On dichotomy above Feder and Vardi's logic

Alexey Barsukov

Aubière, 13.12.2022

Table of Contents

Introduction

Fagin's theorem

ESO $\rightleftarrows$ NP

H O M O M O R P H I S M S

Fagin (1974): NP is captured by existential second order logic (ESO)

LOGIC

C O M P L E X I T Y

NP-complete

P

Ladner's theorem & dichotomy

ESO $\rightleftarrows$ NP

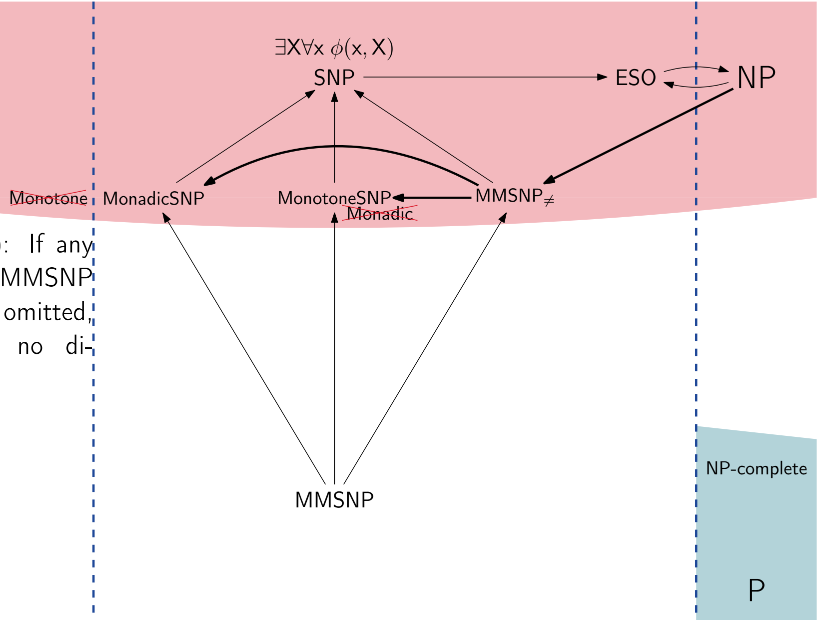
Ladner (1975): If P$\neq$NP, then NP contains problems not from P$\sqcup$NP-complete

A subclass of NP has a dichotomy if every problem of this class is either in P or NP-complete

NP-complete

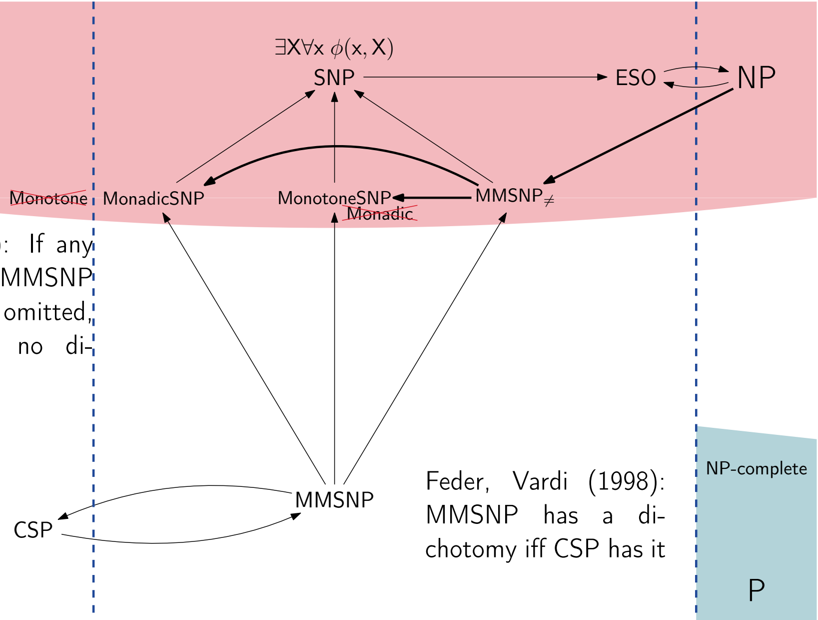
P

Feder and Vardi's logic



F., V. (1998): If any of the three MMSNP properties is omitted, then it has no dichotomy

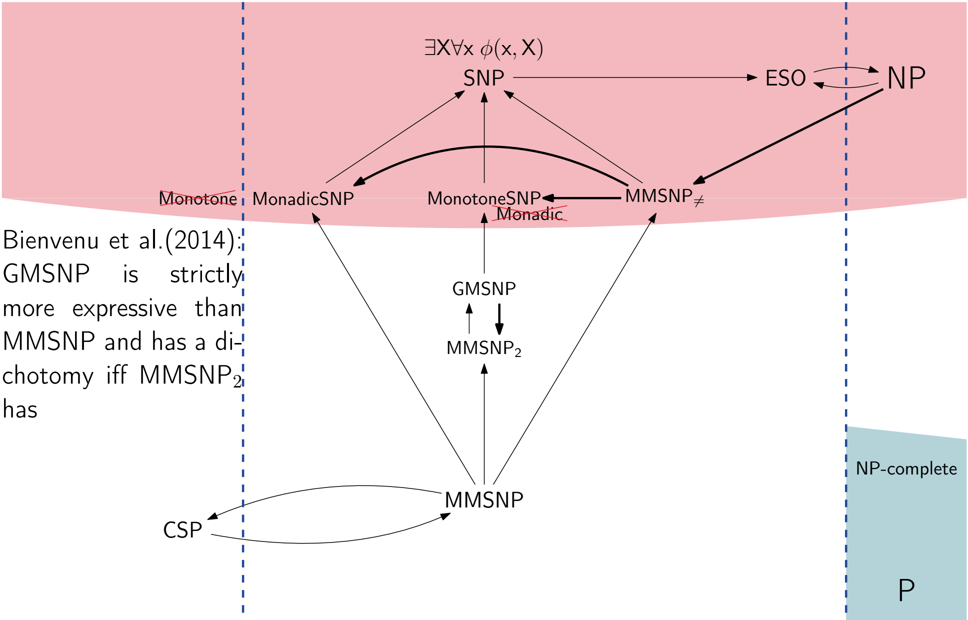
Feder and Vardi's logic



F., V. (1998): If any of the three MMSNP properties is omitted, then it has no dichotomy

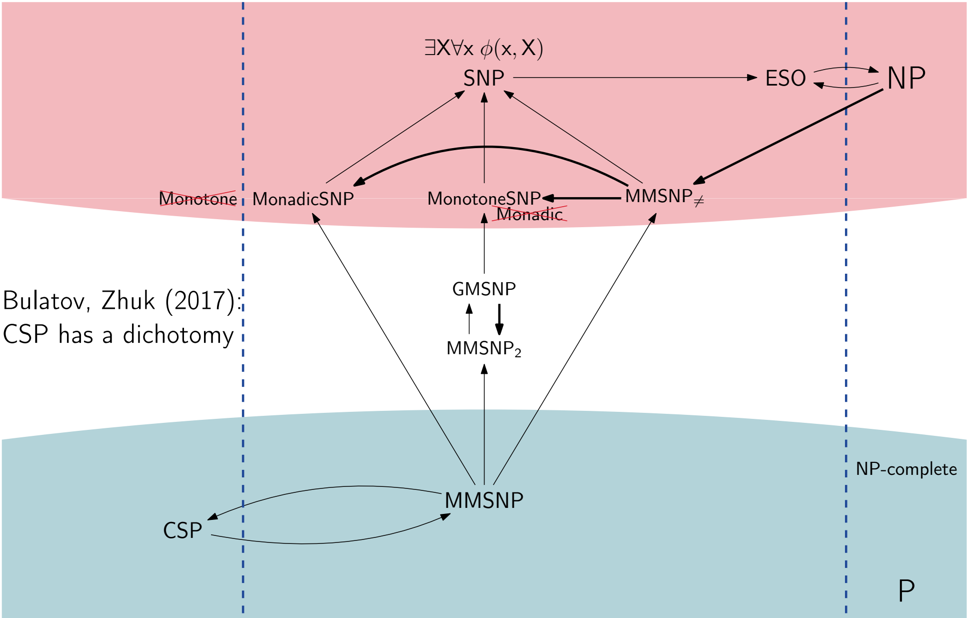
Feder, Vardi (1998): MMSNP has a dichotomy iff CSP has it

