DIAGNOSTICS OF ROBUST IDENTIFICATION OF MODEL

ROBUST 2014

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"ROBUST METHODS FOR NONSTANDARD SITUATIONS, THEIR DIAGNOSTICS AND IMPLEMENTATIONS".

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Content

Notations

The most frequent econometric (statistical) framework is:

Regression model

$$Y_i = X_i'\beta^0 + \varepsilon_i = \sum_{j=1}^{p} X_{ij}\beta_j^0 + \varepsilon_i, \quad i = 1, 2, ..., n$$

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$$Y = X\beta^0 + \varepsilon$$

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Identification of regression model

$$\hat{\beta}^{(n)}(Y, X) \rightarrow R^p,$$

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but we need also $\hat{\sigma}^2_{(n)}(Y, X) \rightarrow R^+$

$\rightarrow$ to be able to establish significance of explanatory variables,

usually we have $\hat{\sigma}^2_{(n)}\left(Y, X, \hat{\beta}^{(n)}\right) \rightarrow R^+$.

Content

Returning to the roots

First of all, let's recall the classical estimators

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2. the Maximum Likelihood $\hat{\beta}^{(ML,n)}$

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4. stressing the importance of normality of disturbances,
5. testing the submodels, etc.

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5. verifying the specification of model (Hausman),
6. studying the stability of model (Chow),
7. testing character of effects (Hausman-Taylor), etc.

A comparison of classical and robust regression analysis

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What about to select one flexible robust estimator and for it to develop the same diagnostics, we are used to for $\hat{\beta}^{(OLS,n)}$ or $\hat{\beta}^{(ML,n)}$?

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What about to select one flexible robust estimator and for it to develop the same diagnostics, we are used to for $\hat{\beta}^{(OLS,n)}$ or $\hat{\beta}^{(ML,n)}$?

But which one?

Returning to the roots

Let’s recall the clasical estimators once again.

What features of estimators we are (“automatically”) used to ?

Equivariance of $\hat{\beta}^{(n)}$

$$\hat{\beta}(Y, X) : M(n, p+1) \to R^p$$

$$\textit{scale-equivariant} : \quad \forall c \in R^+ \quad \hat{\beta}(cY, X) = c\hat{\beta}(Y, X)$$

$$\textit{regression-equivariant} : \quad \forall b \in R^p \quad \hat{\beta}(Y + Xb, X) = \hat{\beta}(Y, X) + b$$

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Examples : $\hat{\beta}^{(OLS,n)} = (X'X)^{-1} X'Y$, $\quad \hat{\beta}^{(L_1,n)} = \underset{\beta \in R^p}{\arg\min} \sum_{i=1}^n |Y_i - X_i'\beta|$,

e. g.

we don't need recalculate estimate when we change (linearly) the units of data.

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Unfortunately, the most popular robust estimators, the *M*-estimators

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They require studentization of residuals!!

Bickel, P. J. (1975): One-step Huber estimates in the linear model. *J. Amer. Statist. Assoc. 70, 428–433.*

To reach *scale-* and *regression-equivariance* of an *M*-estimator, say

$$\hat{\beta}^{(M,\rho,n)} = \arg\min_{\beta \in R^p} \sum_{i=1}^{n} \rho\left(\frac{Y_i - X_i'\beta}{\hat{\sigma}_{(n)}}\right)$$

$\hat{\sigma}_{(n)}$ has to be *scale-equivariant* and *regression-invariant*.

Estimators of scale of disturbances which are sufficiently robust, consistent, scale-equivariant and regression-invariant:

Jurečková, J., P. K. Sen (1993): Regression rank scores scale statistics and studentization in linear models. *Proc. of the Fifth Prague Symposium on Asymptotic Statistics, Physica Verlag, 111-121.*

Croux C., P. J. Rousseeuw (1992): A class of high-breakdown scale estimators based on subranges. *Communications in Statistics - Theory and Methods 21, 1935 - 1951.*

Víšek, J. Á. (2010): Robust error-term-scale estimate. *IMS Collections. Nonparametrics and Robustness in Modern Statistical Inference and Time Series Analysis: Festschrift for Jana Jurečková, Vol. 7(2010), 254 - 267.*

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The common feature of all these estimators -

they are based on (residuals of) a preliminary estimate of regression model

which is already scale- and regression-equivariant.

