4 Polynomial rings, Division of Polynomials, Roots

4.1. Calculate

(a) $(x^4 - 2x^3 - x^2 + x - 2) \bmod x^2 + 2$ in $\mathbb{Z}_5[x]$,

(b) $(x^4 - 2x^3 - x^2 + x - 2) \operatorname{div} x^2 + 2$ in $\mathbb{Z}_5[x]$,

(c) $(x^5 + 2x^3 - 3x - 2) \bmod x - 2$ in $\mathbb{Z}_7[x]$.

4.2. Find all roots of the polynomial

(a) $x^3 + 2x^2 + x \in \mathbb{Z}_3[x]$ in the field $\mathbb{Z}_3$,

(b) $x^2 + 1 \in \mathbb{Z}_3[x]$ in the field $\mathbb{Z}_3$,

(c) $x^2 + 1 \in \mathbb{Z}_5[x]$ in the field $\mathbb{Z}_5$,

(d) $x^6 - 1 \in \mathbb{Z}_7[x]$ in the field $\mathbb{Z}_7$,

(e) $x^6 - 1 \in \mathbb{C}[x]$ in the field $\mathbb{C}$.

4.3. Show that $x^m - 1 \mid x^n - 1$ in $\mathbb{Z}[x]$ if and only if $m \mid n$.

4.4. For a polynomial $f \in \mathbb{C}[x]$, we denote $V(f)$ as the set of all roots of f. Let $f = -8 + 44x - 102x^2 + 129x^3 - 96x^4 + 42x^5 - 10x^6 + x^7 \in \mathbb{C}[x]$. Determine $V(f)$ and find $g \in \mathbb{C}[x]$ such that $V(g) = V(f)$ and g has the smallest possible degree. Derive the general rule.

4.5. Show that there are infinitely many polynomials in $\mathbb{Z}[x]$ such that they define the same polynomial mapping $m : \mathbb{Z}_2 \to \mathbb{Z}_2$.

4.6. Find a linear polynomial in $\mathbb{Z}_4[x]$ such that it has no root in $\mathbb{Z}_4$.

4.7. Let $\mathcal{R}$ be an integral domain. Determine the conditions under which the polynomial $a_1x + a_0$ has a root in $\mathcal{R}$.

4.8. Find $m \in \mathbb{N}$ such that $x^2 - 1$ has 3 roots in Z_m.

4.9. Determine all invertible polynomials in $\mathcal{R}[x]$, assuming that:

- $\mathcal{R}$ is a field,
- $\mathcal{R}$ is a domain.

4.10. Find a ring $\mathcal{R}$ and a nonzero polynomial f such that f has infinitely many roots in $\mathcal{R}$.

4.11. Show that for every finite field $\mathbb{F}$ there exists a polynomial $f \in \mathbb{F}[x]$ such that it has no root in $\mathbb{F}$.