Riccati Transformation Method for Solving Hamilton-Jacobi-Bellman equation

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The presentation is based on papers published by our group:

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Content

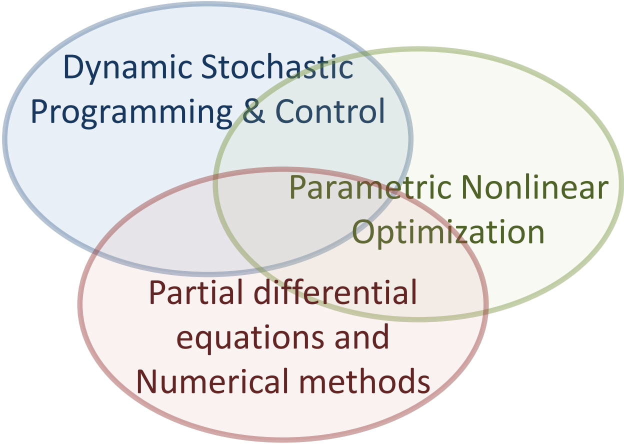
1. Motivation - Optimal portfolio selection problem
2. Principle of Dynamic stochastic optimization

   Hamilton – Jacobi – Bellman equation

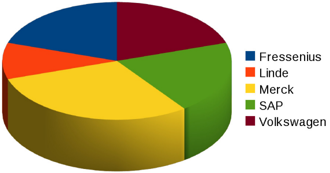
3. Reduction to a quasi-linear parabolic PDE by the Riccati transformation
4. Parametric quadratic optimization and its value function
5. Qualitative properties of the optimal response function
6. Numerical discretization scheme for solving PDEs
7. Applications to the optimal portfolio selection problem and pension savings system

Important interactions between various mathematical fields



Static and Dynamic optimal portfolio selection problem

- Static Markowitz optimal portfolio problem
- Dynamic stochastic optimization problem statement
- Bellman's principle and Hamilton-Jacobi-Bellman equation



Example of optimal asset selection for German DAX30 stock index (2013)

Mathematical formulation of the Markowitz model

$$\max_{\boldsymbol{\theta}\in\mathbb{R}^n} \boldsymbol{\mu}^T\boldsymbol{\theta} \qquad \text{– maximize the mean return}$$

$$s.t. \quad \frac{1}{2}\boldsymbol{\theta}^T\boldsymbol{\Sigma}\boldsymbol{\theta} \leq \frac{1}{2}\sigma^2 \qquad \text{– the variance is prescribed}$$

$$\sum_{i=1}^{n}\theta^i = 1 \qquad \text{– weights sum up to 100\%}$$

$$\boldsymbol{\theta} \geq 0 \qquad \text{– no short positions allowed}$$

Here $\boldsymbol{\mu} \in \mathbb{R}^n, \mu^i = \mathbb{E}(X^i)$ is the vector of mean returns of stochastic asset returns and $\boldsymbol{\Sigma}$ is their covariance matrix, $\boldsymbol{\Sigma}_{ij} = cov(X^i X^j)$

Motivation - Static optimal portfolio selection problem

- its Lagrange function (corresponding to $\min(-\boldsymbol{\mu}^T\boldsymbol{\theta})$)

$$\mathcal{L}(\boldsymbol{\theta}, \varphi, \lambda, \xi) = -\boldsymbol{\mu}^T\boldsymbol{\theta} + \varphi\frac{1}{2}\boldsymbol{\theta}^T\boldsymbol{\Sigma}\boldsymbol{\theta} + \lambda 1^T\boldsymbol{\theta} + \xi^T\boldsymbol{\theta}$$

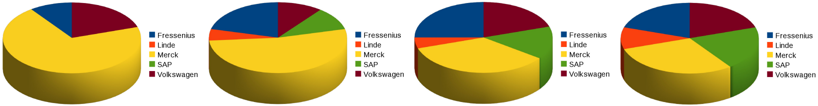
- $\varphi \in \mathbb{R}, \lambda \in \mathbb{R}, \xi \in \mathbb{R}^n, \xi \geq 0$ are Lagrange multipliers
- The same Lagrange function corresponds to the minimization problem:

$$\min_{\boldsymbol{\theta}\in\mathbb{R}^n} -\boldsymbol{\mu}^T\boldsymbol{\theta} + \varphi\frac{1}{2}\boldsymbol{\theta}^T\boldsymbol{\Sigma}\boldsymbol{\theta}$$
$$\sum_{i=1}^{n}\theta^i = 1$$
$$\boldsymbol{\theta} \geq 0$$

provided that the Lagrange multiplier $\varphi > 0$ is fixed.