Preliminary conclusion

We should prefer (robust) estimators which are *"automatically"* scale- and regression-equivarint.

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Rousseeuw, P. J. (1984): Least median of square regression.
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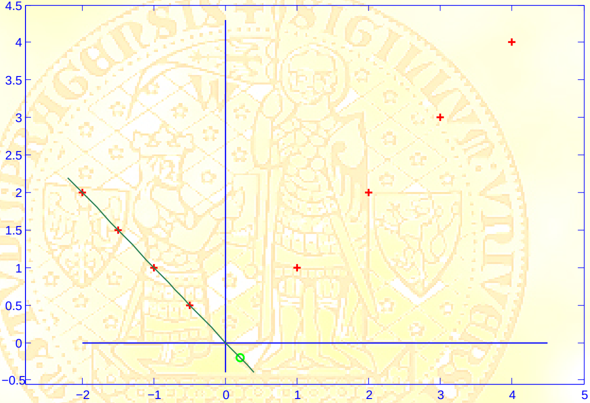
$$\hat{\beta}^{(LTS,n,h)} = \arg\min_{\beta \in R^p} \sum_{i=1}^{h} r^2_{(i)}(\beta)$$

Hampel, F. R. et al. (1986): *Robust Statistics – The Approach Based on Influence Functions.* New York: J.Wiley & Son.

SENSITIVITY OF ANY HIGH-BREAKDOWN-POINT ESTIMATOR
TO A SMALL CHANGE OF DATA

AN (ACADEMIC) EXPLANATION BY A SHIFT OF “INLIER”

Model for the majority of data



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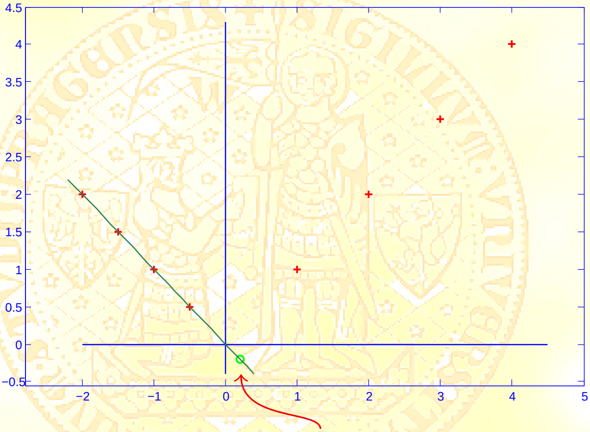
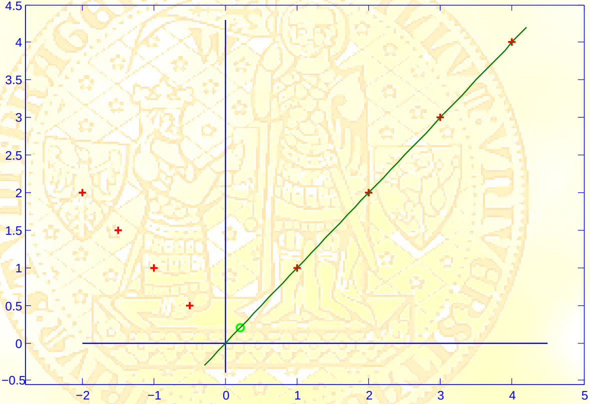
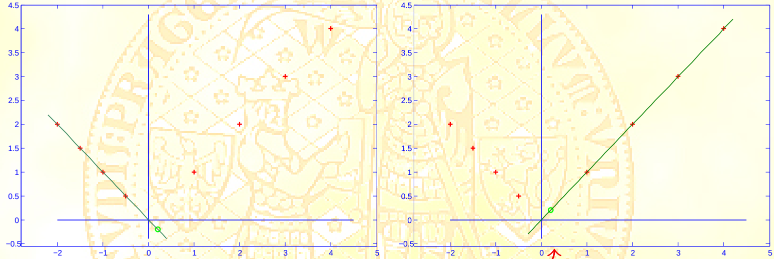


We are going to shift up this green circle " o ".

