Edge-colourings and constraint satisfaction problems

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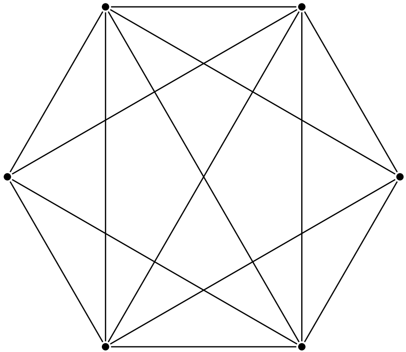
A classic example

No Monochromatic Triangle

Given: a graph (V, E).
Task: to partition E in two classes E_1, E_2 such that neither (V, E_1) nor (V, E_2) contains a triangle.



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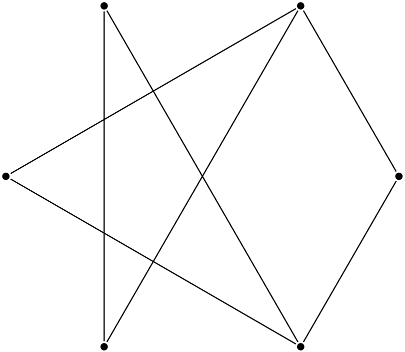

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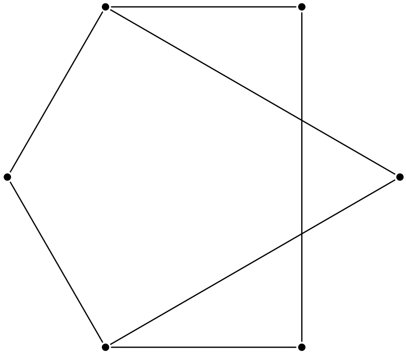
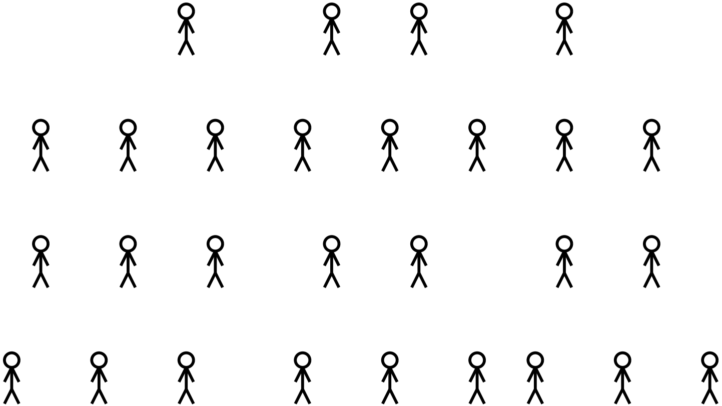


(V, E_2)