- φ can be viewed as a measure of investor's risk-aversion

Motivation - Static optimal portfolio selection problem



$\varphi \approx 1$ $\varphi \approx 4$ $\varphi \approx 6$ $\varphi \approx 8$

Optimal asset selection for German DAX30 stock index for various $\varphi > 0$

Motivation - Dynamic optimal selection problem

Assumptions:

- synthetic stochastic portfolio value $Y_t^{\boldsymbol{\theta}}$ at time t has the same return as the weighted sum of returns of individual asset returns on Y_t^i

$$\frac{\mathrm{d}Y_t^{\boldsymbol{\theta}}}{Y_t^{\boldsymbol{\theta}}} = \sum_{i=1}^{n} \theta_t^i \frac{\mathrm{d}Y_t^i}{Y_t^i}$$

- each individual stochastic process satisfies Itō's SDE:

$$\frac{\mathrm{d}Y_t^i}{Y_t^i} = \mu^i \mathrm{d}t + \sum_{j=1}^{n} \bar{\sigma}^{ji} \mathrm{d}W_t^j \,, \qquad \text{for all } i = 1, \ldots, n,$$

of geometric Brownian motion with mean returns μ^i and mutual covariances $\bar{\sigma}^{ji}$

Motivation - optimal selection problem

- if the portfolio is allowed to have an exogenous non-negative inflow with inflow rate $\varepsilon \geq 0$ and constant interest rate $r \geq 0$ then $Y_t = Y_t^{\boldsymbol{\theta}}$ satisfies SDE:

Stochastic differential equation (SDE) for the portfolio value

$$\mathrm{d}Y_t^{\boldsymbol{\theta}} = (\varepsilon + [r + \mu(\boldsymbol{\theta})]Y_t^{\boldsymbol{\theta}})\mathrm{d}t + \sigma(\boldsymbol{\theta})Y_t^{\boldsymbol{\theta}}\mathrm{d}W_t$$

  - $\mu(\boldsymbol{\theta}) = \boldsymbol{\mu}^T\boldsymbol{\theta}$,
  - $\sigma(\boldsymbol{\theta})^2 = \boldsymbol{\theta}^T\boldsymbol{\Sigma}\boldsymbol{\theta}$ where $\boldsymbol{\Sigma} = \bar{\boldsymbol{\Sigma}}^T\bar{\boldsymbol{\Sigma}}$ with $\bar{\boldsymbol{\Sigma}} = (\bar{\sigma}^{ij})$

- for logarithmic variable $X_t^{\boldsymbol{\theta}} = \ln(Y_t^{\boldsymbol{\theta}})$ we obtain by Itō's lemma

SDE for logarithmic variable

$$\mathrm{d}X_t^{\boldsymbol{\theta}} = \left(\varepsilon e^{-X_t^{\boldsymbol{\theta}}} + r + \mu(\boldsymbol{\theta}) - \frac{1}{2}(\sigma(\boldsymbol{\theta}))^2\right)\mathrm{d}t + \sigma(\boldsymbol{\theta})\mathrm{d}W_t$$

Motivation - optimal selection problem

Individual weights of assets: $\boldsymbol{\theta}_t = (\theta_t^1, ..., \theta_t^n)^T$ belong to the given decision set Δ^n:

$$\Delta^n \subset \{\boldsymbol{\theta} \in \mathbb{R}^n \mid \sum_{i=1}^{n} \boldsymbol{\theta}^i \leq 1\},$$

Examples:

- compact convex simplex

$$\Delta^n = \{\boldsymbol{\theta} \in \mathbb{R}^n \mid \boldsymbol{\theta}^i \geq 0, \sum_{i=1}^{n} \boldsymbol{\theta}^i = 1\} \qquad (or \quad \leq 1)$$

- convex simplex with short positions allowed

$$\Delta^n = \{\boldsymbol{\theta} \in \mathbb{R}^n \mid \sum_{i=1}^{n} \boldsymbol{\theta}^i = 1\}$$

- discrete set (example of a Slovak pension fund system)

$$\Delta^n = \{(0.8, 0.2),\ (0.5, 0.5),\ (0, 1)\}$$

Goal: Find the optimal response strategy

$$\{\boldsymbol{\theta}\} = \{\boldsymbol{\theta}_t \in \Delta^n \,|\, t \in [0, T]\}$$

belonging to a set $\mathcal{A} = \mathcal{A}_{0,T}$ of admissible strategies,

$$\mathcal{A}_{t,T} = \{\{\boldsymbol{\theta}\} \mid \boldsymbol{\theta}_s \in \Delta^n, s \in [t, T]\},$$

and such that $\{\boldsymbol{\theta}\}$ maximizes the expected terminal utility $U(.)$ from the portfolio:

