# Parameter estimation for stochastic partial differential equations of second order

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# Introduction

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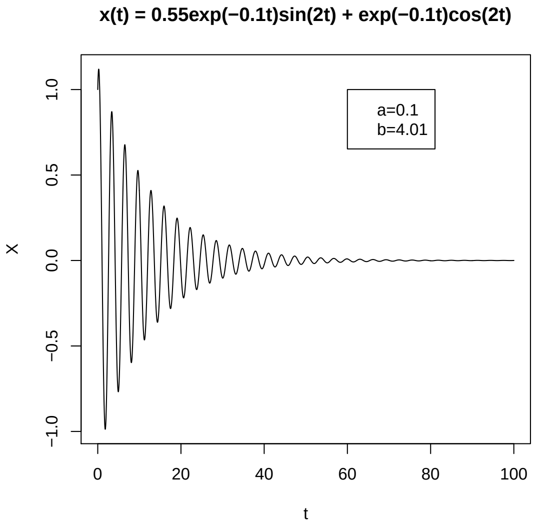
- Solve the characteristic equation $\lambda^2 + 2a\lambda + b = 0$.
- If $a^2 - b < 0$, then $\lambda_{1,2} = -a \pm i\sqrt{b - a^2}$ and the solution is

$$x(t) = C_1 e^{-at} \sin\left(\sqrt{b - a^2}\, t\right) + C_2 e^{-at} \cos\left(\sqrt{b - a^2}\, t\right).$$

# Introduction



# Introduction

- Now we add a noise part and introduce the stochastic differential equation

$$\ddot{x} + 2a\dot{x} + bx = \sigma \dot{B}(t),$$

  where $\dot{B}(t)$ is the formal time derivative of the standard Brownian motion and $\sigma > 0$.

- Since the previous equation is a linear equation, it admits a mild solution, which is called the Ornstein–Uhlenbeck process.

# Introduction

- Consider the following wave equation

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2}(t,\xi) &= bAu(t,\xi) - 2a\frac{\partial u}{\partial t}(t,\xi) + Q^{\frac{1}{2}}\dot{B}(t,\xi),\ (t,\xi) \in \mathbb{R}_+ \times D, \\ u(0,\xi) &= u_1(\xi), \quad \xi \in D, \\ \frac{\partial u}{\partial t}(0,\xi) &= u_2(\xi), \quad \xi \in D, \\ u(t,\xi) &= 0, \quad (t,\xi) \in \mathbb{R}_+ \times \partial D, \end{aligned}$$

  where $D \subset \mathbb{R}^d$ is a bounded domain with a smooth boundary, $a > 0$, $b > 0$ are unknown parameters and the $\dot{B}(t,\xi)$ is the formal time derivative of a space dependent Brownian motion.

- Based on the observation of trajectory of process $\{X(t) = (u(t,\cdot), \frac{\partial u}{\partial t}(t,\cdot))^\top, 0 \le t \le T\}$, the strong consistent estimators of parameters $a$ and $b$ will be proposed.

# Assumptions

- Assume that $\{e_n,\ n \in \mathbb{N}\}$ is an orthonormal basis in $L^2(D)$ and the operator $A : \mathrm{Dom}(A) \subset L^2(D) \to L^2(D)$ is such that
  (i) $Ae_n = -\alpha_n e_n$,
  (ii) $\forall n \in \mathbb{N} \quad \alpha_n > 0$,
  (iii) $\alpha_n \to \infty$ for $n \to \infty$.
- These assumptions cover the case, that if the set $D \subset \mathbb{R}^d$ is open, bounded and with a smooth boundary, then the operator $A = \Delta|_{\mathrm{Dom}(A)}$ and $\mathrm{Dom}(A) = H^2(D) \cap H_0^1(D)$.

# Assumptions

- Assume that the operator $Q$ is a symmetric positive nuclear operator in $L^2(D)$. Then there exists an orthonormal basis $\{e_n', n \in \mathbb{N}\}$ of $L^2(D)$ consisting of eigenvectors of $Q$, that is
  (iv) $Qe_n' = \lambda_n e_n'$,
  (v) $\forall n \in \mathbb{N} \quad \lambda_n > 0$,
  (vi) $\sum_{n=1}^{\infty} \lambda_n < \infty$.
- We consider "non–diagonal case". It means that the eigenvectors $\{e_n', n \in \mathbb{N}\}$ of the operator $Q$ are not necessarily the same as the eigenvectors $\{e_n, n \in \mathbb{N}\}$ of the operator $A$.

# General setting

- This problem may be rewritten as an infinite dimensional stochastic differential equation

$$dX(t) = \mathcal{A}X(t)\,dt + \Phi\,dB(t), \tag{1}$$
$$X(0) = x_0 = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

- To this aim, introduce the Hilbert space $V = \mathrm{Dom}((-A)^{\frac{1}{2}}) \times L^2(D)$ endowed with the norm

$$\left\| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\|_V^2 = \|x_1\|^2_{\mathrm{Dom}((-A)^{\frac{1}{2}})} + \|x_2\|^2_{L^2(D)}$$
$$= \left\|(-A)^{\frac{1}{2}} x_1\right\|^2_{L^2(D)} + \|x_2\|^2_{L^2(D)}. \tag{2}$$

## General setting

- Define the linear operator $\mathcal{A}$:

$$\mathcal{A}x = \mathcal{A}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & I \\ bA & -2aI \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

$$\forall\, x \in \mathrm{Dom}(\mathcal{A}) = \mathrm{Dom}(A) \times \mathrm{Dom}((-A)^{\frac{1}{2}}).$$

- Also define the linear operator $\Phi$ in $V$ as follows

$$\Phi = \begin{pmatrix} 0 & 0 \\ 0 & Q^{\frac{1}{2}} \end{pmatrix}.$$

## Semigroup $S(t)$

- Assume that
$$\forall\, n \in \mathbb{N} \quad a^2 - b\alpha_n < 0. \tag{3}$$
- Under this assumption, the eigenvalues $\{l_n,\ n \in \mathbb{N}\}$ of the operator $\mathcal{A}$ equal to
$$l_n^{1,2} = -a \pm i\sqrt{b\alpha_n - a^2}$$
and the operator $\mathcal{A}$ generates a $C_0$–semigroup in $V$, which has the following form.

