

# Linear SEE's with additive noise of Volterra type

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# The problem

# The stochastic Cauchy problem

$$\begin{aligned}dX_t &= AX_t dt + \Phi dB_t \quad t \in [0, T] \\ X_0 &= x\end{aligned}$$

$U, V$	...	real, separable Hilbert spaces
$A : \text{Dom}(A) \subset V \rightarrow V$	...	generates a $C_0$ -semigroup of operators on $V$
$B = (B_t, t \in [0, T])$	...	infinite-dimensional noise in $U$
$\Phi \in \mathcal{L}(U, V)$	...	diffusion term
$x \in V$	...	initial condition

# Typical Examples

## Stochastic heat equation:

$$\partial_t u = \Delta u + \eta \quad \text{on } [0, T] \times \mathcal{O}$$

$$u(0, \cdot) = f \quad \text{on } \mathcal{O}$$

$$u|_{[0, T] \times \partial \mathcal{O}} = 0$$

## Stochastic wave equation:

$$\partial_{tt}^2 u = \Delta u + \eta \quad \text{on } [0, T] \times \mathcal{O}$$

$$u(0, \cdot) = f_1 \quad \text{on } \mathcal{O}$$

$$\partial_t u(0, \cdot) = f_2 \quad \text{on } \mathcal{O}$$

$$u|_{[0, T] \times \partial \mathcal{O}} = 0$$

- $\mathcal{O} \subset \mathbb{R}^d$  ... bounded domain with smooth boundary
- $f, f_1, f_2$  ... given functions
- $\eta$  ... space-time noise process

# Mild solution

Stochastic evolution equation (SEE):

$$dX_t = AX_t dt + \Phi dB_t \quad t \in [0, T]$$

$$X_0 = x$$

Mild solution:

$$X_t = S(t)x + \int_0^t S(t-r)\Phi dB_r \quad t \in [0, T]$$

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- ▶ give meaning to the convolution integral
- ▶ prove that  $X = (X_t, t \in [0, T])$  has measurable sample paths
- ▶ prove that  $X$  has continuous sample paths

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But how to define the integral when  $B$  is neither a semimartingale, nor a Gaussian process?

# Volterra processes

$K$  ... so-called regular Volterra kernel (see poster for the definition)

## Definition

A stochastic process  $b = (b_t, t \in [0, T])$  is called a Volterra process if

- (i)  $b$  is centered with  $b_0 = 0$
- (ii)  $b$  is  $\varepsilon$ -Hölder continuous for every  $\varepsilon \in (0, \delta)$  for some  $\delta > 0$
- (iii) the covariance of  $b$  is

$$R(s, t) = \int_0^{s \wedge t} K(s, r)K(t, r)dr, \quad s, t \in [0, T].$$



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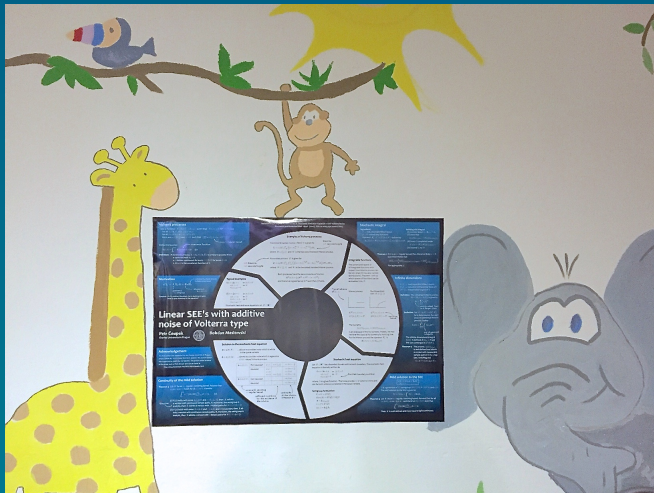
$$R(s, t) = \int_0^{s \wedge t} K(s, r)K(t, r)dr, \quad s, t \in [0, T].$$

Examples:

- ▶ fractional Brownian motion of  $H > \frac{1}{2}$
- ▶ Rosenblatt process

# The poster

# Where to find the poster



# How to read the poster

### Volterra processes

Take a function  $X: [0, T] \rightarrow \mathbb{R}$  such that  $X(t) = 0$  for  $t < 0$  and for all  $f \in C_c^1(\mathbb{R})$ ,  $\int_0^T f(t) dX(t) = 0$ . For all  $s < t < T$ ,  $\text{Cov}(X(s), X(t)) = 0$ . There are  $\alpha \in (0, \frac{1}{2})$  and  $\beta > 0$  such that  $|\text{Cov}(X(s), X(t))| \leq \beta |t-s|^{2\alpha}$  [4].

Define the function  $K(t, s) = \int_0^s (t-u)^{\alpha-1} (s-u)^{\alpha-1} du$ . This is a regular kernel.

**Definition:** A stochastic process  $X = (X_t)_{t \in [0, T]}$  is a Volterra process if it is contained within  $\mathcal{H}_K$ .

- holder continuous for  $\alpha$  if  $\beta > 0$  for some  $\alpha > 0$
- $K(t, s)$  is the covariance function of  $X$ .

### Stochastic Integral

**Ingredients**

- $\mathcal{Y} = \mathbb{H}$ , separable Hilbert space
- $\mathcal{X} = \mathbb{T}$ , real separable functions
- Operator  $\mathcal{K}: \mathcal{X} \rightarrow \mathcal{L}(\mathcal{Y}, \mathcal{Y})$  defined by  $(\mathcal{K}f)(t) = X(t) \int_0^t f(s) ds$

**Building the Integral**

- Consider linear  $\mathcal{H} = \mathcal{H}_K$
- Let  $\mathcal{H} = \mathcal{H}_K$
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**Theorem 1:** If  $\mathcal{K}$  is an  $\alpha$ -regular kernel, then there is a Fokier  $\nu > 0$  such that it holds that  $\int_0^T \int_0^T |\mathcal{K}(t, s)|^2 ds dt < \infty$  for appropriate  $f$ .