Stochastic dynamic optimization problem

$$\max_{\{\boldsymbol{\theta}\} \in \mathcal{A}_{0,T}} \mathbb{E}\left[U(X_T^{\boldsymbol{\theta}}) | X_0^{\boldsymbol{\theta}} = x_0 \right],$$

where U is e.g. ARA terminal utility function $U(x) = -\exp(-ax)$ representing an investor with constant coefficient $a > 0$ of absolute risk aversion

Bellman's principle - Stochastic variant

It is known from the theory of stochastic dynamic programming that the so-called value function

$$V(x,t) := \sup_{\{\boldsymbol{\theta}\}\in\mathcal{A}_{t,T}} \mathbb{E}\left[U(X_T^{\boldsymbol{\theta}}) \mid X_t^{\boldsymbol{\theta}} = x\right]$$

subject to the terminal condition $V(x,T) := U(x)$ can be used for solving the stochastic dynamic optimization problem

- **Bellman's principle**
  (using tower law of conditioned expectations)

$$V(x,t) := \sup_{\{\boldsymbol{\theta}\}\in\mathcal{A}_{t,t+dt}} \mathbb{E}\left[V(X_{t+dt}^{\boldsymbol{\theta}}, t+dt) \mid X_t^{\boldsymbol{\theta}} = x\right]$$

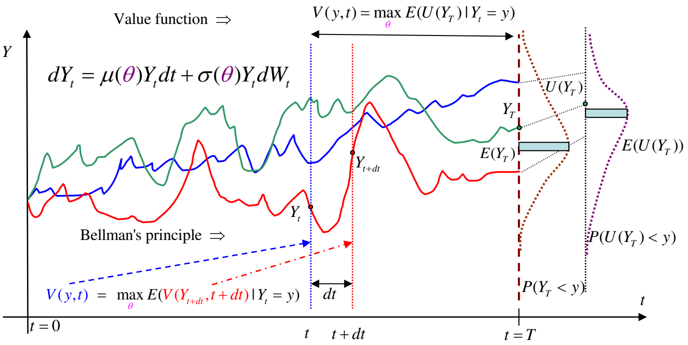
- response $\{\boldsymbol{\theta}\}$ is optimal on the entire interval $[0,T]$
  iff $\{\boldsymbol{\theta}\}$ is optimal on each subinterval $[t,t+dt]$

Bellman's principle - Stochastic variant, a sketch

Stochastic dynamic optimization problem for $Y_t^{\boldsymbol{\theta}} = \exp(X_t^{\boldsymbol{\theta}})$ variable:

- **find a non-anticipative strategy $\{\theta\}$ maximizing**

$$\max_{\{\theta\}} \mathbb{E}(\mathbf{U}(\mathbf{Y_T}) \mid \mathbf{Y_0} = \mathbf{y})$$



Important tool: Itō's lemma

Kiyoshi Itō 伊藤 清 (1915–2008)

- Let $V(x, t)$ be a C^2 smooth function of x, t variables. Suppose that the process $\{X_t, t \geq 0\}$ satisfies SDE:

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t,$$

Then the differential $dV_t = V(X_{t+dt}, t + dt) - V(X_t, t)$ is given by

$$dV_t = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2(x, t)\frac{\partial^2 V}{\partial x^2}\right)dt + \frac{\partial V}{\partial x}dX_t\,,$$

Hamilton-Jacobi-Bellman equation

- **The Bellman principle** can be rewritten as:

$$0 = \sup_{\{\boldsymbol{\theta}\}\in\mathcal{A}_{t,t+dt}} \mathbb{E}\left[V(X^{\boldsymbol{\theta}}_{t+dt}, t+dt) - V(X^{\boldsymbol{\theta}}_t, t) \mid X^{\boldsymbol{\theta}}_t = x\right]$$