# Semigroup $S(t)$

**Lemma**

*For all* $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in V$, *the semigroup* $S(t)$ *equals to*

$$S(t)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} s_{11}(t) & s_{12}(t) \\ s_{21}(t) & s_{22}(t) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

*where*

$$\begin{aligned} s_{11}(t) &= e^{-at}\left(\cos(\beta t) + a\beta^{-1}\sin(\beta t)\right), \\ s_{12}(t) &= e^{-at}\beta^{-1}\sin(\beta t), \\ s_{21}(t) &= e^{-at}\left(-\beta - a^2\beta^{-1}\right)\sin(\beta t), \\ s_{22}(t) &= e^{-at}\beta^{-1}\left(-a\sin(\beta t) + \beta\cos(\beta t)\right). \end{aligned}$$

## Semigroup $S(t)$

- The operator $\beta : L^2(D) \to L^2(D)$ in the previous formulae is defined by $\beta = (-bA - a^2 I)^{\frac{1}{2}}$.
- All operators are defined by their respective series. For example

$$\beta x = \sum_{n=1}^{\infty} \sqrt{b\alpha_n - a^2} \langle x, e_n \rangle_{L^2(D)} e_n,$$

$$\sin(\beta t)x = \sum_{n=1}^{\infty} \sin\left(\sqrt{b\alpha_n - a^2}\, t\right) \langle x, e_n \rangle_{L^2(D)} e_n,$$

where $x \in L^2(D)$ are from their respective domains.

# Covariance operator $Q_\infty^{(a,b)}$

- The equation (1) is a linear equation $\Rightarrow$ there exists a mild solution $X(t)$.
- The semigroup $S(t)$ is exponentially stable $\Rightarrow$ there exists an invariant measure $\mu_\infty^{(a,b)}$, which fullfils $\mu_\infty^{(a,b)} = N\left(0, Q_\infty^{(a,b)}\right)$.
- The covariance operator $Q_\infty^{(a,b)}$ of the limit measure $\mu_\infty^{(a,b)}$ satisfies

$$Q_\infty^{(a,b)} = \int_0^\infty S(t)\Phi\Phi^* S^*(t)\, dt. \tag{4}$$

## Covariance operator $Q_\infty^{(a,b)}$

- We need to compute

$$Q_\infty^{(a,b)} = \int_0^\infty \begin{pmatrix} q_{11}(t) & q_{12}(t) \\ q_{21}(t) & q_{22}(t) \end{pmatrix} dt,$$

where

$$\begin{aligned}
q_{11}(t) &= e^{-2at}\beta^{-1}\sin(\beta t)Q(-A)^{\frac{1}{2}}\beta^{-1}\sin(\beta t)(-A)^{\frac{1}{2}}, \\
q_{12}(t) &= e^{-2at}\beta^{-1}\sin(\beta t)Q\left(\cos(\beta t) - a\beta^{-1}\sin(\beta t)\right), \\
q_{21}(t) &= e^{-2at}\left(\cos(\beta t) - a\beta^{-1}\sin(\beta t)\right)Q\times \\
&\quad \times (-A)^{\frac{1}{2}}\beta^{-1}\sin(\beta t)(-A)^{\frac{1}{2}}, \\
q_{22}(t) &= e^{-2at}\left(\cos(\beta t) - a\beta^{-1}\sin(\beta t)\right)Q\times \\
&\quad \times \left(\cos(\beta t) - a\beta^{-1}\sin(\beta t)\right).
\end{aligned}$$

## Covariance operator $Q_\infty^{(a,b)}$

- The expansion for the operator $q_{11}(t)x_1$ is the following

$$
\begin{aligned}
q_{11}(t)x_1 &= e^{-2at}\beta^{-1}\sin(\beta t)Q\sum_{n=1}^{\infty}\alpha_n \frac{\sin\left(\sqrt{b\alpha_n - a^2}\, t\right)}{\sqrt{b\alpha_n - a^2}} \times \\
&\quad \times \langle x_1, e_n\rangle_{L^2(D)}\, e_n \\
&= e^{-2at}\beta^{-1}\sin(\beta t)\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\alpha_n \frac{\sin\left(\sqrt{b\alpha_n - a^2}\, t\right)}{\sqrt{b\alpha_n - a^2}} \times \\
&\quad \times \langle Qe_n, e_k\rangle_{L^2(D)} \langle x_1, e_n\rangle_{L^2(D)}\, e_k \\
&= e^{-2at}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\alpha_n \frac{\sin\left(\sqrt{b\alpha_n - a^2}\, t\right)}{\sqrt{b\alpha_n - a^2}} \frac{\sin\left(\sqrt{b\alpha_k - a^2}\, t\right)}{\sqrt{b\alpha_k - a^2}} \times \\
&\quad \times \langle Qe_n, e_k\rangle_{L^2(D)} \langle x_1, e_n\rangle_{L^2(D)}\, e_k .
\end{aligned}
$$

## Covariance operator $Q_\infty^{(a,b)}$

### Lemma

*The covariance operator $Q_\infty^{(a,b)}$ equals to*

$$Q_\infty^{(a,b)} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{\langle Qe_n, e_k \rangle_{L^2(D)}}{b^2(\alpha_n - \alpha_k)^2 + 8a^2 b(\alpha_n + \alpha_k)} \times \tag{5}$$

$$\begin{pmatrix} 4a\alpha_n \langle x_1, e_n \rangle_{L^2(D)} e_k + b(\alpha_k - \alpha_n) \langle x_2, e_n \rangle_{L^2(D)} e_k \\ b\alpha_n(\alpha_n - \alpha_k) \langle x_1, e_n \rangle_{L^2(D)} e_k + 2ab(\alpha_n + \alpha_k) \langle x_2, e_n \rangle_{L^2(D)} e_k \end{pmatrix},$$

*for any* $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in V$.

## Covariance operator $Q_\infty^{(a,b)}$

- If we consider "diagonal case", i.e., $Qe_n = \lambda_n e_n$ for orthonormal basis $\{e_n, n \in \mathbb{N}\}$ in $L^2(D)$, the covariance operator $Q_\infty^{(a,b)}$ will take the form

$$Q_\infty^{(a,b)} = \begin{pmatrix} \frac{1}{4ab}\,Q & 0 \\ 0 & \frac{1}{4a}\,Q \end{pmatrix}. \tag{6}$$

## What if $a^2 - b\alpha_n \geq 0$?

- If
$$\exists\, n \in \mathbb{N} \quad a^2 - b\alpha_n > 0, \tag{7}$$
or
$$\exists\, n \in \mathbb{N} \quad a^2 - b\alpha_n = 0, \tag{8}$$
then the eigenvalues of the operator $\mathcal{A}$ are different and the semigroup $S(t)$ has different forms. But the covariance operator $Q_\infty^{(a,b)}$ will remain the same.

# Estimation of parameters

- According to [Maslowski, Pospíšil], some Birkhoff–type ergodic theorem may be applied. Namely

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \|X^{x_0}(t)\|_V^2\,dt = \int_V \|y\|_V^2\,d\mu_\infty^{(a,b)}(y) \tag{9}$$

$$= \mathrm{Tr}\ Q_\infty^{(a,b)}, \tag{10}$$

for any initial condition $x_0 \in V$.