Guarded Monotone SNP without $\neq$

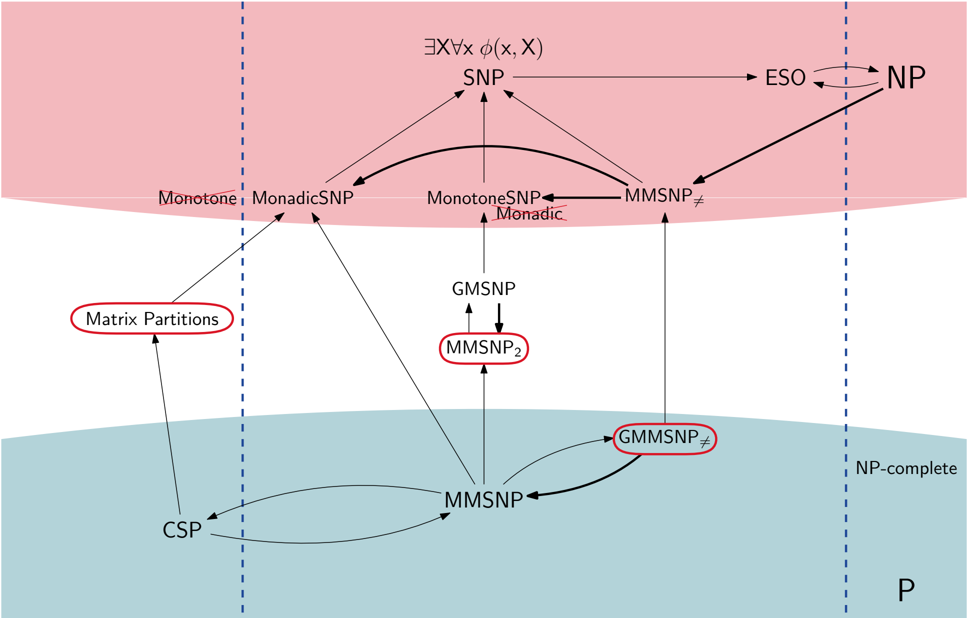


Bienvenu et al.(2014): GMSNP is strictly more expressive than MMSNP and has a dichotomy iff MMSNP_2 has

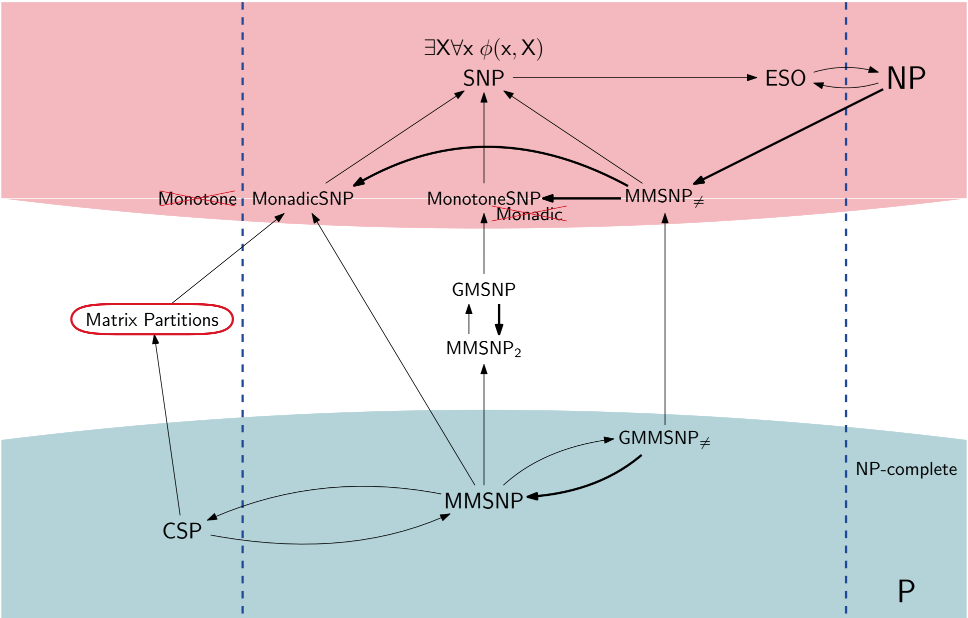
Algebraic Dichotomy for CSP



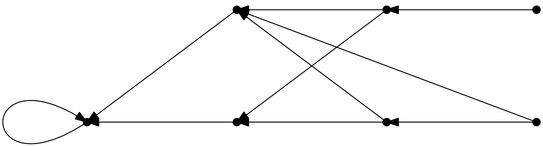
Picture


$\exists X \forall x\ \phi(x, X)$
SNP
ESO
NP
~~Monotone~~
MonadicSNP
MonotoneSNP
~~Monadic~~
$MMSNP_{\neq}$
GMSNP
Matrix Partitions
$MMSNP_2$
$GMMSNP_{\neq}$
NP-complete
MMSNP
CSP
P

Matrix Partition



Matrix Partition



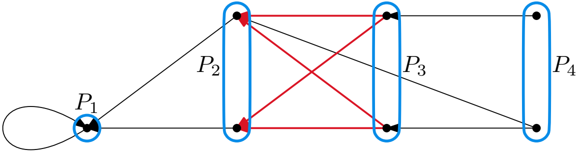
$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \star & \star & 0 \end{bmatrix}$$

Definition ([Feder, Hell, Xie, 2007])

Let M be a square matrix of size m with elements from $\{0, 1, \star\}$. Given an input digraph, split its vertices into disjoint classes $P_1, \ldots, P_m$ such that, for any i, j and any $x \in P_i, y \in P_j$:

- if $\mathsf{M}(i,j) = 0$, then there is no arc between x and y;
- if $\mathsf{M}(i,j) = 1$, then there is an arc between x and y;
- if $\mathsf{M}(i,j) = \star$, then there is no restriction.

