X_{s_-} \rangle^{\alpha} ds}{t} \leq c_1 \;\; a.s., \tag{9}$$

3. in the case of $\alpha = 1$ let a.s.

$$\lim_{t\to\infty} \sup \frac{\int_0^t \langle PX_{s_-}, X_{s_-} \rangle \log \langle PX_{s_-}, X_{s_-} \rangle ds}{t} = 0. \tag{10}$$

Main result

Suppose that

1. $$\lim_{t\to\infty} K_t = k_0 = -R^{-1}B^T P \quad a.s., \tag{11}$$

2. let there exists

$$P_\sigma = \lim_{t\to\infty} \frac{\int_0^t \langle P\sigma_s, \sigma_s\rangle ds}{t} < \infty \tag{12}$$

(which equals $\langle P\sigma, \sigma\rangle$ in the case of constant σ)

Then

$$\lim_{t\to\infty} \frac{J(U,t)}{t} = P_\sigma q, \quad a.s. \tag{13}$$

Proof

1. Using Itôo formula and (6) we obtain

$$\frac{\langle PX_t, X_t\rangle - \langle PX_0, X_0\rangle}{t}$$

$$= -\frac{J(U,t)}{t} + \frac{\int_0^t \int_{\mathbb{R}} \langle P\sigma_s, \sigma_s\rangle |x|^2 d\nu(x) ds}{t}$$

$$-\frac{\int_0^t \langle (k_0 - K_s)X_s, P(k_0 - K_s)X_s\rangle ds}{t}$$

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2. We can send last part to zero *a.s.* using our assumptions, the changing time representations and the strong law of large numbers for Lévy processes.
3. We can prove that assuming (11) the third part tends to zero *a.s.*.
4. Assuming (7), the left side tends to zero *a.s.* also.

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Thank you.