7 Euclidean domains

A domain $\mathcal{R}$ is called *Euclidean* if there is a *Euclidean norm* ν, that is, a function $\nu\colon R \to \mathbb{N}$, which satisfies

1. $\nu(0) = 0$
2. If $a \mid b$, $b \neq 0$, then $\nu(a) \leq \nu(b)$;
3. for all $a, b \in R$, $b \neq 0$, there exists $q, r \in R$ such that $a = bq + r$ and $\nu(r) < \nu(b)$.

7.1. Prove that for any square-free $s \in \mathbb{Z}$, the norm $\nu(a+b\sqrt{s}) = |a^2 - sb^2|$ on the domain $\mathbb{Z}[\sqrt{s}]$ satisfies axioms (1) and (2).

7.2. Using the relationship between the modulus in $\mathbb{C}$ (in Czech: absolutní hodnota) and the norm $\nu(a+bi) = |a^2+b^2| = |a+bi|^2$ of the domain $\mathbb{Z}[i]$, prove for arbitrary $a, b \in \mathbb{Z}[i]$, $b \neq 0$ and $z := \frac{a}{b} \in \mathbb{C}$

(a) that there exists $q \in \mathbb{Z}[i]$ such that $\mid z - q \mid < 1$,

(b) that $|r| < |b|$ and $\nu(r) < \nu(b)$ if $r := a - bq$ for q from (a),

(c) that ν is a Euclidean norm, hence the domain $\mathbb{Z}[i]$ is Euclidean.

7.3. Divide with the remainder α by β in the domain $\mathbb{Z}[i]$ using the Euclidean norm $\nu(a + bi) = |a^2 + b^2|$.

(a) $\alpha = 5 + 7i$, $\beta = 3 - i$,

(b) $\alpha = 3 + 2i$, $\beta = 1 + i$,

7.4. Perform the following computations:

(a) divide with the remainder 4 by $1 - \sqrt{2}i$ in $\mathbb{Z}[\sqrt{2}i]$,

(b) divide with the remainder $1 + 4\sqrt{2}i$ by $3 + \sqrt{2}i$ in $\mathbb{Z}[\sqrt{2}i]$,

(c) $\gcd(6 - 3\sqrt{3}, 3 + \sqrt{3})$ in $\mathbb{Z}[\sqrt{3}]$

7.5. Show that the polynomial $3x^3 + 2x^2 + (4 - 2i)x + (1 + i)$ is irreducible in $\mathbb{Z}[i][x]$.

7.6. Find infinitely many invertible elements in the domain $\mathbb{Z}[\sqrt{3}]$.