Matrix Partition



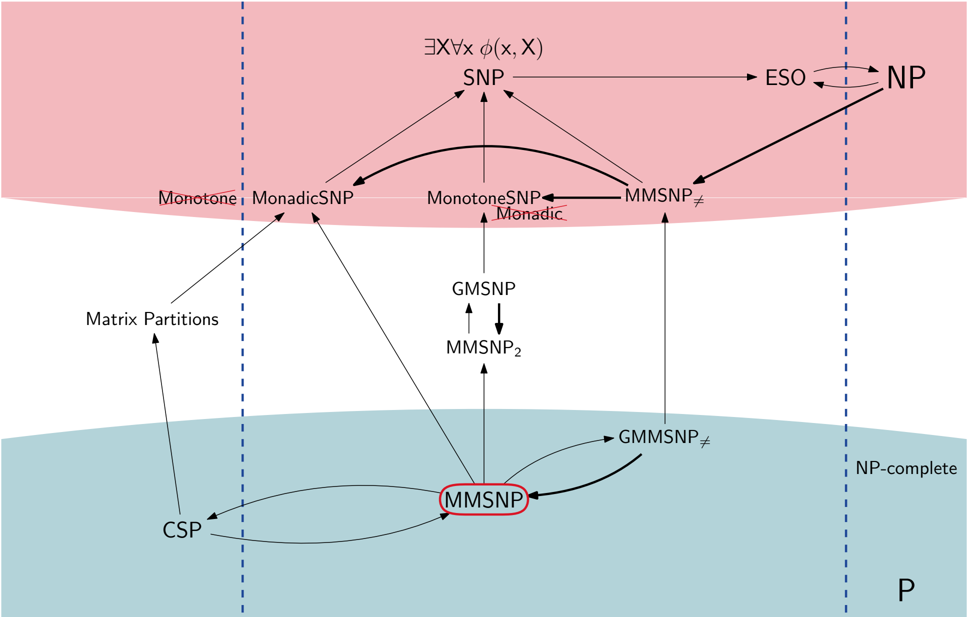
$$\mathsf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \star & \star & 0 \end{bmatrix}$$

Definition ([Feder, Hell, Xie, 2007])

Let M be a square matrix of size m with elements from $\{0, 1, \star\}$. Given an input digraph, split its vertices into disjoint classes $P_1, \ldots, P_m$ such that, for any i, j and any $x \in P_i, y \in P_j$:

- if $\mathsf{M}(i,j) = 0$, then there is no arc between x and y;
- if $\mathsf{M}(i,j) = 1$, then there is an arc between x and y;
- if $\mathsf{M}(i,j) = \star$, then there is no restriction.

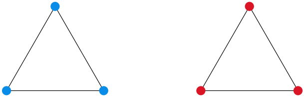
Monotone Monadic SNP without inequalities (MMSNP)

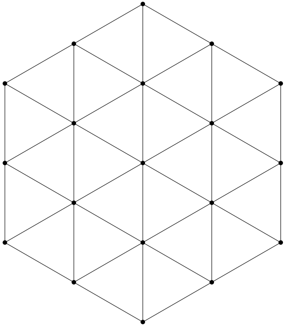


Monotone Monadic SNP without inequalities (MMSNP)

No Monochromatic Triangle

Given a graph, colour its vertices with 2 colours so that the result omits the two following subgraphs.

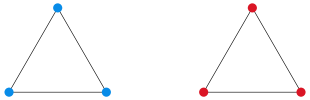


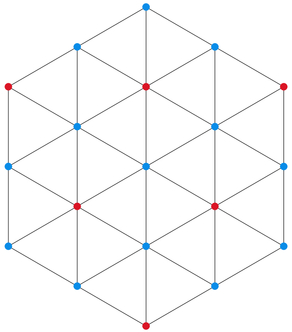


Monotone Monadic SNP without inequalities (MMSNP)

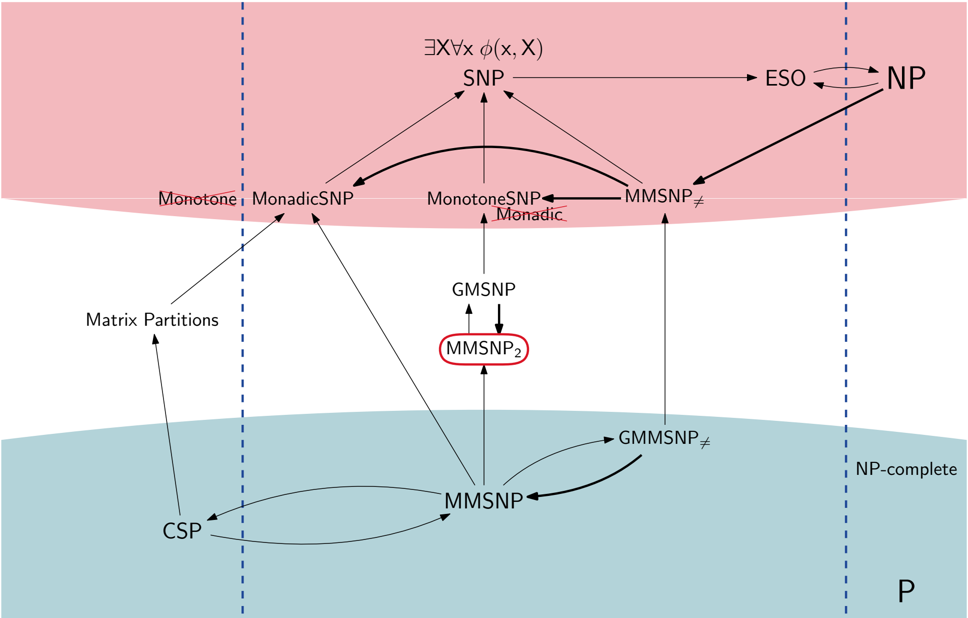
No Monochromatic Triangle

Given a graph, colour its vertices with 2 colours so that the result omits the two following subgraphs.





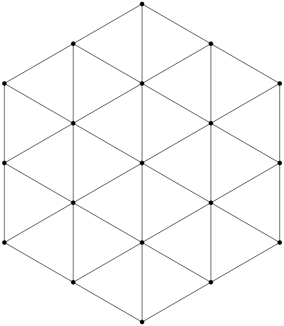
MMSNP_2



$MMSNP_2$

No Monochromatic Edge Triangle

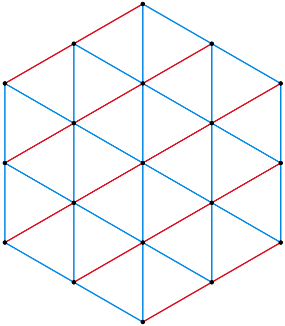
Given a graph, colour its edges with 2 colours so that the result omits the two following subgraphs.



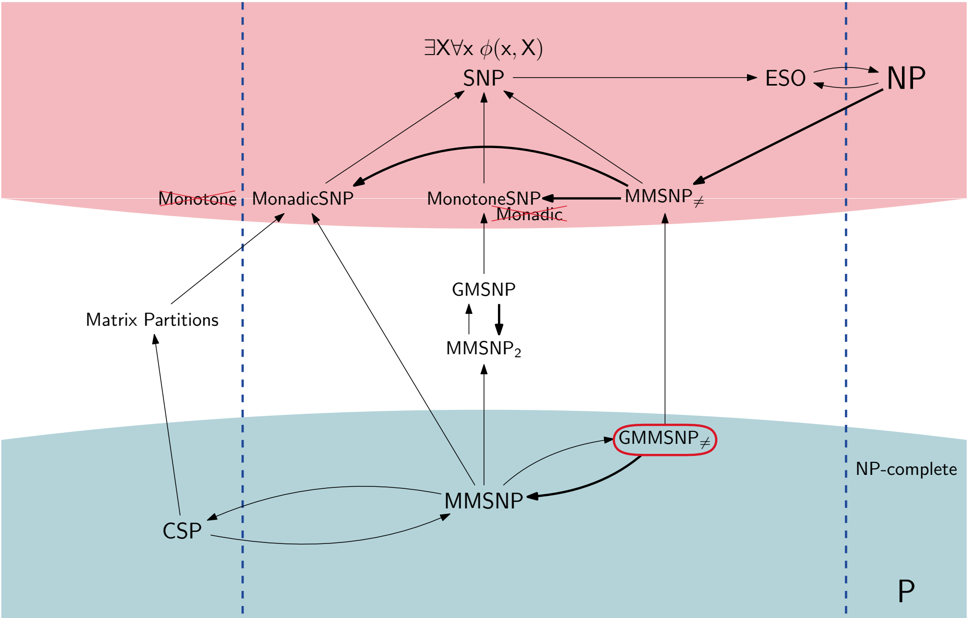
MMSNP$_2$

No Monochromatic Edge Triangle

Given a graph, colour its edges with 2 colours so that the result omits the two following subgraphs.