- Trace of the nuclear operator $Q_\infty^{(a,b)}$ equals to

$$\mathrm{Tr}\ Q_\infty^{(a,b)} = \frac{1}{4ab}\sum_{n=1}^{\infty}\lambda_n + \frac{1}{4a}\sum_{n=1}^{\infty}\lambda_n$$

$$= \frac{b+1}{4ab}\,\mathrm{Tr}\ Q. \tag{11}$$

## Estimation of parameters

**Theorem**

*If we set*

$$I_T = \frac{1}{T}\int_0^T \|X^{x_0}(t)\|_V^2\,dt, \tag{12}$$

*then the processes*

$$\hat{a}_T = \frac{b+1}{4bI_T}\,Tr\,Q, \tag{13}$$

$$\hat{b}_T = \frac{Tr\,Q}{4aI_T - Tr\,Q} \tag{14}$$

*are strongly constistent estimators of the parameters $a$ and $b$, respectively, i.e., $\hat{a}_T \to a$, $\hat{b}_T \to b$, $\mathbb{P} - a.s.$ as $T \to \infty$.*

# New estimators – Introduction

- $$\lim_{T\to\infty} I_T = \lim_{T\to\infty} \frac{1}{T}\int_0^T \|X^{x_0}(t)\|_V^2\, dt = \mathrm{Tr}\ Q_\infty^{(a,b)},$$
  for any initial condition $x_0 \in V$.
- $\mathrm{Tr}\ Q_\infty^{(a,b)} = \frac{1}{4ab}\ \mathrm{Tr}\ Q + \frac{1}{4a}\ \mathrm{Tr}\ Q$.
- $\|X^{x_0}(t)\|_V^2 = \|X_1^{x_0}(t)\|^2_{\mathrm{Dom}((-A)^{\frac{1}{2}})} + \|X_2^{x_0}(t)\|^2_{L^2(D)}$.
- $$I_T = \frac{1}{T}\int_0^T \|X^{x_0}(t)\|_V^2\, dt = \frac{1}{T}\int_0^T \|X_1^{x_0}(t)\|^2_{\mathrm{Dom}((-A)^{\frac{1}{2}})}\, dt + \frac{1}{T}\int_0^T \|X_2^{x_0}(t)\|^2_{L^2(D)}\, dt.$$

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- $\mathrm{Tr}\ Q_\infty^{(a,b)} = \frac{1}{4ab}\ \mathrm{Tr}\ Q + \frac{1}{4a}\ \mathrm{Tr}\ Q$.
- $\|X^{x_0}(t)\|_V^2 = \|X_1^{x_0}(t)\|^2_{\mathrm{Dom}((-A)^{\frac{1}{2}})} + \|X_2^{x_0}(t)\|^2_{L^2(D)}$.
- $$I_T = \frac{1}{T}\int_0^T \|X^{x_0}(t)\|_V^2\,dt = \frac{1}{T}\int_0^T \|X_1^{x_0}(t)\|^2_{\mathrm{Dom}((-A)^{\frac{1}{2}})}\,dt + \frac{1}{T}\int_0^T \|X_2^{x_0}(t)\|^2_{L^2(D)}\,dt.$$

# New estimators – Formulae

- Birkhoff theorem (again) yields, that

$$Y_T := \frac{1}{T}\int_0^T \|X_1^{x_0}(t)\|^2_{\mathrm{Dom}((-A)^{\frac{1}{2}})}\,dt \to \frac{1}{4ab}\,\mathrm{Tr}\;Q, \quad T \to \infty,$$

$$H_T := \frac{1}{T}\int_0^T \|X_2^{x_0}(t)\|^2_{L^2(D)}\,dt \to \frac{1}{4a}\,\mathrm{Tr}\;Q, \quad T \to \infty.$$

# New estimators – Formulae

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$$Y_T := \frac{1}{T}\int_0^T \|X_1^{x_0}(t)\|^2_{\mathrm{Dom}((-A)^{\frac{1}{2}})}\,dt \to \frac{1}{4ab}\,\mathrm{Tr}\,Q, \quad T \to \infty,$$

$$H_T := \frac{1}{T}\int_0^T \|X_2^{x_0}(t)\|^2_{L^2(D)}\,dt \to \frac{1}{4a}\,\mathrm{Tr}\,Q, \quad T \to \infty.$$

- Based on above, new strong consistent estimators may be proposed:

$$\tilde{a}_T = \frac{\mathrm{Tr}\,Q}{4H_T}, \tag{17}$$

$$\tilde{b}_T = \frac{H_T}{Y_T}. \tag{18}$$

# Asymptotic normality

- Are the estimators asymptotically normal?
- $$\sqrt{T}\,(\hat{a}_T - a) \overset{?}{\to} Z_1 \sim N(0, V_1),\ T \to \infty,$$
  $$\sqrt{T}\left(\hat{b}_T - b\right) \overset{?}{\to} Z_2 \sim N(0, V_2),\ T \to \infty.$$
- $$\sqrt{T}\,(\tilde{a}_T - a) \overset{?}{\to} Z_3 \sim N(0, V_3),\ T \to \infty,$$
  $$\sqrt{T}\left(\tilde{b}_T - b\right) \overset{?}{\to} Z_4 \sim N(0, V_4),\ T \to \infty.$$

## Asymptotic normality

- Define the operator $\tilde{R} : V \to L^2(D)$ by

$$\tilde{R}x = \left( \tfrac{2a}{b+1}I \quad I \right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{2a}{b+1}x_1 + x_2, \quad \forall x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in V$$

and note that the adjoint operator of $\tilde{R}$ has the following form

$$\tilde{R}^* : L^2(D) \to V,$$

$$\tilde{R}^* x = \begin{pmatrix} -\frac{2a}{b+1}A^{-1} \\ I \end{pmatrix} x = \begin{pmatrix} -\frac{2a}{b+1}A^{-1}x \\ x \end{pmatrix}, \quad \forall x \in L^2(D).$$

# Asymptotic normality

**Theorem**

*1) The estimator $\hat{a}_T$ is asymptotically normal, i.e., for $T \to \infty$*

$$Law\left(\sqrt{T}\,(\hat{a}_T - a)\right) \xrightarrow{w^*} N\left(0,\ \frac{4a^2}{(Tr\ Q)^2}\ Tr\ \left(Q\tilde{R}Q_{\infty}^{(a,b)}\tilde{R}^*\right)\right).$$

*2) The estimator $\hat{b}_T$ is asymptotically normal, i.e., for $T \to \infty$*

$$Law\left(\sqrt{T}\left(\hat{b}_T - b\right)\right) \xrightarrow{w^*} N\left(0,\ \frac{4b^2(b+1)^2}{(Tr\ Q)^2}\ Tr\ \left(Q\tilde{R}Q_{\infty}^{(a,b)}\tilde{R}^*\right)\right).$$

## Asymptotic normality

- Also define the operator $\tilde{R}_1 : V \to L^2(D)$ by

$$\tilde{R}_1 x = \begin{pmatrix} 0 & I \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_2, \quad \forall x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in V$$

and the operator $\tilde{R}_2 : V \to L^2(D)$ by

$$\tilde{R}_2 x = \begin{pmatrix} 2aI & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 2ax_1, \quad \forall x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in V.$$

