MMSNP and MMSNP2

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Introduction

Overview

Definition

A set of problems has a dichotomy if any of its problems is either P-time or NP-complete.

Theorem (Ladner, 1975)

If P≠NP, then NP has no dichotomy.

$NP = ESO$	no dichotomy
$MMSNP_2$	unknown
$CSP =_p MMSNP$	dichotomy

Overview

Theorem (Fagin, 1974)

NP equals ESO.

Theorem (Feder, Vardi, 1998)

MMSNP has a dichotomy iff CSP has.

Theorem (Bulatov, Zhuk, 2017)

CSP has a dichotomy.

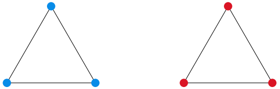
$NP = ESO$	no dichotomy
$MMSNP_2$	unknown
$CSP =_p MMSNP$	dichotomy

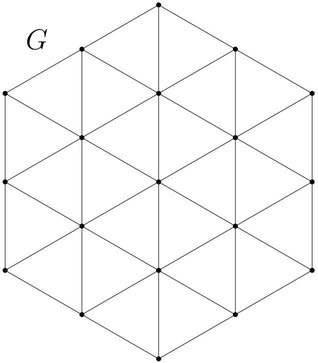
MMSNP

Definition

No Monochromatic Triangle

Given a graph G, colour its vertices with 2 colours so that the result omits the two following subgraphs.





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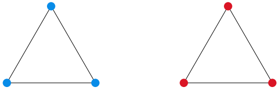
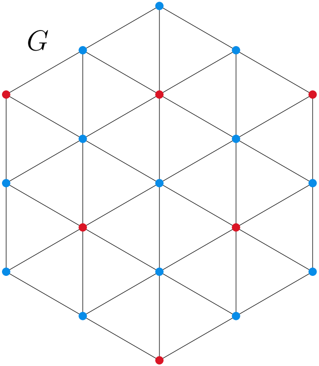
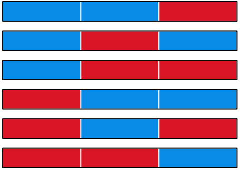




Reduction to CSP

Reduction

- Replace every triangle of the input graph G with a relational triple.
- Check if the resulting structure maps to T, where T is as follows.



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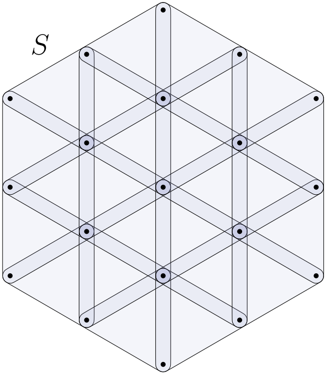


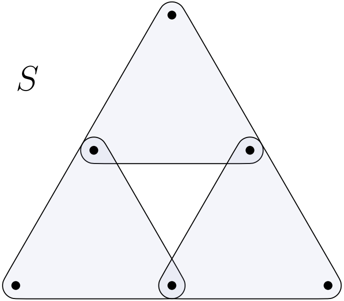
The other direction

Naive approach

- Replace every relational triple of S with a triangle.
- Check if the resulting graph G satisfies the MMSNP sentence.

Obstacle

What to do when S contains implicit triangles?

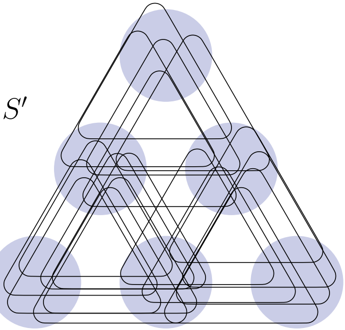


Solution

Lemma (Erdős)

For given structures S, T, and $l > 0$ there exists S' such that

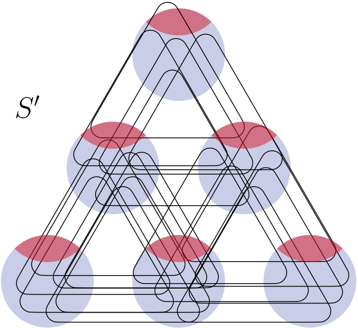
- $S \to T$ iff $S' \to T$;
- S' does not contain cycles of length less than l, *i.e.*, the girth of S' is at least l.