- Apply Itō's lemma for differential $dV_t = V(X^{\boldsymbol{\theta}}_{t+dt}, t+dt) - V(X^{\boldsymbol{\theta}}_t, t)$ of the process $X^{\boldsymbol{\theta}}_t$
- Taking into account $dW_t = W_{t+dt} - W_t$ and $X^{\boldsymbol{\theta}}_t$ are independent we have $\mathbb{E}\left[\partial_x V(X^{\boldsymbol{\theta}}_t, t)dW_t\right] = 0$

$$\Downarrow$$

Hamilton-Jacobi-Bellman PDE

$$\partial_t V + \max_{\boldsymbol{\theta}\in\Delta^n}\left\{\left(\mu(\boldsymbol{\theta}) - \frac{1}{2}\sigma(\boldsymbol{\theta})^2\right)\partial_x V + \frac{1}{2}\sigma(\boldsymbol{\theta})^2\partial_x^2 V\right\} = 0$$

Riccati transformation of the HJB equation

- Riccati transformation
- Transformation of the Hamilton-Jacobi-Bellman equation to a quasi-linear parabolic PDE
- The role of the value function of a quadratic optimization problem

Introduce Riccati transformation

$$\varphi(x,t) = 1 - \frac{\partial_x^2 V(x,t)}{\partial_x V(x,t)}.$$

- notice that the function $a(x,t) \equiv \varphi(x,t) - 1$ can be viewed as the coefficient of absolute risk aversion for the intermediate utility function $V(x,t)$
- HJB equation (if $\partial_x V > 0$)

$$\partial_t V + \max_{\boldsymbol{\theta}\in\Delta^n} \left\{ \left(\mu(\boldsymbol{\theta}) - \frac{1}{2}\sigma(\boldsymbol{\theta})^2 \right) \partial_x V + \frac{1}{2}\sigma(\boldsymbol{\theta})^2 \partial_x^2 V \right\} = 0$$

can be rewritten as:

$$0 = \partial_t V - \alpha(\varphi)\, \partial_x V, \qquad V(x,T) := U(x),$$

where $\alpha(\varphi)$ is the value function of the

Parametric optimization problem:

$$\alpha(\varphi) = \min_{\boldsymbol{\theta}\in\Delta^n} \{ -\mu(\boldsymbol{\theta}) + \frac{\varphi}{2}\sigma(\boldsymbol{\theta})^2 \}\,.$$

Quasilinear parabolic equation for Riccati transformation

Theorem (S. Kilianová & D.Š., 2013)

Suppose that the value function V satisfies HJB Then the Riccati transform function φ is a solution to the Cauchy problem for the quasi-linear parabolic equation:

$$
\begin{aligned}
&\partial_t \varphi + \partial_x^2 \alpha(\varphi) + \partial_x f(\varphi) = 0, \quad x \in \mathbb{R}, t \in [0, T), \\
&\varphi(x, T) = 1 - U''(x)/U'(x), \quad x \in \mathbb{R}.
\end{aligned}
$$

where $f(\varphi) := \varepsilon e^{-x}\varphi + r + (1-\varphi)\alpha(\varphi)$

Parametric quadratic optimization problem

- Value function $\alpha(\varphi)$ as a minimizer of a parametric quadratic optimization problem
- Qualitative properties of the value function $\alpha(\varphi)$
- Counting discontinuities of $\alpha''(\varphi)$
- Generalizations for the case of worst case scenario

Parametric quadratic programming problem

The function $\alpha(\varphi)$ entering quasilinear HJB equation is a value function to the parametric quadratic convex programming problem

Value function

$$\alpha(\varphi) = \min_{\boldsymbol{\theta}\in\Delta^n} \{-\boldsymbol{\mu}^T\boldsymbol{\theta} + \frac{\varphi}{2}\boldsymbol{\theta}^T\boldsymbol{\Sigma}\,\boldsymbol{\theta}\}$$

over a compact (convex) set Δ^n, e.g.:

$$\Delta^n = \{\boldsymbol{\theta} \in \mathbb{R}^n \mid \boldsymbol{\theta}^i \geq 0,\ \sum_{i=1}^n \boldsymbol{\theta}^i = 1\}$$

or

$$\Delta^n = \{\boldsymbol{\theta} \in \mathbb{R}^n \mid \boldsymbol{\theta}^i \geq 0,\ \sum_{i=1}^n \boldsymbol{\theta}^i \leq 1\}$$

Merton model with a risk free asset with interest rate $r > 0$

Properties of the value function $\alpha(\varphi)$

For optimal portfolio selection problem we have

$$\alpha(\varphi) = \min_{\boldsymbol{\theta}\in\Delta^n} \{-\boldsymbol{\mu}^T\boldsymbol{\theta} + \frac{\varphi}{2}\boldsymbol{\theta}^T\Sigma\boldsymbol{\theta}\}$$

Theorem (S. Kilianová & D.Š., 2013)

If $\Sigma \succ 0$ (typical for a covariance matrix) and
Δ^n is compact, convex
then the function: $\varphi \mapsto \alpha(\varphi)$ is:

- *$C^{1,1}$ continuous in $\varphi > 0$;*
- *strictly increasing in φ for $\varphi > 0$*

What do we know about $\alpha(\varphi)$?