# Asymptotic normality

**Theorem**

*1) The estimator $\tilde{a}_T$ is asymptotically normal, i.e., for $T \to \infty$*

$$Law\left(\sqrt{T}\,(\tilde{a}_T - a)\right) \xrightarrow{w^*} N\left(0,\ \frac{4a^2}{(Tr\ Q)^2}\ Tr\ \left(Q\tilde{R}_1 Q_\infty^{(a,b)} \tilde{R}_1^*\right)\right).$$

*2) The estimator $\tilde{b}_T$ is asymptotically normal, i.e., for $T \to \infty$*

$$Law\left(\sqrt{T}\left(\tilde{b}_T - b\right)\right) \xrightarrow{w^*} N\left(0,\ \frac{4b^2}{(Tr\ Q)^2}\ Tr\ \left(Q\tilde{R}_2 Q_\infty^{(a,b)} \tilde{R}_2^*\right)\right).$$

# Asymptotic normality

- Since

$$\mathrm{Tr}\left(Q\tilde{R}Q_{\infty}^{(a,b)}\tilde{R}^{*}\right) = \frac{1}{(b+1)^2}\,\mathrm{Tr}\left(Q\tilde{R}_2 Q_{\infty}^{(a,b)}\tilde{R}_2^{*}\right) + \mathrm{Tr}\left(Q\tilde{R}_1 Q_{\infty}^{(a,b)}\tilde{R}_1^{*}\right),$$

the limiting variance of $\sqrt{T}\,(\tilde{a}_T - a)$ is smaller than the limiting variance of $\sqrt{T}\,(\hat{a}_T - a)$ and the limiting variance of $\sqrt{T}\left(\tilde{b}_T - b\right)$ is smaller than the limiting variance of $\sqrt{T}\left(\hat{b}_T - b\right)$.

# Asymptotic normality

- If we consider "diagonal case", the limiting variances of Gaussian distributions may be specified as

$$\mathrm{Law}\left(\sqrt{T}\,(\hat{a}_T - a)\right) \xrightarrow{w^*}$$

$$\xrightarrow{w^*} N\left(0,\ \frac{1}{(\mathrm{Tr}\,Q)^2}\left(\frac{4a^3}{b(b+1)^2}\,\mathrm{Tr}\left(Q^2(-A)^{-1}\right) + a\,\mathrm{Tr}\,Q^2\right)\right),$$

$$\mathrm{Law}\left(\sqrt{T}\left(\hat{b}_T - b\right)\right) \xrightarrow{w^*}$$

$$\xrightarrow{w^*} N\left(0,\ \frac{1}{(\mathrm{Tr}\,Q)^2}\left(4ab\,\mathrm{Tr}\left(Q^2(-A)^{-1}\right) + \frac{b^2(b+1)^2}{a}\,\mathrm{Tr}\,Q^2\right)\right),$$

$$\mathrm{Law}\left(\sqrt{T}\,(\tilde{a}_T - a)\right) \xrightarrow{w^*} N\left(0,\ a\,\frac{\mathrm{Tr}\,Q^2}{(\mathrm{Tr}\,Q)^2}\right),$$

$$\mathrm{Law}\left(\sqrt{T}\left(\tilde{b}_T - b\right)\right) \xrightarrow{w^*} N\left(0,\ 4ab\,\frac{\mathrm{Tr}\left(Q^2(-A)^{-1}\right)}{(\mathrm{Tr}\,Q)^2}\right),$$

for $T \to \infty$.

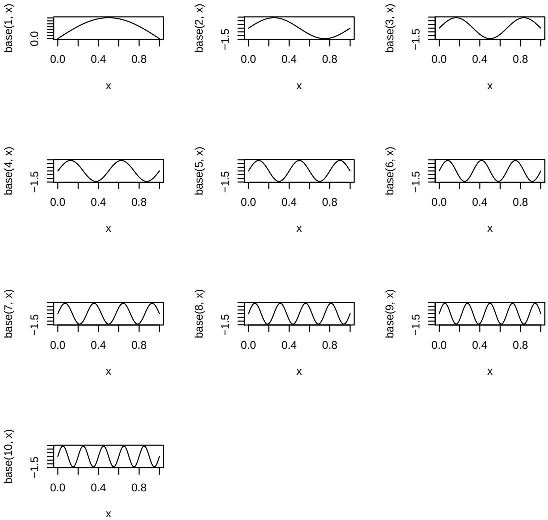
# Simulation – Setup

- $D = (0,1)$ – We consider the wave equation for the oscillating rod. Hence we model it as a function from the space $L^2((0,1))$.
- The choice of the orthonormal basis of the space $L^2((0,1))$ is

$$\{e_n(\xi) = \sqrt{2}\sin(n\pi\xi), n = 1, \ldots, N = 10\},$$

which makes the boundary condition $u(t,0) = 0 = u(t,1)$, for any $t > 0$, always satisfied.

# Simulation – Basis



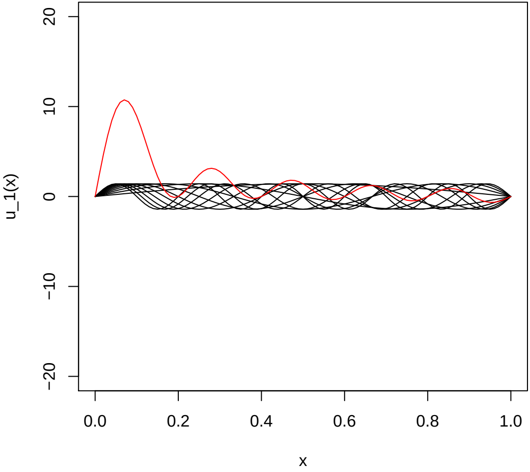
# Simulation – Setup

- $T = 100$ – The length of the time interval which is used.
- $\Delta t = 0.0001$ – The mesh of the partition of the time interval $[0, T]$.
- The intial functions $u_1$ and $u_2$ have the following form

$$u_1(\xi) = \sqrt{2} \sum_{n=1}^{N} \sin(n\pi\xi) = u_2(\xi).$$

This means that $\langle u_1, e_n \rangle_{L^2(D)} = 1 = \langle u_2, e_n \rangle_{L^2(D)}$ for any $n = 1, \dots, N$, i.e., the initial conditions are the same in all $N$ dimensions.

# Simulation – Initial condition



## Simulation – Setup

- $a = 1,\ b = 0.2$ – The setup of the parameters that are to be estimated.
- $-\alpha_n = -n^2\pi^2$ – The eigenvalues of the operator $A$.
- $\lambda_n = \frac{1000}{n^2}$ – The eigenvalues of the operator $Q$.
- We consider the "diagonal case", i.e., the eigenvectors of the operators $A$ and $Q$ coincide and form the basis $\{e_n(\xi), n = 1, \ldots, N\}$.