Solution

Proof

- By construction, $S' \to S$.
- The number of cycles of length $< l$ is small, so we need to remove a few tuples to get rid of them.
- If $S' \to T$, then each "bag" of size N contains at least $\frac{N}{|T|}$ vertices that are mapped to the same vertex in T.



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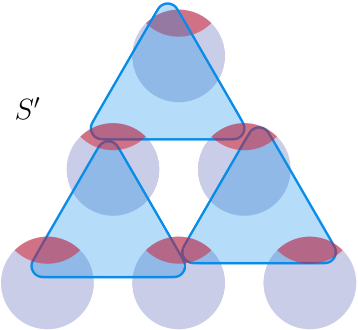
- If $S' \to T$, then each "bag" of size N contains at least $\frac{N}{|T|}$ vertices that are mapped to the same vertex in T.
- Tuples are distributed uniformly, so every triple of "bags" has at least one tuple induced on them.

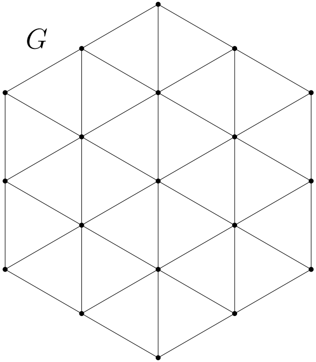


MMSNP2

Definition

No Monochromatic Edge Triangle

Given a graph G, colour its edges with 2 colours so that the result omits the two following subgraphs.



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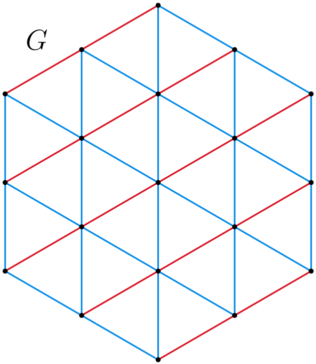
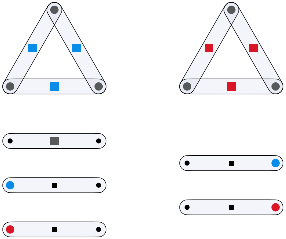
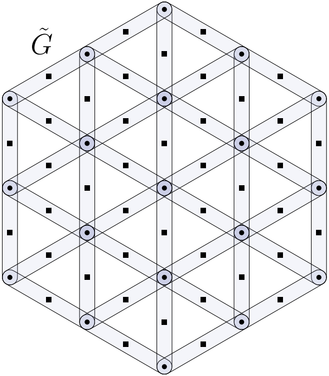


Reduction

Replace every edge with a triple, where the new vertex represents the edge.





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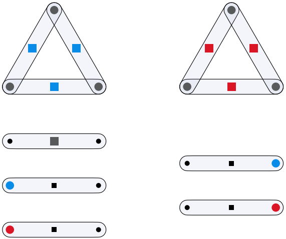

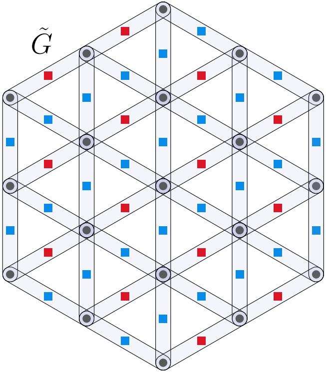

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Reduction

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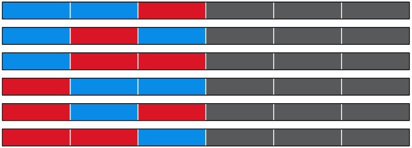

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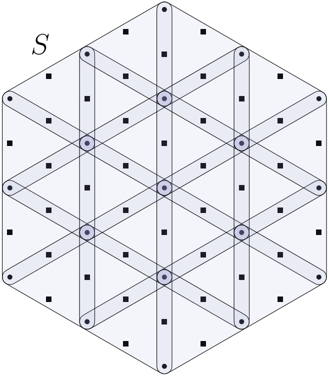
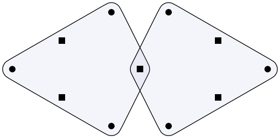
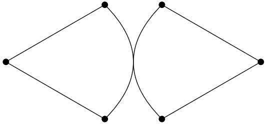


Obstacles for the other direction

- Within S', it is not allowed to join two 6-tuples only by a vertex representing an original edge.