Example: German DAX index, 30 assets:

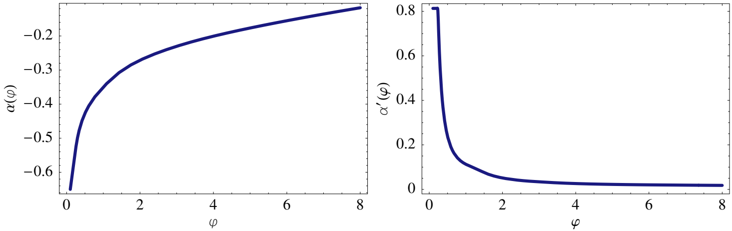


Figure : The function α and its first derivative for the German DAX30 stock index, based on historical data Aug 2010 - Apr 2012.

Properties of the value function $\alpha(\varphi)$

Quadratic parametric optimization problem

$$\alpha(\varphi) = \min_{\boldsymbol{\theta} \in \Delta^n} \{ -\boldsymbol{\mu}^T \boldsymbol{\theta} + \frac{\varphi}{2} \boldsymbol{\theta}^T \Sigma \boldsymbol{\theta} \}$$

Idea of the proof:

- $\alpha'(\varphi)$ it is the diffusion coefficient of the PDE for φ
- $\alpha'(\varphi) = \frac{1}{2} \hat{\boldsymbol{\theta}}^T \Sigma \hat{\boldsymbol{\theta}}$ where $\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}}(\varphi)$ is the unique minimizer
- $0 < \lambda^- \leq \alpha'(\varphi) \leq \lambda^+ < \infty$ for all $\varphi > 0$ and $t \in [0, T]$

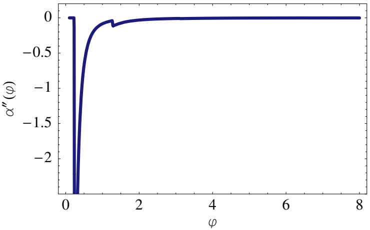
Tools:

- Milgrom-Segal envelope theorem (give C^1 smoothness of α)
- Klatte's results on Lipschitz continuity of the minimizer $\hat{\boldsymbol{\theta}}(\varphi)$

(S. Kilianová & D.Š., 2013)

Role of discontinuities of $\alpha''(\varphi)$

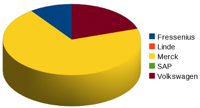
Points of discontinuity of the second derivative:



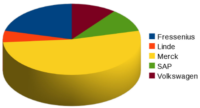
Intervals of φ:	$(0, 0.23)$	$(0.23, 1.27)$	$(1.27, 3.15)$	$(3.15, 6.62)$
Assets with $\theta^i \neq 0$:	$\{23\}$	$\{23, 30\}$	$\{16, 23, 30\}$	$\{16, 23, 27, 30\}$

We know which assets have nonzero weights, without solving PDE

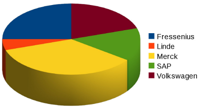
Recall the static Markowitz optimal selection problem for different φ:



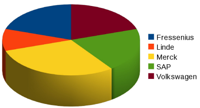
$\varphi \approx 1$



$\varphi \approx 4$

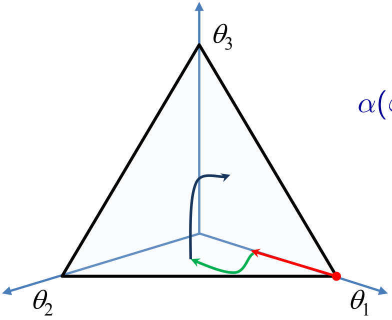


$\varphi \approx 6$



$\varphi \approx 8$

Explanation of discontinuities of $\alpha''(\varphi)$



$$\alpha(\varphi) = \min_{\boldsymbol{\theta} \in \Delta^n} \{ -\boldsymbol{\mu}^T \boldsymbol{\theta} + \frac{\varphi}{2} \boldsymbol{\theta}^T \Sigma \boldsymbol{\theta} \}$$

Minimizer $\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}}(\varphi)$ is only Lipschitz continuous in φ and $\alpha'(\varphi) = \frac{1}{2} \hat{\boldsymbol{\theta}}^T \Sigma \hat{\boldsymbol{\theta}}$