Again model for the majority of data



In both cases the model is for the majority of data



Notice: *The closer* the green circle (“ o ”) is to the *y*-axe,
the smaller shift causes the “switch” of the model.

IT WAS THE (ACADEMIC) EXAMPLE - THERE ARE REAL DATA

HETTMANSPERGER, T. P., S. J. SHEATHER (1992):
A CAUTIONARY NOTE ON THE METHOD OF LEAST MEDIAN SQUARES.
The American Statistician 46, 79–83.

Engine Knock Data ($n = 16$, $p = 4$, $h = 11$)

c	x_1	x_2	x_3	x_4	y
1	13.3	13.9	31	697	84.4
2	13.3	14.1	30	697	84.1
3	13.4	15.2	32	700	88.4
4	12.7	13.8	31	669	84.2
⋮	⋮	⋮	⋮	⋮	⋮
14	12.7	16.1	35	649	93.0
15	12.9	15.1	36	721	93.3
16	12.7	15.9	37	696	93.1

x_1 is spark timing x_2 air/fuel ratio
x_3 intake temperature x_4 exhaust temperature
y engine knock number

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This is the exact value of $\hat{\beta}^{(LTS,n,h)}$!

Data	Interc.	SPARK	AIR	INTK	EXHS.
Correct data ($x_{22} = 14.1$)	35.11	-0.028	2.949	0.477	-0.009
Damaged data($x_{22} = 15.1$)	-88.7	4.72	1.06	1.57	0.068

x_3 intake temperature $\quad$ x_4 exhaust temperature

y engine knock number

Content

The least weighted squares

Residuals $\quad \forall \beta \in R \quad \rightarrow \quad r_i(\beta) = Y_i - X_i'\beta$

Order statistics of squared residuals, i. e.

$$r^2_{(1)}(\beta) \le r^2_{(2)}(\beta) \le \dots \le r^2_{(n)}(\beta)$$

Definition

Let $w_i \in [0, 1]$, $\; i = 1, 2, ..., n$. Then

$$\hat{\beta}^{(LWS,n,w)} = \arg\min_{\beta \in R^p} \sum_{i=1}^n w_i r^2_{(i)}(\beta)$$

will be called *the least weighted squares* (*LWS*).

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The weights are prescribed to the order statistics of squared residuals !!

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$$w_{(1)} \geq w_{(2)} \geq ... \geq w_{(n)}$$

The least median of squares $\hat{\beta}^{(LMS,h,n)}$ as well as the least trimmed squares $\hat{\beta}^{(LTS,h,n)}$ are special cases of the $\hat{\beta}^{(LWS,n,w)}$.

PROS AND CONS OF LWS

Inherited from LTS:

$\sqrt{n}$-consistency (even under heteroscedasticity)

Scale- and affine-equivariance

Pros and cons of LWS

Achieved due to the continuity of weight function:

- Breakdown point adaptable to level and character of contamination
- Quick and reliable algorithm (implemented in MATLAB)
- Modifications for nonstandard situations
  (e. g. instrumental variables, total least squares, fixed or random effects)
- Low sensitivity to the shift and deletion of observation(s)
- More diagnostic tools (D-W, White test, Hausman test, etc.)
- Applicability for panel data
- "Coping automatically" with heteroscedasticity of data

Pros and cons of LWS

Still lacking:

Diagnostics for finite sample size
- i. e. significance of explanatory variables, test of submodels, etc.