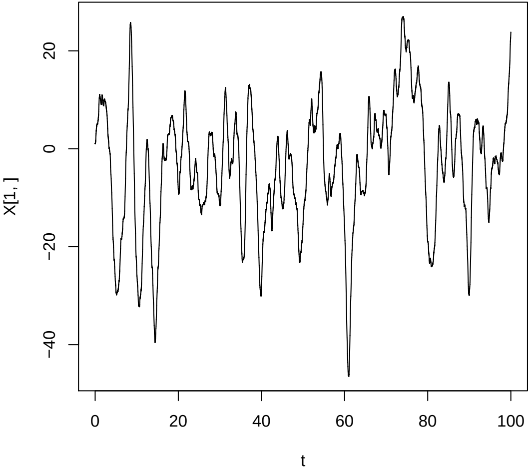
# Simulation – R code

- ```
  T <- 100
  N <- 1000000
  Delta <- T/N
  K <- 10
  t <- seq(0, T, length = N+1)
  X <- matrix(0, K, N+1)
  Y <- matrix(0, K, N+1)
  X0 <- c(rep(1,K))
  Y0 <- c(rep(1,K))
  a <- 1
  b <- 0.2
  ```
  ```

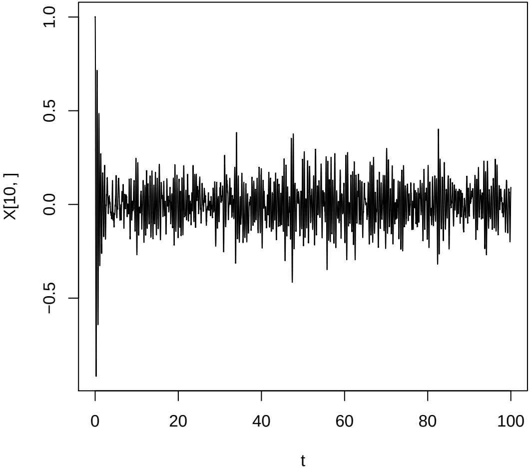
## Simulation – R code

- ```
  sequence <- 1:K
  alpha <- sequence^2*pi^2
  lambda <- 1000/sequence^2
  X[,1] <- X0
  Y[,1] <- Y0
  set.seed(123)
  Z <- matrix(rnorm(K*N), K, N)
  for (i in 2:(N+1)){
  X[,i] <- X[,i-1] + Y[,i-1]*Delta
  Y[,i] <- Y[,i-1] - (b*alpha*X[,i-1] +
  2*a*Y[,i-1])*Delta +
  sqrt(lambda)*Z[,i-1]*sqrt(Delta)
  }
  ```
  ```

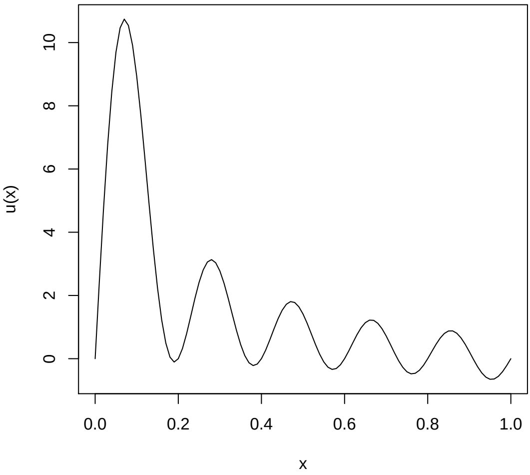
# Simulation – Time evolution – First coordinate



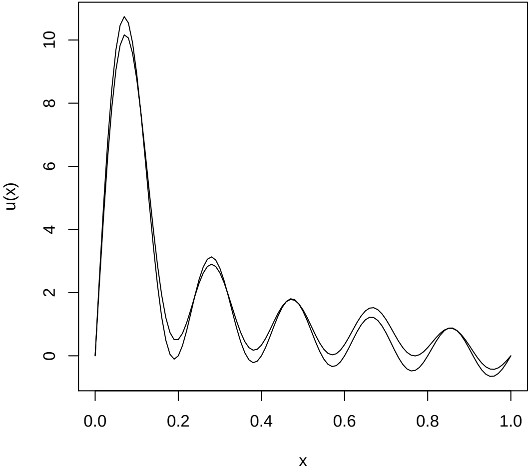
# Simulation – Time evolution – Tenth coordinate



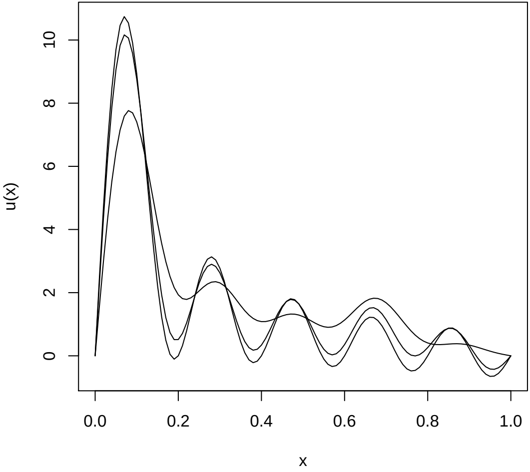
# Simulation – Time evolution



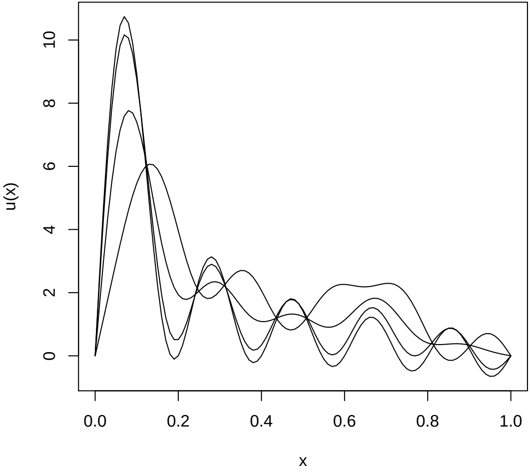
# Simulation – Time evolution, $\Delta t = 0.05$



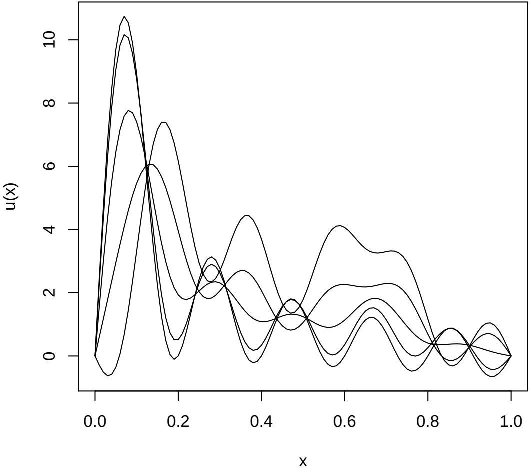
# Simulation – Time evolution, $\Delta t = 0.05$