Increasing φ from $\varphi = 0$ to $\varphi \to \infty$

- small values of φ: only one asset with maximal mean return is active,
  $\theta_1 > 0, \theta_2 = \theta_3 = 0$
- intermediate values of φ: two assets are active,
  $\theta_1 > 0, \theta_2 > 0, \theta_3 = 0$
- large values of φ: all assets are active,
  $\theta_1 > 0, \theta_2 > 0, \theta_3 > 0$

Extensions of the model with different $\alpha(\varphi)$

ROBUST portfolio optimization

$$\alpha(\varphi) = \min_{\boldsymbol{\theta}\in\Delta^n} \max_{(\boldsymbol{\mu},\Sigma)\in\mathcal{K}} \{-\boldsymbol{\mu}^T\boldsymbol{\theta} + \frac{\varphi}{2}\boldsymbol{\theta}^T\Sigma\boldsymbol{\theta}\}$$

- $\mathcal{K}$ is a convex subset or a convex cone of mean return vectors and positive semidefinite covariance matrices
- it corresponds to the "worst case" variance optimization problem
- takes into account uncertainty in the covariance matrix and the mean return

Different behavior of the value function

For a nontrivial connected set $\mathcal{K}$ the value function $\alpha(\varphi)$ can be a linear function on some subintervals (Kilianová & Trnovská (2016))

Qualitative properties of solutions of the HJB equation

Qualitative properties of a solution

Theorem (Kilianová & Ševčovič 2013)

Assume that the terminal condition $\varphi_T(x)$, $x \in \mathbb{R}$, is positive and uniformly bounded for $x \in \mathbb{R}$ and belongs to the Hölder space $H^{2+\lambda}(\mathbb{R})$ for some $0 < \lambda < 1/2$ and α is $C^{1,1}$ smooth.

Then there exists a unique classical solution $\varphi(x,t)$ to

$$\begin{aligned} &\partial_t \varphi + \partial_x^2 \alpha(\varphi) + \partial_x f(\varphi) = 0, \quad x \in \mathbb{R}, t \in [0, T), \\ &\varphi(x, T) = \varphi_T(x). \end{aligned}$$

Moreover, $\partial_t \varphi, \partial_x \varphi$ is $\lambda/2$-Hölder continuous and Lipschitz continuous, and $\alpha(\varphi(.,.)) \in H^{2+\lambda, 1+\lambda/2}(\mathbb{R} \times [0, T])$.

- By the parabolic comparison principle we have:
  $0 < \varphi(x,t) \leq \sup \varphi(., T)$, for all t, x
- Existence of a unique classical solution to the Cauchy problem follows from Ladyzhenskaya, Solonnikov and Uralceva theory. It is based on regularization of the diffusion function α and solving the equation for $\eta = \alpha(\varphi)$

Numerical approximation scheme

Idea of finite volumes numerical discretization

General form of PDE to be solved (backward in time):

$$\partial_t\varphi + \partial_x^2 A(\varphi, x, t) + \partial_x B(\varphi, x, t) + C(\varphi, x, t) = 0,$$

- function values $\varphi(x_i, \tau^j)$ are approximated at grid points x_i by φ_i^j where $\tau^j = T - jk \quad$ (k is the time step)
- the first derivative is approximated at the dual mesh point $x_{i+\frac{1}{2}}$ = mid point of $[x_i, x_{i+1}]$,
  ($h = x_{i+1} - x_i$ is the spatial step)
- PDE equation is integrated over the dual volume $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$

Semi-implicit scheme: tridiagonal system

$$\underbrace{-\frac{k}{h^2}D^j_{i+\frac{1}{2}}\varphi^{j+1}_{i+1} + (1+\frac{k}{h^2}(D^j_{i+\frac{1}{2}} + D^j_{i-\frac{1}{2}}))\varphi^{j+1}_i - \frac{k}{h^2}D^j_{i-\frac{1}{2}}\varphi^{j+1}_{i-1}}_{\mathcal{A}(\varphi^j)\varphi^{j+1}}$$

$$= \underbrace{\frac{k}{h}(I^j_i + E^j_{i+\frac{1}{2}} - E^j_{i-\frac{1}{2}} + F^j_{i+\frac{1}{2}} - F^j_{i-\frac{1}{2}})}_{\mathcal{B}(\varphi^j)} + \varphi^j_i\,,$$

$$D^j_{i\pm\frac{1}{2}} = A'_\varphi(\varphi,x,\tau)\big|_{\varphi^j_{i\pm\frac{1}{2}},x_{i\pm\frac{1}{2}},\tau^j} \qquad E^j_{i\pm\frac{1}{2}} = A'_x(\varphi,x,\tau)\big|_{\varphi^j_{i\pm\frac{1}{2}},x_{i\pm\frac{1}{2}},\tau^j}$$

$$F^j_{i\pm\frac{1}{2}} = B(\varphi,x,\tau)\big|_{\varphi^j_{i\pm\frac{1}{2}},x_{i\pm\frac{1}{2}},\tau^j} \qquad I^j_i = hC(\varphi^j_i, x_i, \tau^j)$$

