10 Groups

10.1. Decide whether the following sets with binary operation can form a group. If so, describe the neutral element and inverses.

(a) $(\mathbb{Z}, +)$ and $(\mathbb{Z}, \cdot)$

(b) $(\mathbb{Q}^+, +)$; $(\mathbb{Q}^+, \cdot)$; $(\mathbb{Q}, +)$; $(\mathbb{Q}, -)$

(c) Positive rationals with $a \circ b := a^b$

(d) Let $(R, +, -, 0, *, 1)$ be a ring: $(R, +)$ and $(R, *)$

10.2. Write the following permutations as the product of independent cycles and for each permutation σ determine σ^{-1} and σ^{2020}:

(a) $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 5 & 1 & 2 \end{pmatrix} \in \S_5$,

(b) $\tau = (4\,6\,5\,1\,2) \in \S_6$,

(c) $\sigma = (1\,5\,6)(2\,3\,8\,4\,7) \in \S_8$,

(d) $\rho = (4\,3\,5) \circ (5\,1\,2) \in \S_5$.

10.3. Show that if $\pi(a) = b$ and $\sigma = \tau\pi\tau^{-1}$, then $\tau(b)$ is right after the element $\tau(a)$, i.e. $\sigma(\tau(a)) = \tau(b)$, and determine $\pi\tau\pi^{-1}$ and $\tau\pi\tau^{-1}$ for the permutations π and τ from example 10.2.

Recall that the operation $\pi^\tau = \tau\pi\tau^{-1}$ is called *conjugation of the permutation* π *by the permutation* τ and the permutation π^τ is then *conjugated* with π.

10.4. Prove that the conjugation relation is an equivalence.

10.5. Find all permutations of α on the set $\{1, 2, 3, 4\}$ for which $\alpha \circ (1\,2\,3) \circ \alpha^{-1} = (1\,2\,4)$.

10.6. Determine the order of the following elements:

(a) 4 and 15 in $\mathbb{Z}_{75}$,

(b) 7 and 9 in $\mathbb{Z}_{20}^*$,

(c) 4 and 15 in $\mathbb{Z}$,

(d) $(1\,2\,3\,4)(5\,6\,7)(8\,9)$, $(1\,2)(5\,6\,8\,9)$ in $\S_9$ and $\mathbf{A}_{2020}$.