5 Irreducible elements, Unique factorization domains

5.1. Let $\mathcal{F}$ be a field and $f \in F[x]$ of degree $\deg(f) \leq 3$. Prove that f is irreducible iff it has no root or $\deg(f) = 1$.

5.2. Find all irreducible polynomials over $\mathbb{Z}_2$ of degree at most 3.

5.3. Let $f(x) = x^3 + 2x + 3 \in \mathbb{Z}_5[x]$. Describe all invertible elements of the ring $\mathbb{Z}_5[x]$ and all polynomials associated with f. Decide whether $x + 3$ divides f.

5.4. Decide which of the polynomials $2x + 6$, $x^2 - 6$, and $3x^2 + 4x + 1$ are irreducible in (a) $\mathbb{Z}[x]$, (b) $\mathbb{Q}[x]$, (c) $\mathbb{R}[x]$.

5.5. Calculate in the domains $\mathbb{C}[x]$, $\mathbb{R}[x]$, $\mathbb{Q}[x]$, $\mathbb{Z}_3[x]$ and $\mathbb{Z}_5[x]$ the irreducible decompositions of the polynomial $x^3 - 2$.

For $s \in \mathbb{Z}$ square-free, consider a subdomain of $\mathbb{Q}$ defined as $\mathbb{Z}[\sqrt{s}] = \{a + b\sqrt{s} \mid a, b \in \mathbb{Z}\}$. For this ring, we define a norm $\nu : \mathbb{Z}[\sqrt{s}] \to \mathbb{N}$:

$$\nu(a + b\sqrt{s}) =| a^2 - sb^2 |$$

5.6. Prove for $\alpha, \beta \in \mathbb{Z}[\sqrt{s}]$

(a) $\nu(\alpha \cdot \beta) = \nu(\alpha)\nu(\beta)$,

(b) $\nu(\alpha) = 1$ iff α is invertible, and find a formula for computing the inverse.

(c) if $\alpha \mid \beta$ then $\nu(\alpha) \mid \nu(\beta)$,

(d) if $\nu(\alpha)$ is prime, then α is irreducible.

5.7. In the domain $\mathbb{Z}[\sqrt{3}]$

(a) explain why $2 + \sqrt{3}, 2 - \sqrt{3} \in \mathbb{Z}[\sqrt{3}]^*$,

(b) find inverses to $2 + \sqrt{3}$ and $2 - \sqrt{3}$,

(c) explain why $1 + \sqrt{3}, 4 - \sqrt{3}$ are irreducible.

5.8. Consider the domain $\mathbb{Z}[\sqrt{5}]$. Show that

(a) $2, \sqrt{5} + 1, \sqrt{5} - 1$ are irreducible,

(b) 2 is not associated with $\sqrt{5} \pm 1$,

(c) $\mathbb{Z}[\sqrt{5}]$ is not a UFD.

5.9. Calculate the irreducible decompositions of 2, 3, 5 and 6 in the domain $\mathbb{Z}[i]$.

5.10. Find the greatest common divisors in $\mathbb{Z}[i]$

(a) $7 + i$ and $5 - 3i$,

(b) $4 + 2i$ and $3 + i$.