Tridiagonal system of linear equations

$$\mathcal{A}\varphi^{j+1} = \mathcal{B}(\varphi^j) + \varphi^j \qquad \mathcal{A} = \mathcal{A}(\varphi^j)$$

for the unknown vector $\varphi = (\varphi^{j+1}_1, \cdots, \varphi^{j+1}_N) \in \mathbb{R}^N$

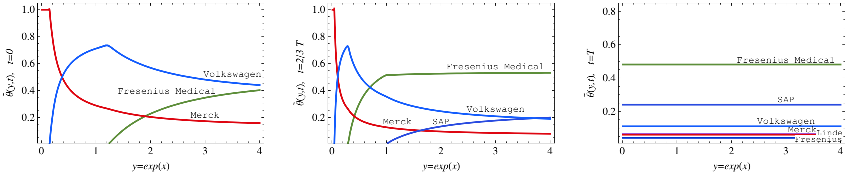
Computational results and applications to portfolio optimization

Optimal portfolio selection problem for

- DAX30 index with a compact simplex Δ^n
- Slovak pension system with
  - discrete decision set $\Delta^n, n = 2$

Example: investment problem for the DAX 30 Index

- Utility function: $U(x) = \frac{1}{1-a} e^{(1-a)x}$, with risk aversion $a = 9$, inflow and interest rate $\varepsilon = 1$, $r = 0$, time horizon $T = 10$
- $\varphi(x, T) = 1 - U''(x)/U'(x) \equiv a \quad \Rightarrow 0 < \varphi(x, t) \leq 9$ for all (x, t)
- Decision set: $\Delta^n = \{\boldsymbol{\theta} \in \mathbb{R}^n \mid \boldsymbol{\theta}^i \geq 0,\ \sum_{i=1}^n \boldsymbol{\theta}^i = 1\}$ $n = 30$.



$\varphi \in (0,9]$:	$\varphi \in$	active assets
1 - Adidas	$(0, 0.23)$	$\{23\}$
15 - Fresenius	$(0.23, 1.27)$	$\{23, \mathbf{30}\}$
16 - Fres Medical	$(1.27, 3.15)$	$\{\mathbf{16}, 23, 30\}$
21 - Linde	$(3.15, 6.62)$	$\{16, 23, \mathbf{27}, 30\}$
23 - Merck	$(6.62, 7.96)$	$\{16, \mathbf{21}, 23, 27, 30\}$
27 - SAP	$(7.96, 8.98)$	$\{\mathbf{15}, 16, 21, 23, 27, 30\}$
30 - Volkswagen	$(8.98, \ldots)$	$\{\mathbf{1}, 15, 16, 21, 23, 27, 30\}$

Kilianová & Ševčovič (2013)

In case of $n = 2$ assets the asymptotic analysis of PDE yields the first order approximation of the optimal stocks θ^1 to bonds $\theta^2 = 1 - \theta^1$ proportion :

$$\hat{\theta}_1(x,t) = C_0 + C_1 \frac{1}{a}\left[1 + \varepsilon \exp(-x)\frac{1 - e^{-\delta(T-t)}}{\delta}\right] + O(\varepsilon^2).$$

it means that the optimal stock to bonds proportion θ is a decreasing function with respect to time t as well as to the amount $y = \exp(x) > 0$ of yearly saved salaries, i.e.