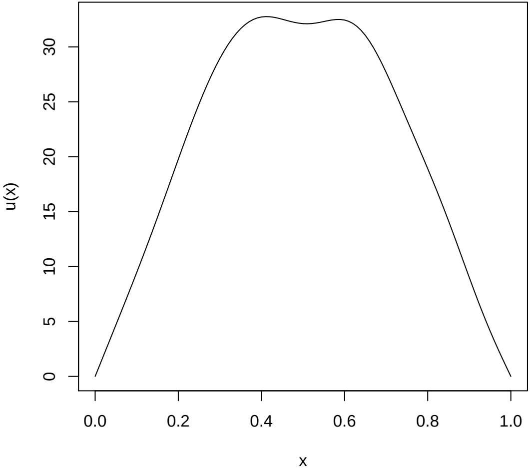
# Simulation – Time evolution, $\Delta t = 0.05$



# Simulation – Time evolution, $\Delta t = 0.05$



# Simulation – Time evolution, $T = 100$



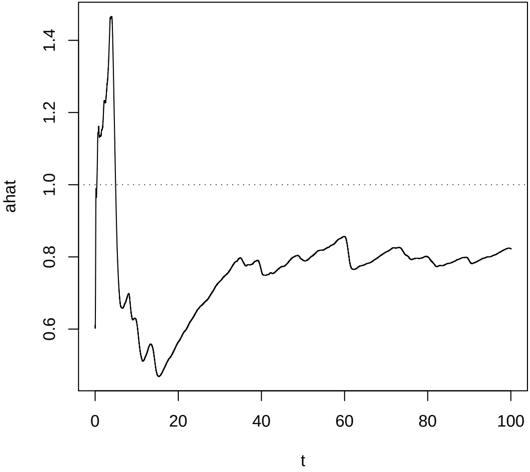
# Implementation of estimators – R code

- ```
  I <- numeric(N+1)
  ahat <- numeric(N+1)
  bhat <- numeric(N+1)
  I[1] <- sum(alpha*X[,1]^2) + sum(Y[,1]^2)
  ahat[1] <- sum(lambda)*(b+1)/(4*b*I[1])
  bhat[1] <- sum(lambda)/(4*a*I[1] - (sum(lambda)))
  for (i in 2:(N+1)){
  I[i] <- (I[i-1]*(i-1) + sum(alpha*X[,i]^2) +
  sum(Y[,i]^2))/i
  ahat[i] <- sum(lambda)*(b+1)/(4*b*I[i])
  bhat[i] <- sum(lambda)/(4*a*I[i] - sum(lambda))
  }
  Tr <- sum(lambda)*(b+1)/(4*a*b)
  Tr
  I[N+1]
  ahat[N+1]
  bhat[N+1]
  ```
  ```

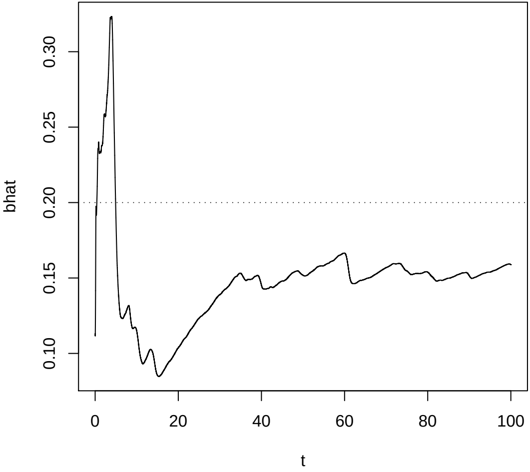
# Simulation – Results

- From the generated trajectory, we obtained these results:
  - $I_T = 2827.041$
  - $\mathrm{Tr}\ Q_\infty^{(a,b)} = \frac{b+1}{4ab} \sum_{n=1}^{N=10} \lambda_n = 2324.652$
  - $\hat{a}_T = 0.8223$, $\hat{b}_T = 0.1588$
  - The limiting variance of the random variable $\sqrt{T}(\hat{a}_T - a)$ equals to 1.0466.
  - The limiting variance of the random variable $\sqrt{T}(\hat{b}_T - b)$ equals to 0.0776.

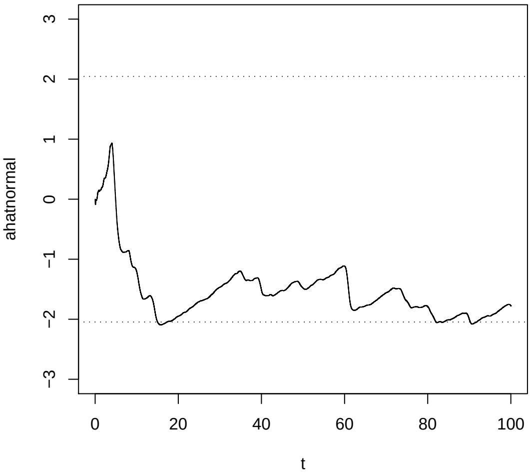
# Estimator $\hat{a}_T$



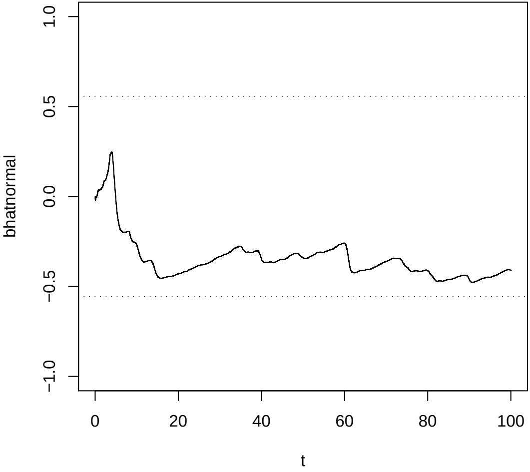
# Estimator $\hat{b}_T$



# The values of $\sqrt{T}(\hat{a}_T - a)$



# The values of $\sqrt{T}(\hat{b}_T - b)$



## Simulation – Results

- The results for the family of estimators $(\tilde{a}_T, \tilde{b}_T)$ are the following:

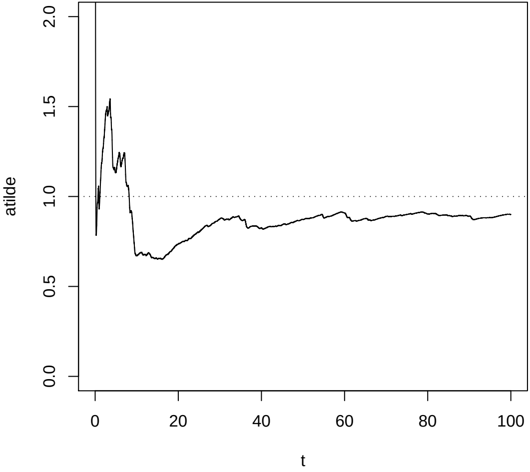
$$Y_T = 2396.570, \quad \frac{1}{4ab} \sum_{n=1}^{N=10} \lambda_n = 1937.210,$$

$$H_T = 430.471, \quad \frac{1}{4a} \sum_{n=1}^{N=10} \lambda_n = 387.442,$$

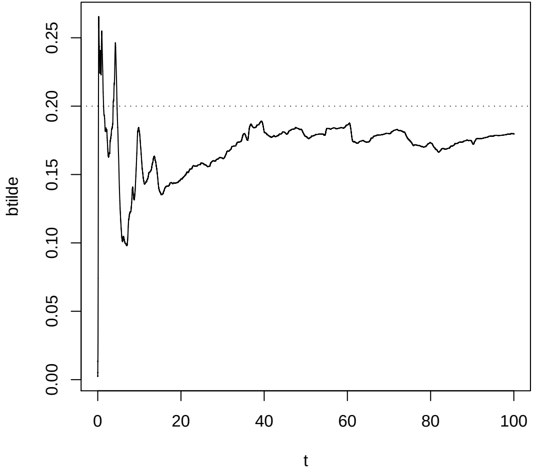
$$\tilde{a}_T = 0.9000, \quad \tilde{b}_T = 0.1796.$$

- The limiting variance of the random variable $\sqrt{T}(\tilde{a}_T - a)$ equals to 0.4505.
- The limiting variance of the random variable $\sqrt{T}(\tilde{b}_T - b)$ equals to 0.0343.

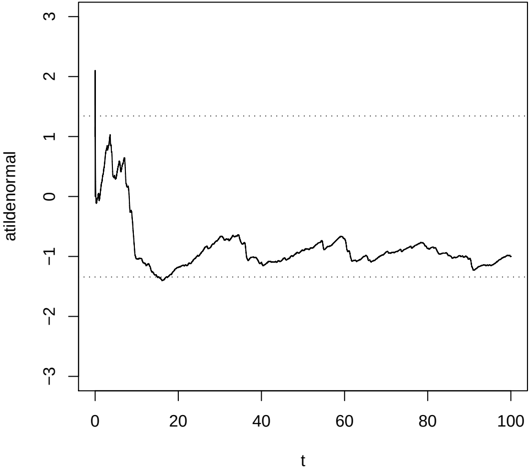
# Estimator $\tilde{a}_T$



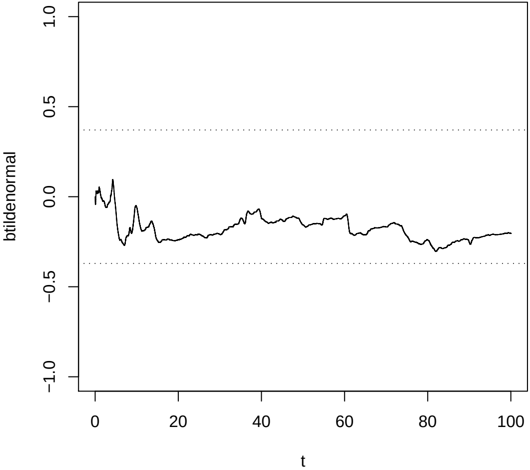
# Estimator $\tilde{b}_T$



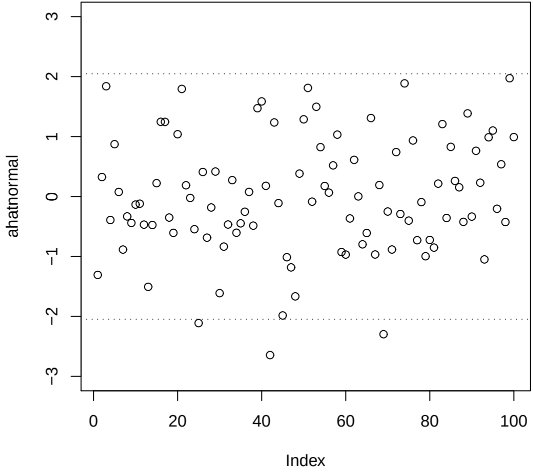
# The values of $\sqrt{T}(\tilde{a}_T - a)$



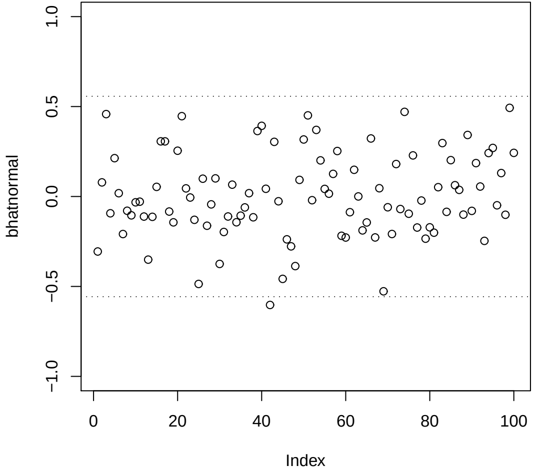
# The values of $\sqrt{T}(\tilde{b}_T - b)$



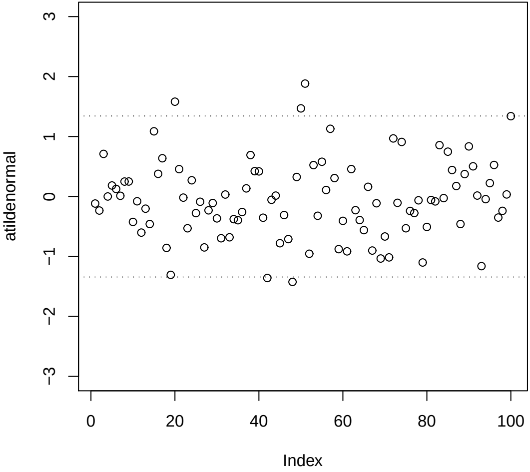
