3 CRT, Rings, Integral Domains and Fields

3.1. Find all $x \in \mathbb{Z}$ satisfying

(a) $x \equiv 2 \pmod 3, \quad x \equiv 4 \pmod 7, \quad x \equiv 3 \pmod 8.$

(b) $2x + 1 \equiv 2 \pmod 3, \quad 3x + 2 \equiv 3 \pmod 4, \quad 4x + 3 \equiv 2 \pmod 5.$

(c) $10x \equiv 6 \pmod{32}, \quad 3x \equiv 1 \pmod 5.$

3.2. Find all $x \in \mathbb{Z}$ such that

(a) $x^2 \equiv 1 \pmod 3, \quad x^2 \equiv 1 \pmod 7.$

(b) $x^2 \equiv -1 \pmod{66}.$

(c) $x^2 \equiv -1 \pmod{65}.$

3.3. Using the Chinese remainder theorem, calculate 12^{100} $(mod\ 30)$.

3.4. Let $\mathcal{Z}^2 = (\mathbb{Z}^2, +, -, \cdot, (0,0))$, where $(a,b) \pm (c,d) = (a \pm c, b \pm d)$ and $(a,b) \cdot (c,d) = (a \cdot c, b \cdot d)$. Sketch the proof that $\mathcal{Z}^2$ is a commutative ring with identity which is not a domain.

3.5. Describe the quotient fields of the domains

(a) integers $\mathbb{Z}$,

(b) real polynomials $\mathbb{R}[x]$

(c)* arbitrary field F.

3.6. Let $\mathbb{Z}[i] = \{a + ib \colon a, b \in \mathbb{Z}\}$ (the Gaussian integers). Prove that

(a) $\mathbb{Z}[i]$ forms a subring of the field of complex numbers $\mathbb{C}$,

(b) $(\mathbb{Z}[i], +, -, \cdot, 0)$ is a domain,

(c)* $\mathbb{Q}[i] = \{a + ib \colon a, b \in \mathbb{Q}\} \subseteq \mathbb{C}$ is a quotient field of $\mathbb{Z}[i]$.

3.7. (a)* Show that the empty set cannot be a ring.

(b)* Discuss the one-element ring. Is it a domain? Is it a field?