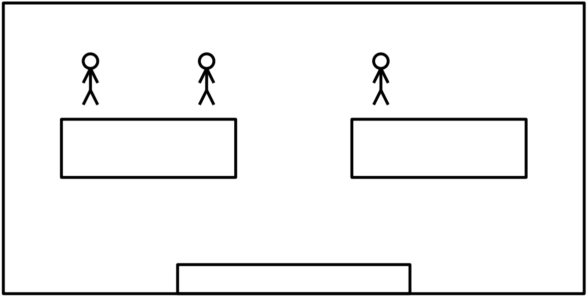
A “real life” example



The participants of AAA

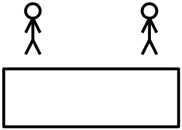
A “real life” example

The AAA classroom

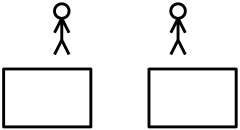


Two desks, three people

A “real life” example



$\Longrightarrow$ always together



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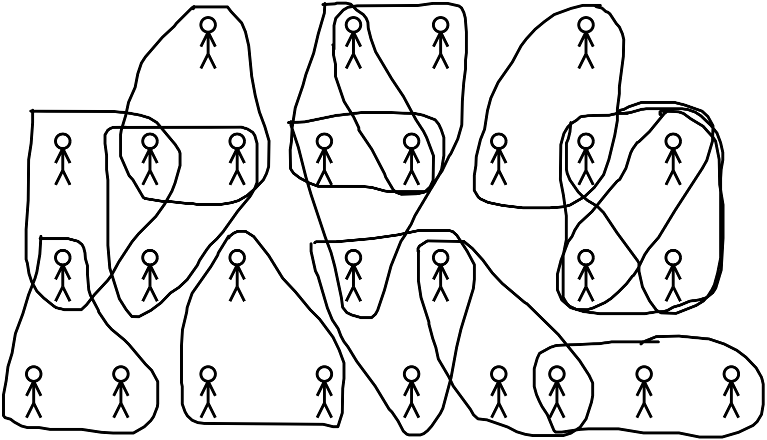

Choose who sits together

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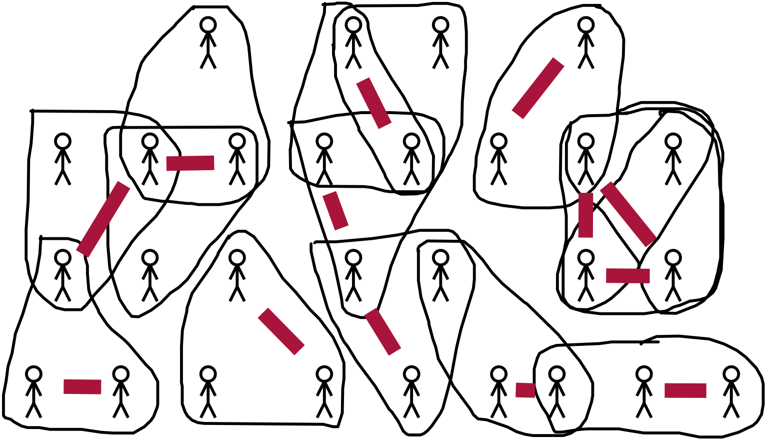

Choose who sits together

A formal definition of GMSNP

Given: a finite relational structure (e.g., a graph or a ternary AAA)

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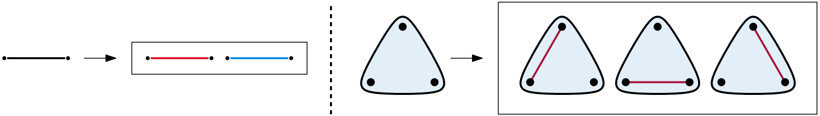
Task: assign to every relational tuple one of the several colours



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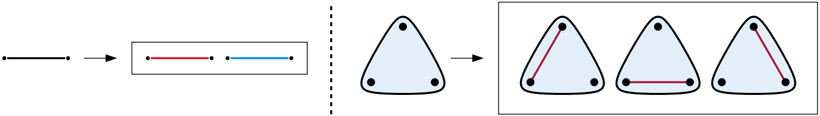
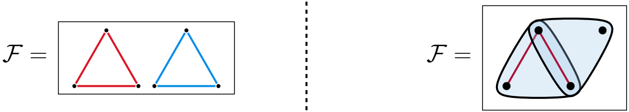


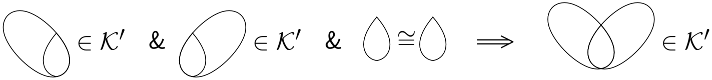
s.t. the result is $\mathcal{F}$-free, i.e., it contains no hom-images of structures from a fixed finite forbidden family $\mathcal{F}$.