Practical conclusions for policy makers

- higher amount of saved yearly salaries y
- closer time t to retirement T
- higher saver's risk aversion a
  $\Rightarrow$ **lower amount of risky assets (stocks) in the portfolio**
- higher defined yearly contribution ε
  $\Rightarrow$ **higher amount of risky assets (stocks) in the portfolio**

Slovak pension system - Second pillar limitations

Optimal weight decision set $n = 2$

Δ^n is the three element discrete set of funds

growth fund $\theta^{(s)} = 0.8,\ \theta^{(b)} = 0.2$
balanced fund $\theta^{(s)} = 0.5,\ \theta^{(b)} = 0.5$
conservative fund $\theta^{(s)} = 0,\ \theta^{(b)} = 1$

$$\Delta^n = \{\ (0.8, 0.2),\ \ (0.5, 0.5),\ \ (0, 1)\ \}$$

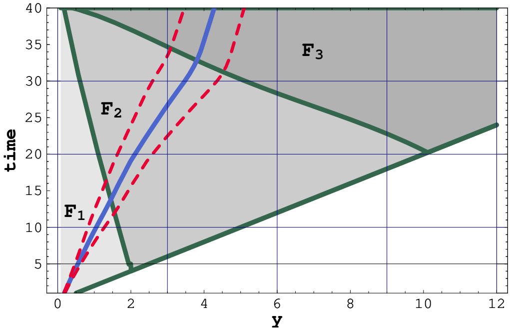
In general, if Δ^n is a discrete set then the function $\alpha(\varphi)$

$$\alpha(\varphi) = \min_{\boldsymbol{\theta}\in\Delta^n}\{-\boldsymbol{\mu}^T\boldsymbol{\theta} + \frac{\varphi}{2}\boldsymbol{\theta}^T\Sigma\boldsymbol{\theta}\}$$

need not be even C^1 smooth, and $\boldsymbol{\theta}(\varphi)$ is not continuous

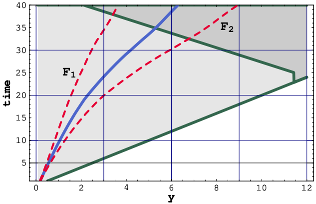
Example: $n = 1, \boldsymbol{\mu} = 1, \Sigma = 1, \Delta = \{0, 1\}$
$\Longrightarrow \alpha(\varphi) = \min\{-1 + \varphi/2,\ 0\}$ is only Lipschitz continuous !!!

Computational results

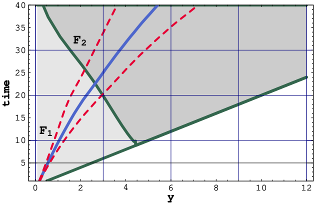


- risk aversion $a = 9$ for utility function $U(x) = \frac{1}{1-a} e^{(1-a)x}$
- $n = 2$ (Stocks and Bonds) with $\Delta^n = \{(0.8, 0.2),\ (0.5, 0.5),\ (0, 1)\}$ for the Slovak pension system
- Conservative – F3, Balanced – F2, Growth – F1 fund
- Results of 10000 Monte-Carlo simulations of the path y_t. Mean $E(y_t)$ and $\pm\sigma(y_t)$

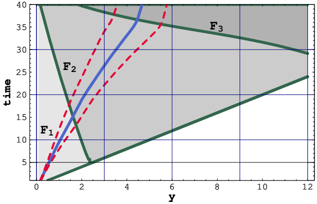
Dynamic stochastic accumulation model
Computational results – Various risk aversions



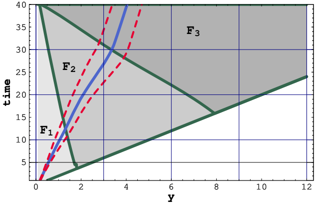
$a = 3$



$a = 8$



$a = 8$



$a = 10$

- Higher risk aversion $\Rightarrow$ earlier transition to less risky funds and lower expected terminal value of savings

Kilianová, Melicherčík, Ševčovič (2006)

Conclusions

- HJB equation arises dynamic stochastic portfolio selection problem
- HJB equation can be transformed to a quasilinear PDE and solved numerically
- important role is played by smoothness properties of the value function of a convex quadratic optimization problem

Thank you for your attention

1 S. Kilianová and D. Ševčovič: *A Method of Solving Hamilton-Jacobi-Bellman Equation for Constrained Optimal Investment Problem via Riccati Transformation*, ANZIAM Journal (55) 2013, 14-38.

2 Z. Macová, D. Ševčovič: *Weakly nonlinear analysis of the Hamilton -Jacobi -Bellman equation arising from pension savings management*, International Journal of Numerical Analysis and Modeling, 7(4) 2010, 619-638

3 S. Kilianová, I. Melicherčík, D. Ševčovič: Dynamic Accumulation Model for the Second Pillar of the Slovak Pension System, Czech Journal of Economics and Finance, 56 (11-12), 2006, 506-521

4 S. Kilianová and M. Trnovská: Robust Portfolio Optimization via solution to the Hamilton-Jacobi-Bellman Equation, Int. Journal of Computer Mathematics, 93, 2016, 725-734.

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