Ergodic Control for Lévy-driven linear stochastic equations in Hilbert spaces.

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Contents.

Functional analysis notions.

1. $\mathbb{H}$, $\mathbb{Y}$ real separable Hilbert spaces.
2. $\{e_i; i \in I\}$, $I \neq \emptyset$, $I \subset \mathbb{N}$ an (arbitrary) orthonormal basis of $\mathbb{H}$.
3. $A \in \mathcal{L}(\mathbb{H}, \mathbb{Y})$ the Hilbert-Schmidt operator iff

$$|A|^2_{HS(\mathbb{H},\mathbb{Y})} = \sum_{i \in I} |Ae_i|^2_{\mathbb{Y}} < \infty,$$

4. A^* the adjoint operator of $A \in \mathcal{L}(\mathbb{H}, \mathbb{Y})$ iff "$\langle Ax, y\rangle_{\mathbb{Y}} = \langle x, A^* y\rangle_{\mathbb{H}}$"
    1. ($\mathbb{D}(A^*)$ is the set of $y \in \mathbb{Y}$ for which exists $z \in \mathbb{H}$ such that for all $x \in \mathbb{D}(A)$: $\langle Ax, y\rangle_{\mathbb{Y}} = \langle x, z\rangle_{\mathbb{H}}$ and then $A^* y = z$),
5. The operator $A : \mathbb{H} \to \mathbb{H}$ selfadjoint iff $A^* = A$.

1. $S : \mathbb{R}_+ \to \mathcal{L}(\mathbb{H})$ the C_0-semi-group:
   1. $S(t+s) = S(t)S(s)$, $t, s \geq 0$, $S(0) = I$,
   2. $S(t)x \to x$ strongly, $t \to 0_+$, $x \in \mathbb{H}$,
2. A is the infinitesimal generator of the C_0-semi-group S iff

$$\frac{S(t)x - x}{t} \to Ax, \quad t \to 0_+,$$

   for $x \in \mathbb{D}(A)$ (the domain of A), where $\mathbb{D}(A)$ is the set of all $x \in \mathbb{H}$ for which the limit exists.
   1. $S(t)x$ is the solution to the equation $\dot{y} = Ay$, $y(0) = x \in \mathbb{D}(A)$.
   2. S analytic if S can be extended to $\{z \in \mathbb{C}, |\arg(z)| < \theta\}$ for some $\theta \in (0, \frac{\pi}{2})$.

Stochastic basis $(\Omega, \mathcal{A}, \mathcal{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$.

1. L Lévy process (in $\mathbb{H}$):
   1. indexed $\mathbb{R}_+$,
   2. stationary independent increments,
   3. $L(0) = 0_{\mathbb{H}}$,
   4. stochastically continuous.
2. L Cylindrical Lévy process (in $\mathbb{H}$) if for all $h \in \mathbb{H}$ is $\langle L, h\rangle_{\mathbb{H}}$ Lévy process in $\mathbb{R}$.
3. In our special case L has
   1. weak second moments (for all $h \in \mathbb{H}$ has $\langle L, h\rangle_{\mathbb{H}}$ finite second moments),
   2. characteristic function

$$\mathbb{E}e^{i\langle L(t),h\rangle_{\mathbb{H}}} = e^{\int_{\mathbb{H}}(e^{i\langle z,h\rangle_{\mathbb{H}}}-1-i\langle z,h\rangle_{\mathbb{H}})d\nu(z)}, \quad h \in \mathbb{H},$$

   where ν is cylindrical measure (measure on cylindrical sets).

1. We define the stochastic integral of simple process of Hilbert-Schmidt operators

$$\Phi(t) = \sum_{k=0}^{n-1} \Phi_k \mathbf{I}_{(t_k, t_{k+1}]}(t)$$

as

$$I(\Phi)(t) = \sum_{k=0}^{n-1} \Phi_k (L(\min\{t_{k+1}, t\}) - L(\min\{t_k, t\})),$$

where $0 \leq t_0 < \ldots < t_n \leq T$.

2. For general process of Hilbert-Schmidt operators Φ such that $\mathbf{E} \int_0^T |\Phi(s)|_{HS}^2 ds < \infty$ we can approximate $I(\Phi)(t)$ by $I(\Phi_k)(t)$, $k \in \mathbb{N}$, in $\mathbf{E}|\cdot|_{\mathbb{H}}^2$ where Φ_k, $k \in \mathbb{N}$, approximate Φ in $\mathbf{E} \int_0^T |\cdot|_{HS}^2 ds$.

Controlled SEE.