GMSNP seen as a CSP

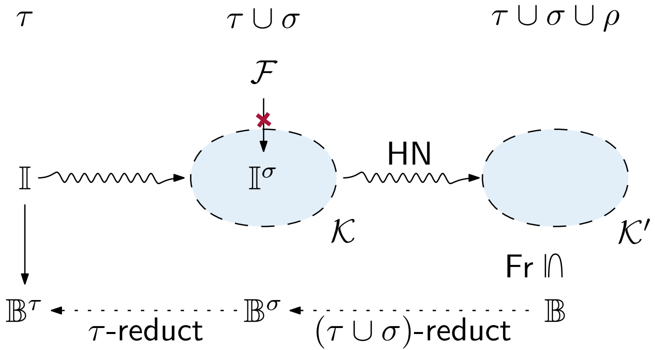
Let $\mathcal{K} :=$ be the class of all finite $\mathcal{F}$-free structures (all solutions).
Hubička, Nešetřil: there is a class $\mathcal{K}'$ obtained from $\mathcal{K}$ by adding finitely many new relations, $\mathcal{K}'$ is closed under taking substructures (HP) and has the amalgamation (AP) and Ramsey properties.

AP :



Fraïssé: if $\mathcal{K}'$ is closed under disjoint unions, has HP and AP, then there is a homogeneous structure $\mathbb{B}$ such that $\text{Age}(\mathbb{B}) = \mathcal{K}'$.

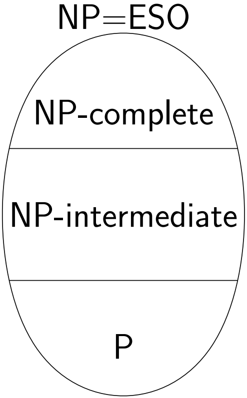
GMSNP seen as a CSP



Observation

An input $\mathbb{I}$ has an $\mathcal{F}$-free σ-expansion ($\mathbb{I} \in$ GMSNP($\mathcal{F}$)) if and only if $\mathbb{I}$ homomorphically maps to $\mathbb{B}^\tau$ ($\mathbb{I} \in$ CSP($\mathbb{B}^\tau$)).

The dichotomy question



Ladner: If P $\neq$ NP, then NP has problems that are neither in P nor NP-complete.

Fagin: The problems in NP are precisely those that are described by sentences in Existential Second-Order logic (ESO).

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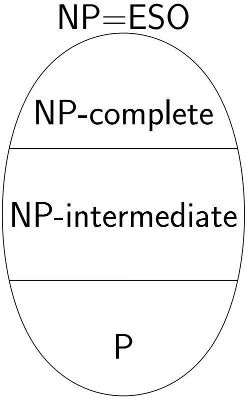

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Question

For a given logic $\mathcal{L} \subset$ ESO, is $\mathcal{L}$ a subset of (P $\cup$ NP-complete)?

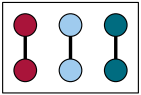
The dichotomy question

finite CSP $\subset$ MMSNP $\subset$ GMSNP $\subset$ CSPs of first-order reducts of finitely bounded homogeneous structures $\subset$ Monotone SNP

Given: a finite relational structure.

Task: assign to every vertex one of the several colours

such that the result is $\mathcal{F}$-free

$\mathcal{F} =$ 

The dichotomy question

finite CSP ⊂ MMSNP ¦ ⊂ GMSNP ⊂ CSPs of first-order reducts of finitely bounded homogeneous structures ⊂ ¦ Monotone SNP

Feder, Vardi: Every problem in MMSNP is P-time equivalent to a finite CSP.

Zhuk, Bulatov: Finite CSPs have a dichotomy that is characterized by algebraic properties of the template.

The dichotomy question

finite CSP $\subset$ MMSNP $\subset$ GMSNP $\subset$ CSPs of first-order reducts of finitely bounded homogeneous structures $\subset$ Monotone SNP

$\mathbb{A}$ is **homogeneous** if every isomorphism between its finite substructures extends to an automorphism of $\mathbb{A}$.

$\mathbb{A}$ is **finitely bounded** if for some finite family $\mathcal{F}$

$\forall\, \mathbb{B}$ finite $(\mathbb{B} \subset \mathbb{A} \Leftrightarrow \forall\, \mathbb{F} \in \mathcal{F} \;\; \mathbb{F} \not\to \mathbb{B})$ (Age($\mathbb{A}$) is $\mathcal{F}$-free)

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$\mathbb{B}$ is a **first-order reduct** of $\mathbb{A}$ if $\mathbb{B}$ has the same domain as $\mathbb{A}$ and if every relation of $\mathbb{B}$ is first-order definable in $\mathbb{A}$.