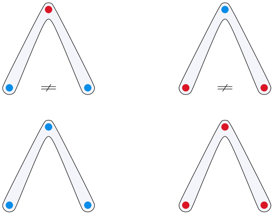
MMSNP with guarded inequalities

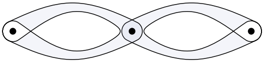


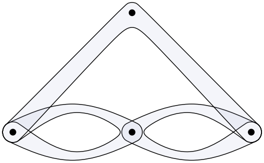
MMSNP with guarded inequalities

Example

Given a structure, colour its vertices with 2 colours so that the result omits the following substructures.



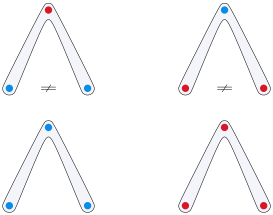


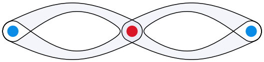


MMSNP with guarded inequalities

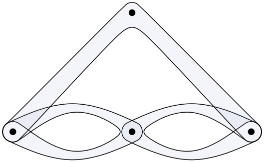
Example

Given a structure, colour its vertices with 2 colours so that the result omits the following substructures.





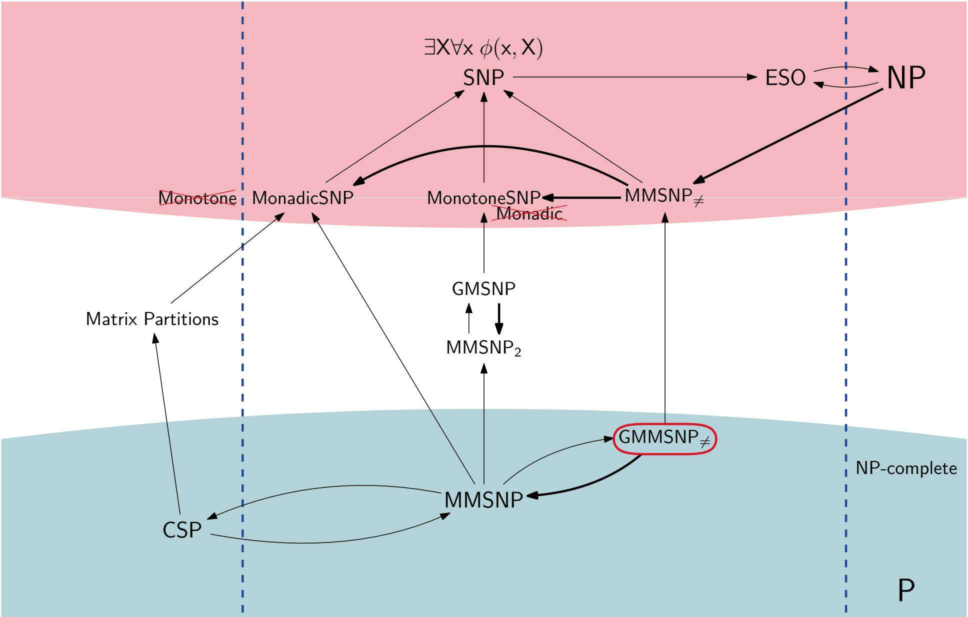
accept



reject

MMSNP with guarded inequalities

GMMSNP$_{\neq}$

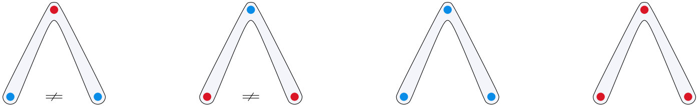


$\exists X \forall x\ \phi(x, X)$
SNP
ESO
NP
~~Monotone~~
MonadicSNP
MonotoneSNP
~~Monadic~~
MMSNP$_{\neq}$
GMSNP
Matrix Partitions
MMSNP$_2$
GMMSNP$_{\neq}$
NP-complete
MMSNP
CSP
P

Main result

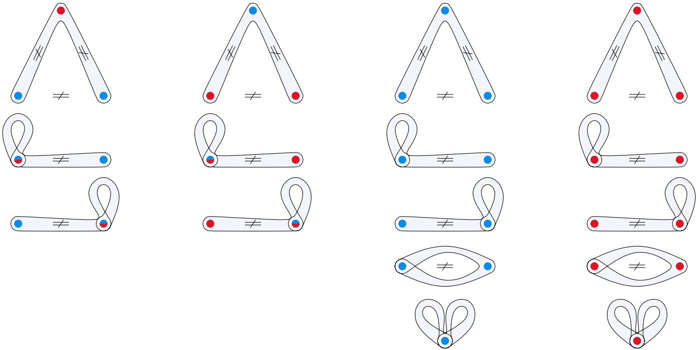
Theorem (B., Kanté, Madelaine)

For any GMMSNP$_{\neq}$ *sentence* Φ *there exists an* MMSNP *sentence* Ψ *such that the problems SAT*(Φ) *and SAT*(Ψ) *are* P*-time equivalent.*



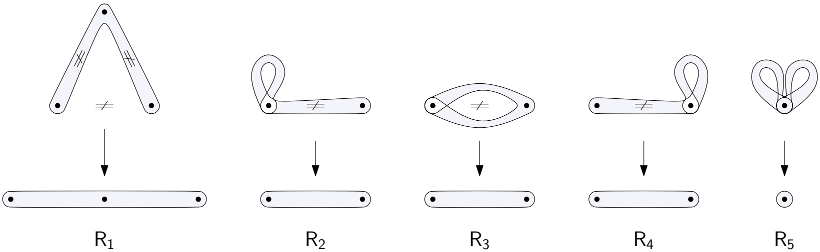
Construction

1. Substitute Φ with an equivalent GMMSNP$_{\neq}$ sentence Φ', where any two distinct variables within the same atom must be unequal.



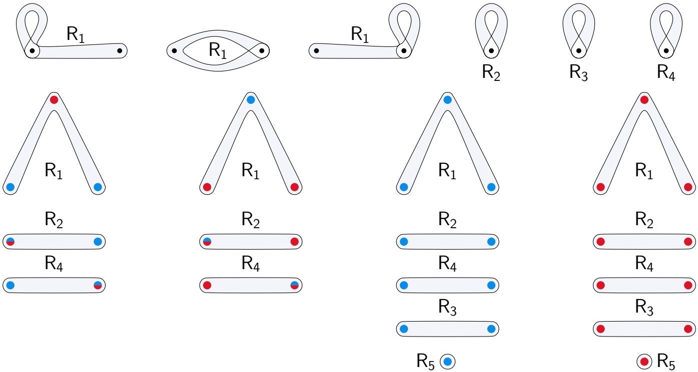
Construction

2. The input signature of Ψ corresponds to all possible equivalence relations on a k-element set, where k is the arity of the $\text{GMMSNP}_{\neq}$ relation.



Construction

3. Restrict $R_1, \ldots, R_5$ to appear only on pairwise distinct vertices. Replace every atom of Φ' with a corresponding atom of the new signature.