# The values of $\sqrt{T}(\hat{a}_T - a)$ – Overall



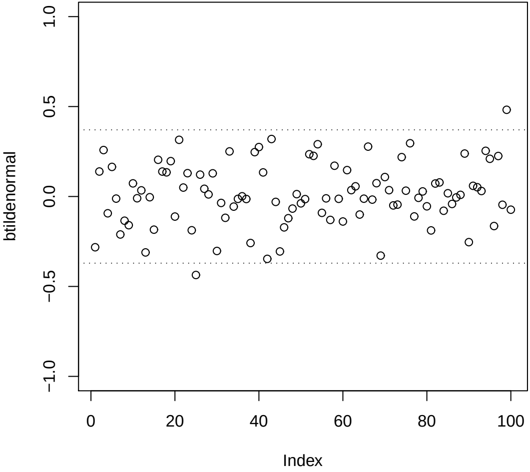
# The values of $\sqrt{T}(\hat{b}_T - b)$ – Overall



## The values of $\sqrt{T}(\tilde{a}_T - a)$ – Overall



# The values of $\sqrt{T}(\tilde{b}_T - b)$ – Overall



# Simulation – Conclusion

- Due to smaller limiting variances, the family of the estimators $(\tilde{a}_T, \tilde{b}_T)$ is better than the family $(\hat{a}_T, \hat{b}_T)$ not only in conrete example, but it is better overall.
- The limiting variances are accurate. It really seems that $95,45\%$ of all data lie within the $95,45\%$ confidence intervals.
- Data from the previous Figures seems to be normally distributed as prescribed. The Wilk–Shapiro test of normality did not reject the hypothesis of normality on 5%–significance level. (The $p$–values of the tests were actually over 25%.)

# Estimation of parameters – Observational window

- Fix $z \in V$, $z \neq 0$. Birkhoff theorem yields

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \langle X^{x_0}(t), z\rangle_V^2 \, dt = \left\langle Q_\infty^{(a,b)} z, z \right\rangle_V, \quad \mathbb{P} - a.s.$$

  for any $x_0 \in V$.
- Remind that $V = \mathrm{Dom}((-A)^{\frac{1}{2}}) \times L^2(D)$.
- The orthonormal basis of $L^2(D)$ consists of vectors $\{e_n, n \in \mathbb{N}\}$.
- The orthonormal basis of $\mathrm{Dom}((-A)^{\frac{1}{2}})$ consists of vectors $\{f_n, n \in \mathbb{N}\}$, where $f_n = \frac{e_n}{\sqrt{\alpha_n}}$.
- Consider "diagonal case".

## Estimation of parameters – Observational window

- Let $z = \begin{pmatrix} 0 \\ z_2 \end{pmatrix}$, where $z_2 \in L^2(D)$. Then the strongly consistent estimator of parameter $a$ is

$$\bar{a}_{T,z_2} = \frac{\langle Qz_2, z_2 \rangle_{L^2(D)}}{\frac{4}{T} \int_0^T \langle X_2^{x_0}(t), z_2 \rangle_{L^2(D)}^2 \, dt}.$$

- Let $z = \begin{pmatrix} z_1 \\ 0 \end{pmatrix}$, where $z_1 \in \mathrm{Dom}((-A)^{\frac{1}{2}})$. Then the strongly consistent estimator of parameter $b$ is

$$\bar{b}_{T,z_1} = \frac{\langle Qz_1, z_1 \rangle_{\mathrm{Dom}((-A)^{\frac{1}{2}})}}{\frac{4a}{T} \int_0^T \langle X_1^{x_0}(t), z_1 \rangle_{\mathrm{Dom}((-A)^{\frac{1}{2}})}^2 \, dt}.$$

## Estimation of parameters – Observational coordinate

- Let $z = \begin{pmatrix} 0 \\ e_k \end{pmatrix}$, where $e_k \in L^2(D)$. Then the strongly consistent estimator of parameter $a$ is