Conjecture (Bodirsky, Pinsker): CSPs of FORoFBHS have a dichotomy characterized by algebraic properties of the template.

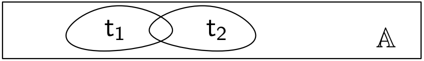
The dichotomy question

finite CSP $\subset$ MMSNP $\subset$ GMSNP $\subset$ CSPs of first-order reducts of finitely bounded homogeneous structures $\subset$ Monotone SNP

Given: a finite relational structure $\mathbb{A}$.

Task: assign a colour to each k-element subset of $\mathbb{A}$ (k is fixed)

s.t. the colours assigned to intersecting subsets are compatible.



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Feder, Vardi: Every problem in NP is P-time equivalent to a problem in Monotone SNP.

The containment question

Given: two decision problems Φ and Ψ.
Task: to check whether every YES instance of Φ is a YES instance of Ψ, denoted $\Phi \subseteq \Psi$.

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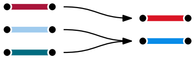
Observation

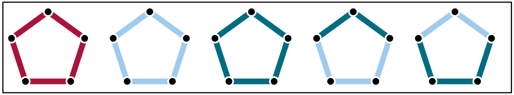
- In general, containment is undecidable.
- For $\mathbb{A}, \mathbb{B}$ finite or FORoFBHS, we have CSP$(\mathbb{A}) \subseteq$ CSP$(\mathbb{B})$ if and only if $\mathbb{A} \to \mathbb{B}$.

Containment is decidable for GMSNP

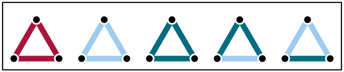
r: {colours of Φ} $\rightarrow$ {colours of Ψ} is a **recolouring** from Φ to Ψ



if the preimage $r^{-1}(\mathcal{F}_\Psi)$ has no $\mathcal{F}_\Phi$-free structures



$\mathcal{F}_\Phi$



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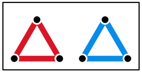
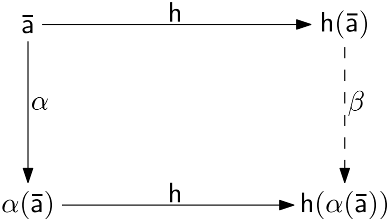


$\mathcal{F}_\Psi$

recolouring $\Rightarrow$ containment

Containment is decidable for GMSNP

A mapping h: $\mathbb{A} \to \mathbb{B}$ is **canonical** if for every n and every $\bar{a} \in A^n$ and every automorphism $\alpha \in \mathrm{Aut}(\mathbb{A})$ there is $\beta \in \mathrm{Aut}(\mathbb{B})$ s.t.



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Bodirsky, Pinsker, Tsankov:

$\mathbb{B}^{\tau}_{\Phi}$ has a homogeneous Ramsey expansion $\mathbb{B}_{\Phi}$ $\Longrightarrow$ h can be made canonical w.r.t. $\mathrm{Aut}(\mathbb{B}_{\Phi})$ and $\mathrm{Aut}(\mathbb{B}_{\Psi})$

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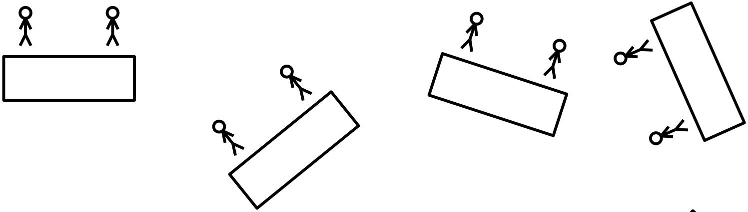
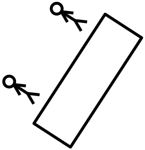
containment $\Rightarrow$ recolouring





Thank You!