Conclusion for GMMSNP$_{\neq}$

Theorem (B., Kanté, Madelaine)

GMMSNP$_{\neq}$ *is strictly more expressive than* MMSNP*.*

Proof (sketch)

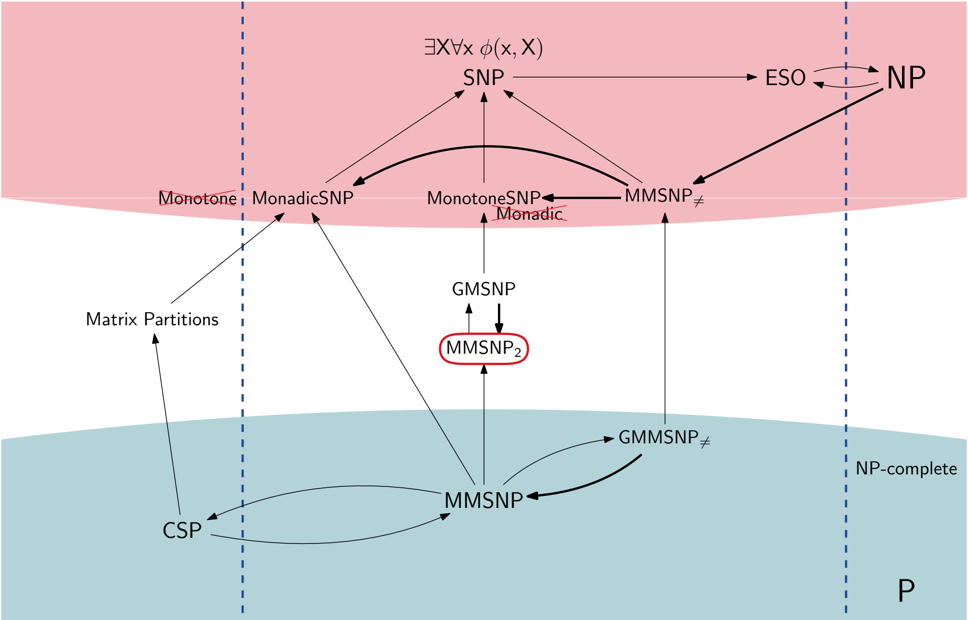
Pick any problem of GMMSNP$_{\neq}$ that is not closed under inverse homomorphisms.

Conclusion

We have found a logic that strictly contains MMSNP and that also has a dichotomy.

MMSNP_2

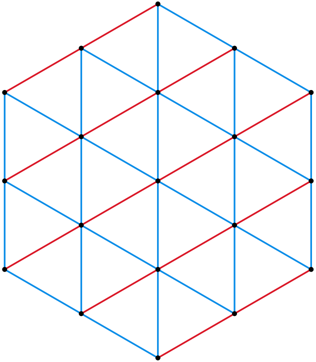
MMSNP$_2$


$\exists X \forall x\ \phi(x, X)$
SNP
ESO
NP
~~Monotone~~
MonadicSNP
MonotoneSNP
~~Monadic~~
MMSNP$_{\neq}$
GMSNP
Matrix Partitions
MMSNP$_2$
GMMSNP$_{\neq}$
NP-complete
MMSNP
CSP
P

Definition of MMSNP_2

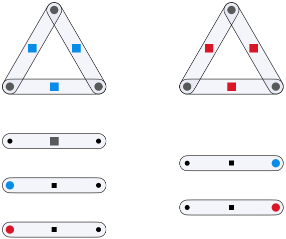
Definition ([Madelaine, 2009])

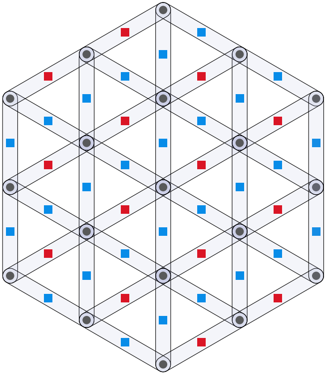
Given a graph, colour its vertices and edges with a fixed number of colours so that no coloured graph of some fixed finite family can map to it.



Reduction to MMSNP

Replace every edge with a triple, where the new third vertex represents the edge colour.

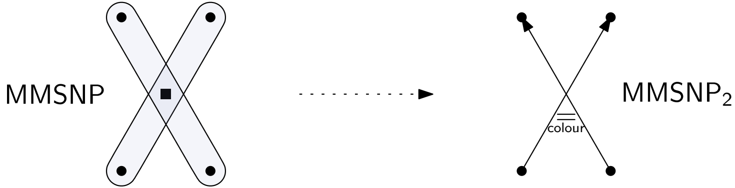




How to go back?

Question

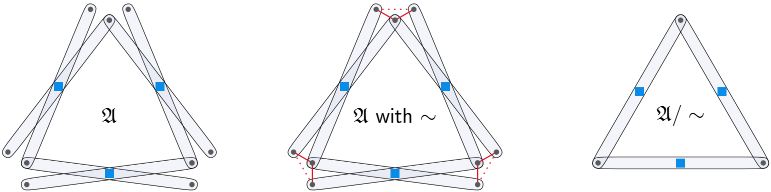
If an MMSNP input instance has two tuples with a common edge-colour vertex, then what should it correspond to in the MMSNP_2 world?



Infinite MMSNP

Answer

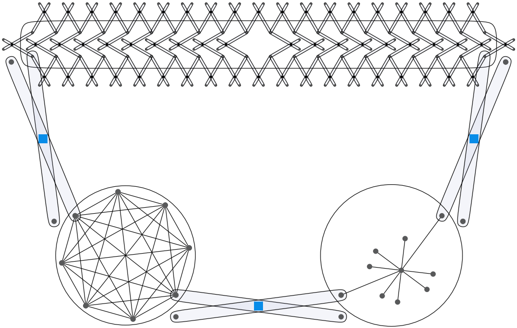
- Say that $x \sim y$ if these vertices are on the same position within a pair of duplicated tuples.
- It suffices to forbid every (infinitely many) coloured structure $\mathfrak{A}$ such that $\mathfrak{A}/\sim$ contains an original forbidden structure.



Infinite MMSNP

Question

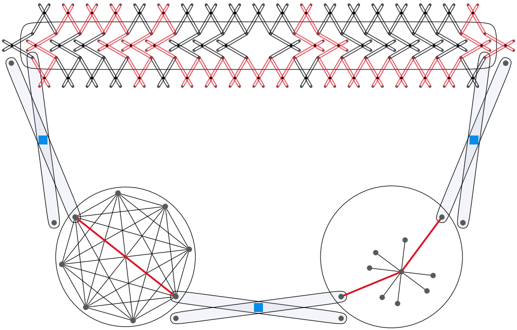
Every $\sim$-equivalence class is based on a connected graph, where edges are provided by $\mathbb{X}$. These graphs may be very complex and large. Is there a simpler family that is also sufficient?



Infinite MMSNP

Answer

Yes. It is sufficient to forbid those and only those structures, where each such graph is a tree with every its leaf being incident to a pair of tuples that connects two graphs.



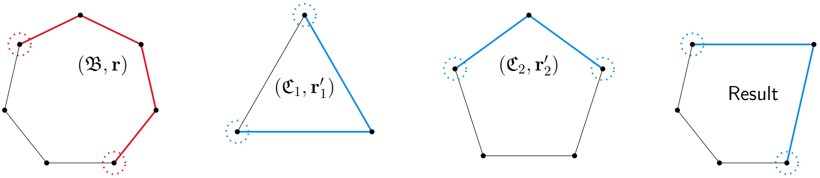
Regularity

Definition ([Hubička, Nešetřil, 2015])

$(\mathfrak{B}, \mathbf{r})$ is called a piece of $\mathfrak{A}$ if $\mathfrak{B}$ is a connected induced substructure and $\mathbf{r}$ is the set of elements of $\mathfrak{B}$ that are incident to tuples not from $\mathfrak{B}$.