Determination of model

Content

Significance of explanatory variable - for the Least Weighted Squares (LWS)

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The first one:

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The second one:

The classical derivation for significance of *individual explanatory variable for OLS* $\hat{\beta}^{(OLS,n)}(Y,X)$ can be generalised for $\hat{\beta}^{(OLS,n,W,\pi)}(\tilde{Y},\tilde{X})$.

Deriving form of $\hat{\beta}^{(LWS,n)}$

$\underline{\forall \ \ (\omega \in \Omega) \ \ \exists \ \ (\pi = \pi(\omega) = \{\pi_1(\omega), \pi_2(\omega), ..., \pi_n(\omega)\})}$ so that

$$\hat{\beta}^{(LWS,n,w)}(\omega) = \arg\min_{\beta \in R^p} \sum_{i=1}^{n} w_i \left(Y_{\pi_i} - X'_{\pi_i}\beta\right)^2$$

$$= \arg\min_{\beta \in R^p} \sum_{i=1}^{n} \left(w_i^{\frac{1}{2}} Y_{\pi_i} - w_i^{\frac{1}{2}} X'_{\pi_i}\beta\right)^2 = \arg\min_{\beta \in R^p} \sum_{i=1}^{n} \left(\tilde{Y}_i - \tilde{X}'_i\beta\right)^2 .$$

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Put $\tilde{W} = diag\left\{w_1^{\frac{1}{2}}, w_2^{\frac{1}{2}}, ..., w_n^{\frac{1}{2}}\right\}$, $\tilde{Y} = \tilde{W} Y_\pi$, $\tilde{X} = \tilde{W} X_\pi$ and $\tilde{e} = \tilde{W} e_\pi$ and consider the model

$$\tilde{Y} = \tilde{X}\beta^0 + \tilde{\varepsilon} \quad \text{with} \quad \mathcal{L}(\tilde{\varepsilon}) = \mathcal{N}\left(0, \sigma^2 \tilde{W}^2\right).$$

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Then

$$\underline{\hat{\beta}^{(LWS,n,w)}(\omega)} = \arg\min_{\beta \in R^p} \left\{\left(\tilde{Y} - \tilde{X}\beta\right)' \left(\tilde{Y} - \tilde{X}\beta\right)\right\}$$

$$= \left(\tilde{X}'\tilde{X}\right)^{-1} \tilde{X}'\tilde{Y} = \underline{\hat{\beta}^{(OLS,n,W,\pi)}(\tilde{Y}, \tilde{X})}.$$

Deriving form of $\hat{\beta}^{(LWS,n,w)}$

Fix $\pi = \{\pi_1, \pi_2, ..., \pi_n\}$ and put

$$B(\pi) = \left\{\omega \in \Omega \ : \ \hat{\beta}^{(LWS,n,w)}(Y, X) = \hat{\beta}^{(OLS,n,W,\pi)}(\tilde{Y}, \tilde{X})\right\}$$

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$$\bigcup_{over \;\; all \;\; \pi' s} B(\pi) = \Omega$$

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Then: $$\pi^{(1)} \neq \pi^{(2)} \implies B(\pi^{(1)}) \cap B(\pi^{(2)}) = \emptyset \quad a.s.$$

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- Conditional p-value
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Significance of explanatory variable - classical OLS case

Let's recall the simplest classical framework for finite-sample diagnostics:

Regression model

$$Y_i = X_i'\beta^0 + \varepsilon_i, \quad i = 1, 2, ..., n \qquad \text{or} \qquad Y = X\beta^0 + \varepsilon$$

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Conditions :

$\{(X_i', \varepsilon_i)'\}_{i=1}^{\infty}$ i.i.d., $F_{X,\varepsilon}(x, v) = F_X(x) \cdot F_\varepsilon(v)$, $F_\varepsilon(v) = \mathcal{N}(0, \sigma^2)$,

$Q = E[X_1 \cdot X_1']$ is regular.