1. $$dX_t^U = (AX_t^U + BU_t)dt + \Phi dL_t, \quad X_0^U = x, \qquad (1)$$

   where
   1. A the infinitesimal generator of the exponentially stable analytic semi-group S such that for fixed $\beta \geq 0$ and $\delta \in (0, \frac{1}{2})$: $\Phi^*(-A^* + \beta \mathbf{I})^{-\frac{1}{2}+\delta}$ is Hilbert-Schmidt.
   2. $B \in \mathcal{L}(\mathbb{Y}, \mathbb{D}_A^{\epsilon-1} = \mathcal{D}((-A + \beta \mathbf{I})^{\epsilon-1}))$, where $\epsilon \in (0, 1)$.
   3. U $\mathcal{F}$-progressively measurable control from $\mathbb{L}^{p,loc}(\mathbb{R}_+, \mathbb{Y})$ for fixed $p > \max\{2, \frac{1}{\epsilon}\}$ ($\mathcal{U}$ denotes the space of controls).
2. X^U Strong solution:
   1. $\int_0^T |AX_s^U|_{\mathbb{H}} ds < \infty$ **P**-s.j.,
   2. $X_t^U \in \mathbb{D}(A)$ **P**-s.j.,
   3. $$X_t^U = x + \int_0^t (AX_s^U + BU_s)ds + \Phi L_t, \quad t > 0.$$
   4. Assumptions for A too restrictive. Less strict concepts of solutions (avoiding A) needed.

Controlled SEE.

1. X^U Mild solution:

$$X_t^U = x + \int_0^t S(t-s)BU_s ds + \int_0^t S(t-s)\Phi dL_s, \quad t \geq 0. \tag{2}$$

2. X^U Weak solution:

$$\langle a, X_t^U \rangle_{\mathbb{H}}$$

$$= \langle a, x\rangle_{\mathbb{H}} + \int_0^t \langle A^* a, X_s^U \rangle_{\mathbb{H}} ds + \int_0^t \langle B^* a, U_r \rangle_{\mathbb{H}} dr + \langle a, \Phi L_t \rangle_{\mathbb{H}},$$

   1. $a \in \mathcal{D}(A^*)$, $t \in \mathbb{R}_+$.

3. In our case, X^U weak solution iff X^U mild solution and both exist and are unique in the space $\mathbb{L}_{\mathcal{F}}^{2,loc}(\mathbb{R}_+, \mathbb{H})$.

Ergodic control

1. $$J(U,T) = \int_0^T (\langle QX_s^U, X_s^U\rangle_{\mathbb{H}} + \langle RU_s, U_s\rangle_{\mathbb{Y}})ds, \qquad (3)$$
    1. $Q \in \mathcal{L}(\mathbb{H})$ symmetric positive semi-definite operator,
    2. $R \in \mathcal{L}(\mathbb{Y})$ symmetric positive definite operator.
2. The issue is to find $C \in \mathbb{R}$ (optimal cost) such that for all $U \in \mathcal{L}$ **P**-a.s. (or in mean)
$$\liminf_{t\to\infty} \frac{J(U,t)}{t} \geq C,$$
and $U_0 \in \mathcal{L}$ (optimal control) such that **P**-a.s. (or in mean)
$$\lim_{t\to\infty} \frac{J(U_0,t)}{t} = C.$$
3. Denote V solution to the stationary Riccati equation
$$VA + A^*V + Q - VBR^{-1}B^*V = 0.$$

Main results.

1. (Itoo formula.) Let $Tr(V\Phi\Phi^*) < \infty$, $V \in \mathcal{L}(\mathbb{H}, \mathcal{D}_{A^*}^{1-\epsilon})$ is non-negative and self-adjoint on $\mathbb{H}$. Then
$$\left\langle X^U(t), VX^U(t) \right\rangle_{\mathbb{H}} - \langle x, Vx \rangle_{\mathbb{H}}$$
$$= 2\int_0^t h(X^U(s))ds + \int_0^t 2\left\langle B^* VX^U(s), U(s) \right\rangle_{\mathbb{H}} ds$$
$$+ \int_{\mathbb{H}} \left\langle \Phi^* VX^U(s_-), dL(s) \right\rangle_{\mathbb{H}} + \sum_{s \le t} \left| \Delta V^{\frac{1}{2}} \Phi L(s) \right|_{\mathbb{H}}^2, \quad a.s.$$
2. (Optimal control and optimal cost.) Suppose that
$$\frac{\langle VX_t^U, X_t^U \rangle_{\mathbb{H}}}{t} \to 0, \quad t \to \infty, \quad a.s., \tag{4}$$
$$\lim_{t\to\infty} \sup \frac{\int_0^t |X_s^U|_{\mathbb{H}}^2 ds}{t} < \infty \quad a.s. \tag{5}$$
Then $C = Tr(V\Phi\Phi^*)$ and $U_0 = -R^{-1}B^*VX$.

Example.

1.

$$w_{tt}(t,x) - \Delta w_t(t,x) + \Delta^2 w(t,x) = \mathbb{I}_{x=x_0} u(t) + l(t,x), \quad (t,x) \in \mathbb{R}_+ \tag{6}$$

   1.

   $$w(0,x) = w_0, \quad w_t(0,x) = w_1, \quad x \in G,$$

   2.

   $$w(t,x) = w_t(t,x) = 0, \quad (t,x) \in \mathbb{R}_+ \times \partial G,$$

   where $G \subset \mathbb{R}^n$, $n \in \{1,2,3\}$, $x_0 \in G$, l formally represents Lévy noise.

2.

$$J(u,T) = \int_0^T (|w(t)|^2_{H^2(G)} + |w_t(t)|^2_{\mathbb{L}(G)} + |u(t)|^2) dt,$$

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Thank you.