Definition ([Hubička, Nešetřil, 2015])

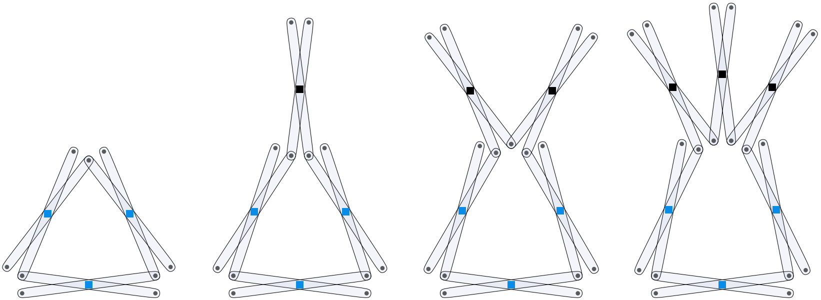
A class of structures $\mathcal{A}$ is called regular if there is a constant $c \in \mathbb{N}$ such that, for any $\mathfrak{A} \in \mathcal{A}$ and any piece $(\mathfrak{B}, \mathbf{r})$ of $\mathfrak{A}$ there is a piece $(\mathfrak{C}, \mathbf{r}')$ of another structure of $\mathcal{A}$ such that $|\mathfrak{C}| < c$ and that replacing $(\mathfrak{B}, \mathbf{r})$ with $(\mathfrak{C}, \mathbf{r}')$ gives another structure from $\mathcal{A}$.



Regular MMSNP

Theorem (B., Kanté, Madelaine)

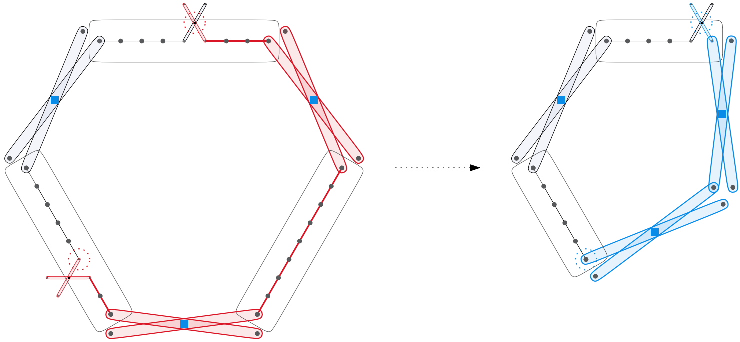
Any MMSNP_2 *problem is* P*-time equivalent to a infinite* MMSNP *problem, where the infinite class of forbidden structures is regular.*



Regular MMSNP

Proof (sketch)

Every piece is as the red part on the left. It can be replaced by a rooted structure of bounded size as on the right, and the result is another structure from the class, where the paths are shorter.



Conclusion for MMSNP_2

Theorem ([Bodirsky, Madelaine, Mottet, 2018])

The class of ω-categorical CSP that are described by MMSNP *sentences has a dichotomy.*

Conclusion

- If one manages to extend this result onto ω-categorical CSP that are described by regular MMSNP sentences, then it will immediately imply a dichotomy for MMSNP_2.
- In any case, now we understand better how MMSNP_2 and MMSNP are related.

References

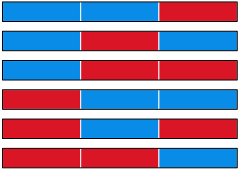
- Tomás Feder and Pavol Hell and Wing Xie
  Matrix Partitions with Finitely Many Obstructions
  Electron. J. Comb., 2007, 10.37236/976
- Florent R. Madelaine
  Universal Structures and the logic of Forbidden Patterns
  Log. Methods Comput. Sci., 2009, 10.2168/LMCS-5(2:13)2009
- Jan Hubička and Jaroslav Nešetřil
  Universal Structures with Forbidden Homomorphisms
  Logic Without Borders, 2015, 10.1515/9781614516873.241
- Manuel Bodirsky and Florent R. Madelaine and Antoine Mottet
  A universal-algebraic proof of the complexity dichotomy for Monotone Monadic SNP
  Logic in Computer Science, 2018, 10.1145/3209108.3209156

Thank you!

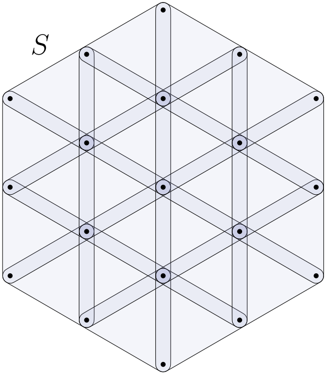
Reduction of MMSNP to CSP

Reduction

- Replace every triangle of the input graph with a relational triple.
- Check if the resulting structure S maps to T, where T is as follows.



T



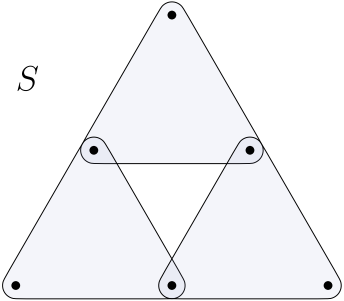
The other direction

Naive approach

- Replace every relational triple of S with a triangle.
- Check if the resulting graph satisfies the MMSNP sentence.

Obstacle

What to do when S contains implicit triangles?

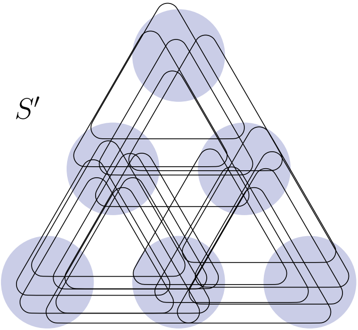


Solution

Lemma (Erdős)

For given structures S, T, and $\ell > 0$ there exists S' such that

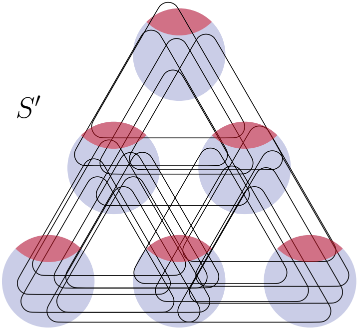
- $S \to T$ iff $S' \to T$;
- S' does not contain cycles of length less than ℓ, i.e., the girth of S' is at least ℓ.



Solution

Proof

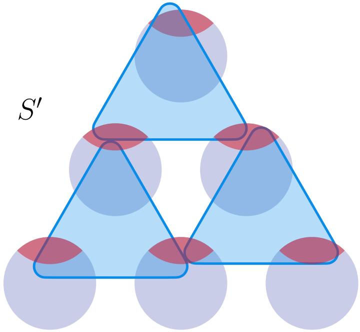
- By construction, $S' \to S$.
- The number of cycles of length $< \ell$ is small, so we need to remove a few tuples to get rid of them.
- If $S' \to T$, then each "bag" of size N contains at least $\frac{N}{|T|}$ vertices that are mapped to the same vertex in T.



Solution

Proof

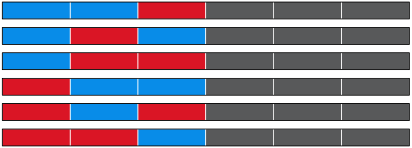
- By construction, $S' \to S$.
- The number of cycles of length $< \ell$ is small, so we need to remove a few tuples to get rid of them.
- If $S' \to T$, then each "bag" of size N contains at least $\frac{N}{|T|}$ vertices that are mapped to the same vertex in T.
- Tuples are distributed uniformly, so every triple of "bags" has at least one tuple induced on them.