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Test is based on

$$\mathcal{L}\left(\frac{\hat{\beta}_\ell^{(OLS,n)} - \beta_\ell^0}{s_n \cdot c_{\ell,\ell}}\right) = \mathcal{L}\left(t_\ell\right) = t_{n-p}$$

(Fisher-Cochran theorem)

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Let's also recall

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Notice the orthogonality of the projection matrices in (1) and (2).

Recalling the classical regression for $\tilde{Y}, \tilde{X}$

The orthogonality of the projection matrices in (1) and (2)

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Recalling the classical regression for $\tilde{Y}, \tilde{X}$

Recalling that we consider the model

$$\tilde{Y} = \tilde{X}\beta^0 + \tilde{\varepsilon} \;\text{ with }\; \mathcal{L}(\tilde{\varepsilon}) = \mathcal{N}\left(0, \sigma^2 \tilde{W}^2\right),$$

we have from

$$\hat{\beta}^{(OLS,n,W,\pi)}(\tilde{Y},\tilde{X}) - \beta^0 = \tilde{X}\left(\tilde{X}'\tilde{X}\right)^{-1}\tilde{X}'\tilde{\varepsilon},$$

$$E\left\{\hat{\beta}^{(OLS,n,W,\pi)}(\tilde{Y},\tilde{X}) - \beta^0\right\} = 0$$

and

$$\operatorname{cov}\left\{-\beta^0\right\} = \left(\tilde{X}'\tilde{X}\right)^{-1}\tilde{X}'\tilde{W}^2\tilde{X}\left(\tilde{X}'\tilde{X}\right)^{-1}$$

$$= (X'WX)^{-1}\cdot\sum_{i=1}^{n} w_i^2\cdot\tilde{X}_i\cdot\tilde{X}_i'\,(X'WX)^{-1}.$$

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Then

$$\mathcal{L}\left(\frac{\hat{\beta}_\ell^{(OLS,n,W,\pi)}(\tilde{Y},\tilde{X}) - \beta_\ell^0}{\sigma d_{n,\ell}(w,X)}\right) = \mathcal{N}(0,1).$$

Establishing the result

We can show (similarly as in the OLS-regression),

$$\mathcal{L}\left(\sigma^{-2}\tilde{r}'\left(\hat{\beta}^{(OLS,n,W,\pi)}(\tilde{Y},\tilde{X})\right)\cdot\tilde{r}\left(\hat{\beta}^{(OLS,n,W,\pi)}(\tilde{Y},\tilde{X})\right)\right)$$

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We conclude

THEOREM

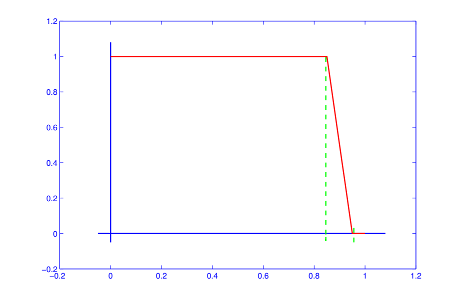
$$\mathcal{L}\left(\frac{\hat{\beta}^{(OLS,n,W,\pi)}(\tilde{Y},\tilde{X})-\beta^0}{d_{n,\ell}(w,X)}\cdot\left[\frac{\sum_{i=1}^n w_i(1-d_{ii})}{RSS}\right]^{\frac{1}{2}}\right) = t_{generalized}\left(n-p\right).$$

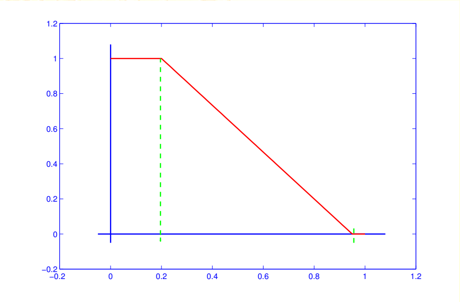
Content

Function generating the weights

Under low contamination, the intuitively optimal (left)
and really optimal (right) weight function
(in the *sense of mean square error* of the estimates of regression coefficients).