$$\bar{a}_{T,k} = \frac{\lambda_k}{\frac{4}{T}\int_0^T \langle X_2^{x_0}(t), e_k \rangle_{L^2(D)}^2 \, dt}.$$

- Let $z = \begin{pmatrix} f_j \\ 0 \end{pmatrix}$, where $f_j \in \mathrm{Dom}((-A)^{\frac{1}{2}})$. Then the strongly consistent estimator of parameter $b$ is

$$\bar{b}_{T,j} = \frac{\lambda_j}{\frac{4a}{T}\int_0^T \langle X_1^{x_0}(t), f_j \rangle_{\mathrm{Dom}((-A)^{\frac{1}{2}})}^2 \, dt}.$$

# Estimation of parameters – Observational coordinate

- If we use $\bar{a}_T$ instead of $a$ and set $j = k$, we may go even further

$$\bar{b}_{T,k,k} = \frac{\frac{1}{T}\int_0^T \langle X_2^{x_0}(t), e_k\rangle^2_{L^2(D)}\,dt}{\frac{1}{T}\int_0^T \langle X_1^{x_0}(t), f_k\rangle^2_{\mathrm{Dom}((-A)^{\frac{1}{2}})}\,dt}$$

# Asymptotic normality– Observational coordinate

Theorem

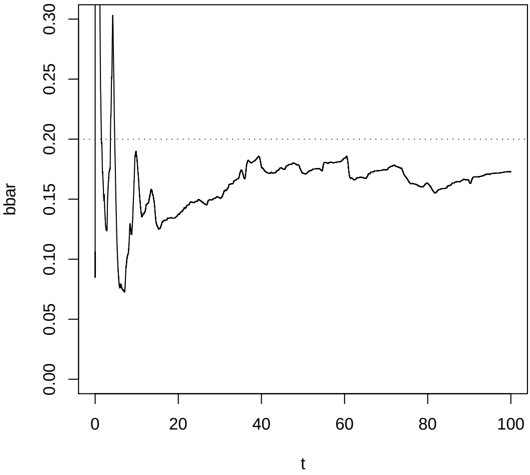
*1) The estimator $\bar{a}_{T,k}$ is asymptotically normal, i.e., for $T \to \infty$*

$$Law\left(\sqrt{T}\left(\bar{a}_{T,k} - a\right)\right) \xrightarrow{w^*} N\left(0,\ a\right).$$

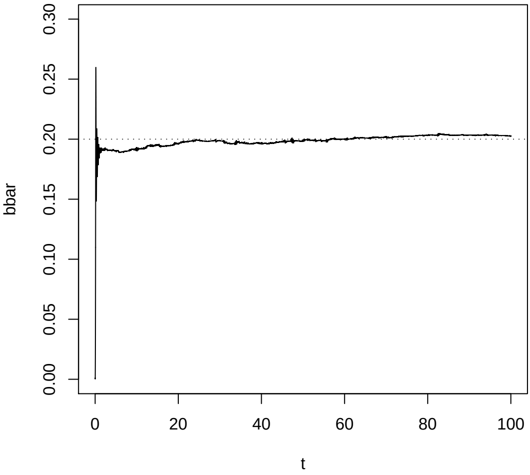
*2) The estimator $\bar{b}_{T,k,k}$ is asymptotically normal, i.e., for $T \to \infty$*

$$Law\left(\sqrt{T}\left(\bar{b}_{T,k,k} - b\right)\right) \xrightarrow{w^*} N\left(0,\ \frac{4ab}{\alpha_k}\right).$$

# Estimator $\bar{b}_{T,1,1}$



# Estimator $\bar{b}_{T,10,10}$



# Parameter estimation – Conclusion

- Rewrite the stochastic partial differential equation to the form

$$dX(t) = \mathcal{A}X(t)\,dt + \Phi\,dB(t), \quad X(0) = x_0.$$

- Compute the semigroup $S(t)$.
- Check if the semigroup $S(t)$ is exponentially stable.
- Compute the covariance operator $Q_\infty$.
- Use suitable ergodic theorem.
- Derive the estimator from the limiting property.
- Try to prove asymptotic normality for the estimator.

# References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- T. Koski, W. Loges, *On identification for distributed parameter systems*, Stochastic Processes – Mathematics and Physics II, Proceedings of the 2nd BiBoS Symposium (1985), 152–159.
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- Y. A. Kutoyants, *Statistical inference for ergodic diffusion processes*, Springer, London, 2004.
- B. Maslowski, J. Pospíšil, *Ergodicity and parameter estimates for infinite–dimensional fractional Ornstein–Uhlenbeck process*, Applied Mathematics and Optimization **57** (2008), no. 3, 401–429.