Reduction of MMSNP$_2$ to CSP

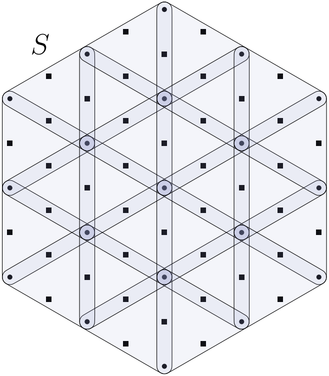
Reduction

- Replace every triangle of the input graph G with a relational 6-tuple.
- Check if the resulting structure maps to T, where T is as follows.



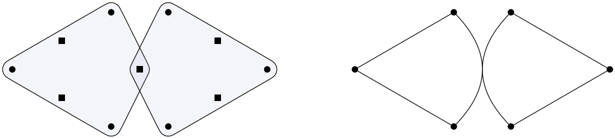
T

S

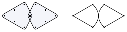


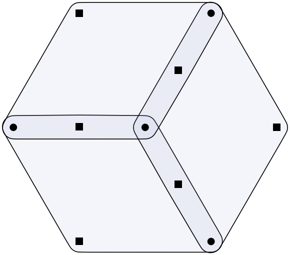
Obstacles for the other direction

- Within S', it is not allowed to join two 6-tuples only by a vertex representing an original edge.

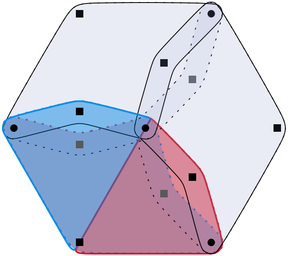


Obstacles for the other direction

- Within S', it is not allowed to join two 6-tuples only by a vertex representing an original edge. 
- Joining 6-tuples only by vertices that represent original vertices is not sufficient.

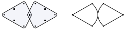


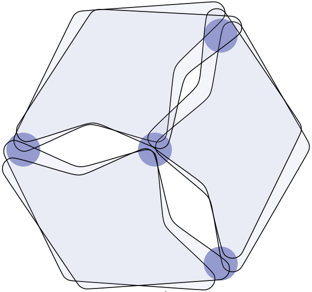
S



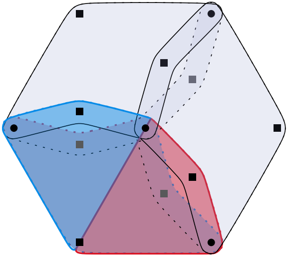
T

Obstacles for the other direction

- Within S', it is not allowed to join two 6-tuples only by a vertex representing an original edge. 
- Joining 6-tuples only by vertices that represent original vertices is not sufficient.



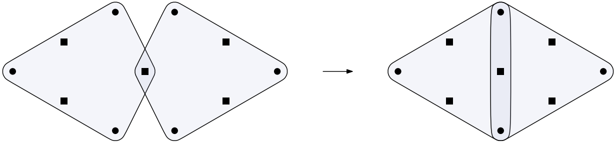
S'



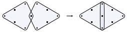
T

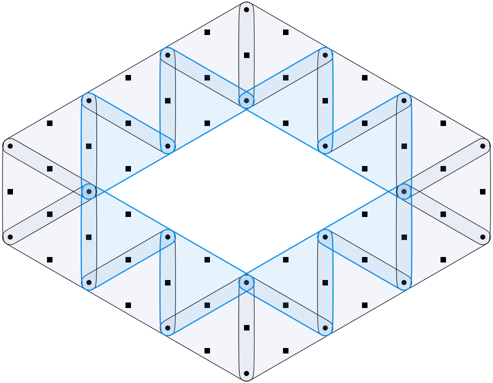
How can we help ourselves?

- We can provide that, if two tuples in S share an edge-vertex, then they share the whole implicit edge.



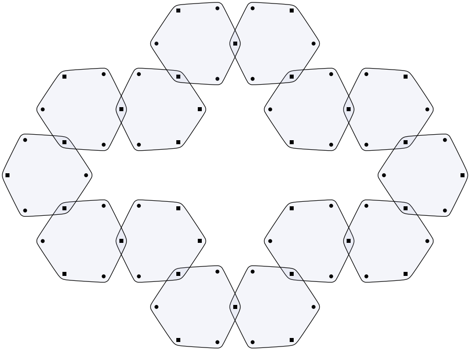
How can we help ourselves?

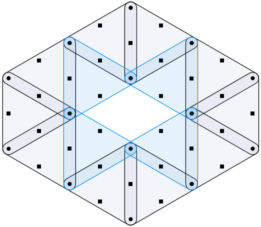
- We can provide that, if two tuples in S share an edge-vertex, then they share the whole implicit edge. 
- Within S', we are allowed to join two tuples by an implicit edge, and it will not reduce the girth down to 2.



How to construct S' right?

We can apply the same Erdős' method as for MMSNP. But then we need to identify vertices within S' in order to replace it later with a graph. This procedure reduces the girth of S'.





How to construct S' right?

- Change the measure function for vertices of S' from the Lemma of Erdős, *e.g.*, consider the degrees of vertices.
- Consider the layer configuration of T and its cycles and to construct S' depending on them.
- To solve a weaker problem: bounded-degree input, S' having exponential size with respect to S, etc.

Questions about MP apart from dichotomy

1. To show that any generalization of MP to an arbitrary relational signature is P-time equivalent to MP on digraphs.

Questions about MP apart from dichotomy

1. To show that any generalization of MP to an arbitrary relational signature is P-time equivalent to MP on digraphs.
2. To find a logic that is equivalent to MP. Similarly, as MMSNP $\leftrightarrow$ CSP.

Questions about MP apart from dichotomy

1. To show that any generalization of MP to an arbitrary relational signature is P-time equivalent to MP on digraphs.
2. To find a logic that is equivalent to MP. Similarly, as MMSNP $\leftrightarrow$ CSP.
3. To determine when a MP problem has finitely many minimal obstructions.

Definition

A directed graph G is called a minimal obstruction of a problem MP(M) if $G \notin$ MP(M) and, for any induced subgraph $G' \subsetneq G$, $G' \in$ MP(M).

Theorem (Atserias'08)

CSP(H) is definable in first-order logic iff it has finitely many minimal obstructions.

CSP Inclusion into Digraphs

$$\begin{tikzcd}
\mathrm{CSP}(R_1, \ldots, R_n) \arrow[dd, bend right] \\
\mathrm{CSP}(R) \arrow[u] \\
\mathrm{CSP}(\text{digraphs}) \arrow[u]
\end{tikzcd}$$

Theorem (Feder, Vardi'98) (Bulin et al.'15)

CSP over an arbitrary finite signature is P-time equivalent to CSP on digraphs.

Results for the MP case

$$\begin{tikzcd}
\mathsf{MP}_\star(\mathsf{R}_1,\ldots,\mathsf{R}_n) \arrow[r, leftrightarrow] \arrow[dd, bend right] & \mathsf{MP}(\mathsf{R}_1,\ldots,\mathsf{R}_n) \\
\mathsf{MP}_\star(\mathsf{R}) \arrow[r, leftrightarrow] \arrow[u] & \mathsf{MP}(\mathsf{R}) \\
\mathsf{MP}_\star \arrow[r, leftrightarrow] & \mathsf{MP}
\end{tikzcd}$$

The difference between $\mathsf{MP}_\star$ and MP is the presence of $\star$-graphs in the input.