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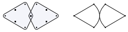
- Within S', it is not allowed to join two 6-tuples only by a vertex representing an original edge. 
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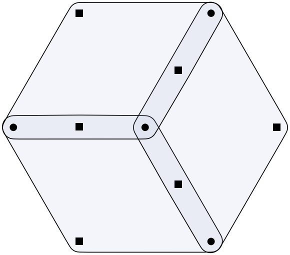
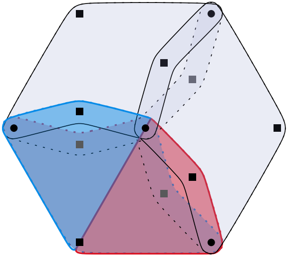


S



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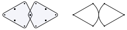
- Within S', it is not allowed to join two 6-tuples only by a vertex representing an original edge. 
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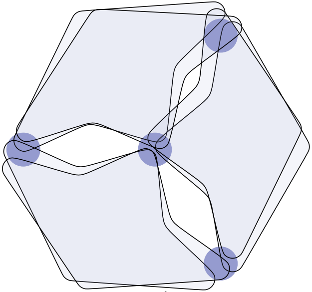
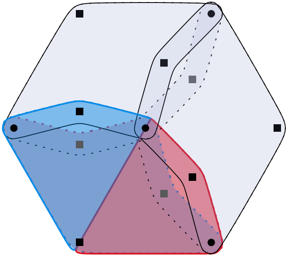


S'



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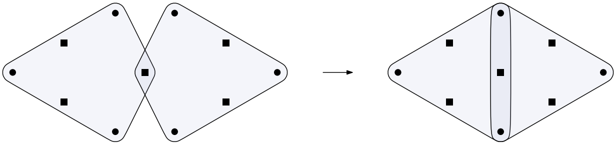
How can we help ourselves?

- We can provide that, if two tuples in S share an edge-vertex, then they share the whole implicit edge.



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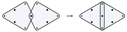
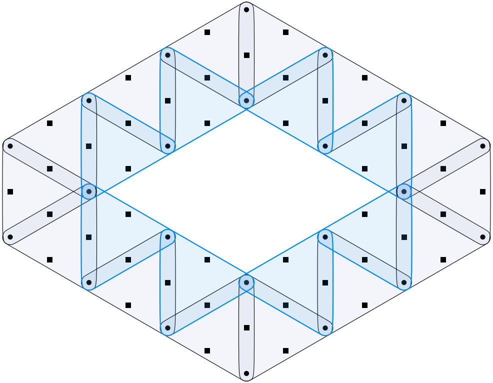
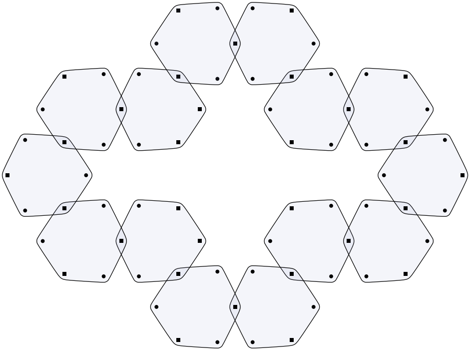
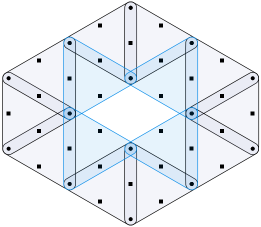
- We can provide that, if two tuples in S share an edge-vertex, then they share the whole implicit edge. 
- Within S', we are allowed to join two tuples by an implicit edge, and it will not reduce the girth down to 2.



How to construct S' right?

We can apply the same Erdős' method as for MMSNP. But then we need to identify vertices within S' in order to replace it later with a graph. This procedure reduces the girth of S'.





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- To solve a weaker problem: bounded-degree input, S' having exponential size with respect to S, etc.

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Thank You!