$w_\ell = w\left(\frac{\ell-1}{n}\right)$ Contamination : 4% outliers





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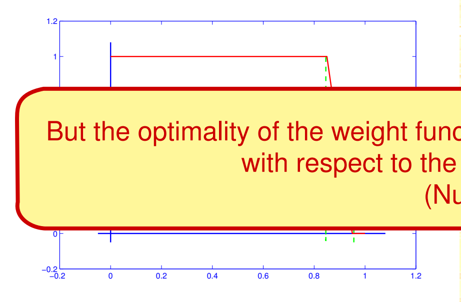

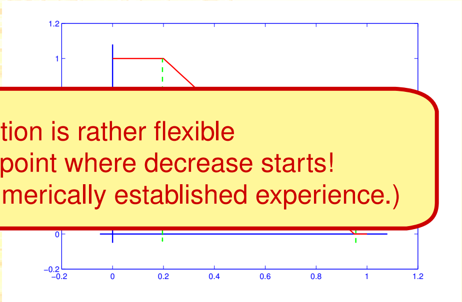

But the optimality of the weight function is rather flexible
with respect to the point where decrease starts!
(Numerically established experience.)

Content

Framework of simulations

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4. We have done it for $n = 20, 30, ..., 190$.

TABLE 1

The simulated quantiles for 5%.

n	20	30	40	50	60	70
$\hat{t}^{LWS}_{0.975}(n)$	2.148 $_{(0.047)}$	2.087 $_{(0.040)}$	2.056 $_{(0.046)}$	2.027 $_{(0.045)}$	2.017 $_{(0.046)}$	2.012 $_{(0.045)}$
$t_{0.975}(n)$	2.085	2.043	2.022	2.009	2.000	1.995

n	80	90	100	110	120	130
$\hat{t}^{LWS}_{0.975}(n)$	2.008 $_{(0.040)}$	1.999 $_{(0.041)}$	1.992 $_{(0.040)}$	1.991 $_{(0.041)}$	1.990 $_{(0.040)}$	1.988 $_{(0.040)}$
$t_{0.975}(n)$	1.990	1.987	1.984	1.982	1.980	1.978

n	140	150	160	170	180	190
$\hat{t}^{LWS}_{0.975}(n)$	1.986 $_{(0.043)}$	1.989 $_{(0.041)}$	1.975 $_{(0.035)}$	1.974 $_{(0.035)}$	1.973 $_{(0.035)}$	1.973 $_{(0.035)}$
$t_{0.975}(n)$	1.977	1.976	1.975	1.974	1.974	1.973

TABLE 2

The simulated quantiles for 1%.

n	20	30	40	50	60	70
$\hat{t}^{LWS}_{0.995}(n)$	2.999 (0.100)	2.825 (0.082)	2.766 (0.080)	2.702 (0.085)	2.688 (0.077)	2.678 (0.079)
$t_{0.995}(n)$	2.845	2.748	2.705	2.678	2.661	2.651

n	80	90	100	110	120	130
$\hat{t}^{LWS}_{0.995}(n)$	2.659 (0.067)	2.644 (0.075)	2.633 (0.077)	2.627 (0.063)	2.629 (0.070)	2.626 (0.071)
$t_{0.995}(n)$	2.640	2.632	2.625	2.619	2.614	2.612

n	140	150	160	170	180	190
$\hat{t}^{LWS}_{0.995}(n)$	2.619 (0.072)	2.621 (0.073)	2.609 (0.079)	2.609 (0.070)	2.620 (0.078)	2.602 (0.078)
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By the way,

0.995-upper quantile of the standard normal distribution is equal to 2.575.

THANKS FOR ATTENTION