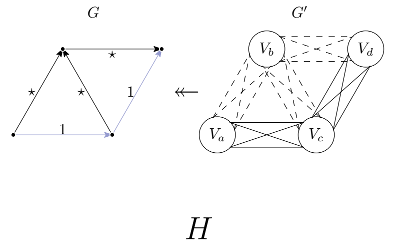
Theorem (B., Kanté '21)

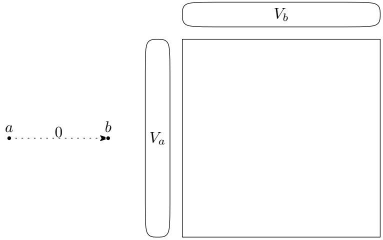
For any finite signature σ, $\mathsf{MP}(\sigma) \leftrightarrow \mathsf{MP}_\star(\sigma)$.

Theorem (B., Kanté '21)

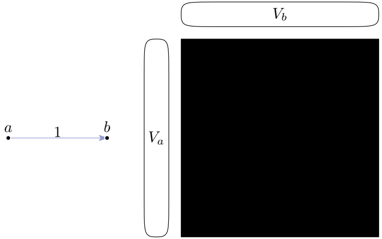
For any finite signature σ, there exists a signature $\tilde{\sigma} = \{\mathsf{R}\}$ such that $\mathsf{MP}(\tilde{\sigma}) \to \mathsf{MP}(\sigma)$.

MP $\leftrightarrow$ MP$_\star$

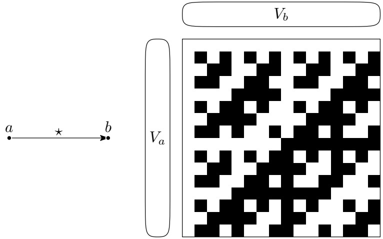




All-zero matrix

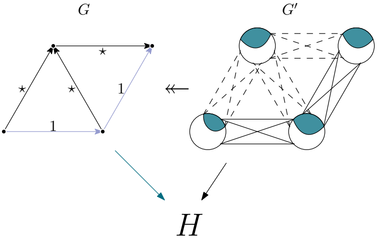


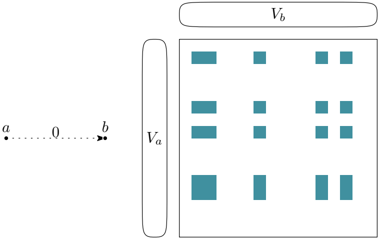
All-one matrix



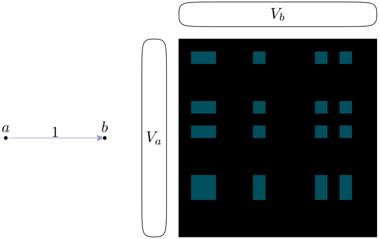
Hadamard matrix

MP $\leftrightarrow$ MP$_\star$

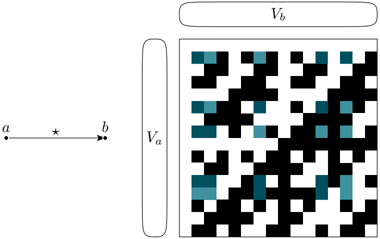




All-zero matrix



All-one matrix



Hadamard matrix

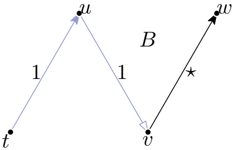
$\mathsf{MP}_\star(\mathsf{R}) \to \mathsf{MP}_\star(\mathsf{R}_1, \ldots, \mathsf{R}_n)$

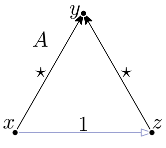
Lemma

$\mathsf{MP}_\star(A)$ reduces in P-time to $\mathsf{MP}_\star(\tilde{A})$.

Proof

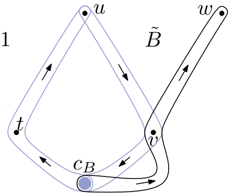
$$\mathsf{R}_1(t,u) = 1$$
$$\mathsf{R}_1(v,w) = \star$$
$$\mathsf{R}_2(u,v) = 1$$

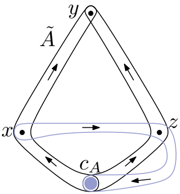




$$\mathsf{R}_1(x,y) = \star$$
$$\mathsf{R}_1(x,z) = \star$$
$$\mathsf{R}_2(x,z) = 1$$

$$\mathsf{R}(c_B, c_B, c_B) = 1$$
$$\mathsf{R}(c_B, t, u) = 1$$
$$\mathsf{R}(c_B, v, w) = \star$$
$$\mathsf{R}(u, v, c_B) = 1$$





$$\mathsf{R}(c_A, c_A, c_A) = 1$$
$$\mathsf{R}(c_A, x, y) = \star$$
$$\mathsf{R}(c_A, z, y) = \star$$
$$\mathsf{R}(x, z, c_A) = 1$$

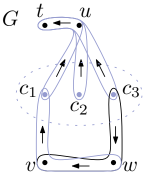
$\mathsf{MP}_\star(\mathsf{R}) \to \mathsf{MP}_\star(\mathsf{R}_1, \dots, \mathsf{R}_n)$

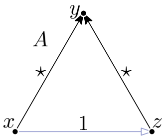
Lemma

$\mathsf{MP}_\star(\tilde{A})$ reduces in P-time to $\mathsf{MP}_\star(A)$.

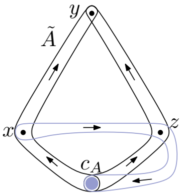
Proof

$$\mathsf{R}(w, v, c_1) = 1$$
$$\mathsf{R}(c_3, w, v) = \star$$
$$\forall i\colon \mathsf{R}(c_i, u, t) = 1$$
$$\forall i, j, k\colon \mathsf{R}(c_i, c_j, c_k) = 1$$





$$\mathsf{R}_1(x, y) = \star$$
$$\mathsf{R}_1(x, z) = \star$$
$$\mathsf{R}_2(x, z) = 1$$



$$\mathsf{R}(c_A, c_A, c_A) = 1$$
$$\mathsf{R}(c_A, x, y) = \star$$
$$\mathsf{R}(c_A, z, y) = \star$$
$$\mathsf{R}(x, z, c_A) = 1$$

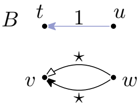
$\mathsf{MP}_\star(\mathsf{R}) \to \mathsf{MP}_\star(\mathsf{R}_1, \ldots, \mathsf{R}_n)$

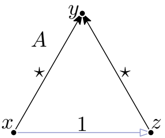
Lemma

$\mathsf{MP}_\star(\tilde{A})$ reduces in P-time to $\mathsf{MP}_\star(A)$.

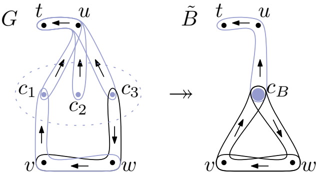
Proof

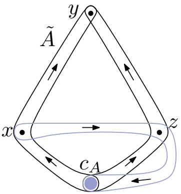
$$\mathsf{R}(w, v, c_1) = 1$$
$$\mathsf{R}(c_3, w, v) = \star$$
$$\forall i\colon \mathsf{R}(c_i, u, t) = 1$$
$$\forall i, j, k\colon \mathsf{R}(c_i, c_j, c_k) = 1$$





$$\mathsf{R}_1(x, y) = \star$$
$$\mathsf{R}_1(x, z) = \star$$
$$\mathsf{R}_2(x, z) = 1$$





$$\mathsf{R}(c_A, c_A, c_A) = 1$$
$$\mathsf{R}(c_A, x, y) = \star$$
$$\mathsf{R}(c_A, z, y) = \star$$
$$\mathsf{R}(x, z, c_A) = 1$$