### Integrable functions

The constructed space  $\mathcal{H}$  of integrable functions with respect to a Volterra process can be very large (it may also contain distributions). Theorem 1 tells us which spaces of functions can be embedded into  $\mathcal{H}$ .

### Examples of Volterra processes

**Fractional Brownian motion (fBm)**  $X$  is given by  $X_t = (t^\alpha) \int_0^t (t-s)^{\alpha-1} dW_s$  where  $\alpha \in (\frac{1}{2}, 1)$  and  $W$  is the two-sided standard Wiener process.

**Gaussian  $\alpha$ -stable Lévy flight**

**Random process  $X$  is given by**  $X_t = \int_0^t \int_0^s (t-u)^{\alpha-1} (s-u)^{\alpha-1} dW_u dW_s$  where  $\alpha \in (\frac{1}{2}, 1)$  and  $W$  is the two-sided standard Wiener process.

Both processes have the same covariance function  $K^{\alpha, \alpha}(t, s) = \frac{1}{2} (t^\alpha + s^\alpha - |t-s|^\alpha)$  and there is a regular kernel  $K^\alpha$  such that (1) holds.

### Motivation

$dX_t = A_t X_t dt + \Phi_t dW_t$

**Question:** If the random disturbance, the  $\Phi$  is a martingale, how to give meaning to the product  $\Phi_t dW_t$ ?

### Stochastic heat equation

Let  $\mathcal{O} \subset \mathbb{R}^d$ .  $\Delta u = \partial_x^2 u = 0$  or  $\partial_t^2 u = 0$

**Typical examples**

- $\Delta u = \partial_x^2 u = 0$  or  $\partial_t^2 u = 0$
- $\partial_t^2 u = \partial_x^2 u = 0$
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Stochastic heat and wave equations on  $\mathcal{O} \subset \mathbb{R}^d$ .

### Infinite dimensions

$\mathcal{H} = \mathbb{H}$ , separable Hilbert spaces

**Definition:** The infinite-dimensional process  $X = (X_t)_{t \in [0, T]}$  is given by  $X_t = \sum_{k=1}^{\infty} \langle X, e_k \rangle e_k$

**Definition:** Let  $G: [0, T] \rightarrow \mathcal{L}(\mathcal{H}, \mathcal{H})$  be a deterministic function and  $\mathcal{H}$  a Hilbert space. The process  $X$  is called a Volterra process, before  $G(t) = \int_0^t \int_0^s G(t, s) ds dt$

**Theorem 3:** The process  $(X_t)_{t \in [0, T]}$  is well-defined and admits a version with measurable sample paths if  $\mathcal{H}$  is separable,  $\nu$ -regular and  $\int_0^T \int_0^T \|G(t, s)\|_{\mathcal{L}(\mathcal{H}, \mathcal{H})}^2 ds dt < \infty$ .

### Linear SEE's with additive noise of Volterra type

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### Solution to the stochastic heat equation

$\mathcal{O} \subset \mathbb{R}^d$  ... allows to consider noise which is white in the space variable

$\mathcal{O} \subset \mathbb{R}^d$  ... allows to consider noise which is a genuine  $L^2(\mathbb{T})$ -valued Volterra process

$\mathcal{O} \subset \mathbb{R}^d$	non-Gaussian	$\alpha \in (\frac{1}{2}, 1)$	$\mathcal{O} \subset \mathbb{R}^d$	$\mathcal{O} \subset \mathbb{R}^d$
$\mathcal{O} \subset \mathbb{R}^d$	Gaussian	$\alpha \in (\frac{1}{2}, 1)$	$\mathcal{O} \subset \mathbb{R}^d$	$\mathcal{O} \subset \mathbb{R}^d$
$\mathcal{O} \subset \mathbb{R}^d$	non-Gaussian	$\alpha \in (\frac{1}{2}, 1)$	$\mathcal{O} \subset \mathbb{R}^d$	$\mathcal{O} \subset \mathbb{R}^d$
$\mathcal{O} \subset \mathbb{R}^d$	Gaussian	$\alpha \in (\frac{1}{2}, 1)$	$\mathcal{O} \subset \mathbb{R}^d$	$\mathcal{O} \subset \mathbb{R}^d$

noise with  $\alpha$ -regular  $\alpha$ -regular kernel

applied conditions for the existence of the solution

conditions all the solution (Theorem 4)

### Stochastic heat equation

Let  $\mathcal{O} \subset \mathbb{R}^d$  be a bounded domain with smooth boundary. The stochastic heat equation is formally written as  $\Delta u = \partial_x^2 u = 0$  or  $\partial_t^2 u = 0$

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where  $f$  is a given function. The noise process  $\nu$  is Volterra in time and can be both white or correlated in the space variable.

**Semigroup Formulation:**

- $\mathcal{O} = \mathbb{T}$
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- $\mathcal{O} = \mathbb{T}$

**Theorem 5:** Let  $\mathcal{K}$  be an  $\alpha$ -regular, vanishing kernel. Assume that for all  $t \in [0, T]$ ,  $\mathcal{K}(t, s) \in \mathcal{L}(\mathcal{H}, \mathcal{H})$  and there is a  $\nu > 0$  such that  $\int_0^T \int_0^T \|\mathcal{K}(t, s)\|_{\mathcal{L}(\mathcal{H}, \mathcal{H})}^2 ds dt < \infty$ . Then  $X$  is well-defined and mean-square right continuous.

Thank you for your 10 minutes and please, stop by!