Investigating potential dichotomies above Feder and Vardi's logic MMSNP

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Introduction

Constraint Satisfaction Problems

Definition

A constraint satisfaction problem (CSP) is a triple $\langle V, D, C \rangle$ where:

- $V = \{v_1, \ldots, v_n\}$ is the set of variables.
- $D = \{d_1, \ldots, d_s\}$ is the set of the domain values.
- C is the set of constraints. Any constraint is of the form $\langle x_1, \ldots, x_k, R \rangle$ where $x_1, \ldots, x_k \in V$ and R is a k-ary relation defined on D.

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Definition

A solution to a CSP is a map $s\colon V \to D$ such that, for any $\langle x_1, \ldots, x_n, R\rangle \in C$, $R(s(x_1), \ldots, s(x_n))$ is satisfied.

Constraint Satisfaction Problems

Example

- Variables: the regions of a map.
- Domain: green, violet, blue, grey.
- Constraints: any two contiguous departments cannot have the same colour.

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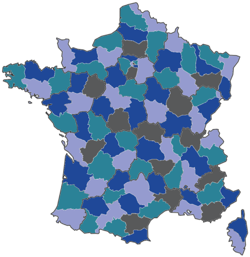
$\rightarrow$



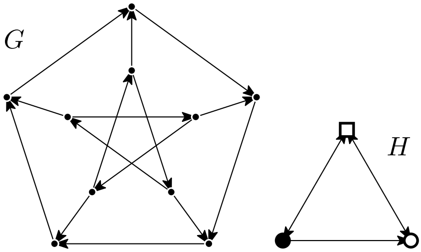
Constraint Satisfaction Problems

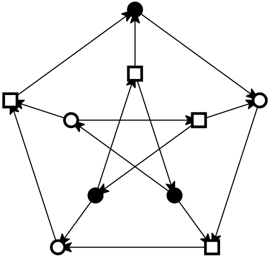
Example

- Variables: the vertices of G.
- Domain: the vertices of H.
- Constraints: the edges of G.

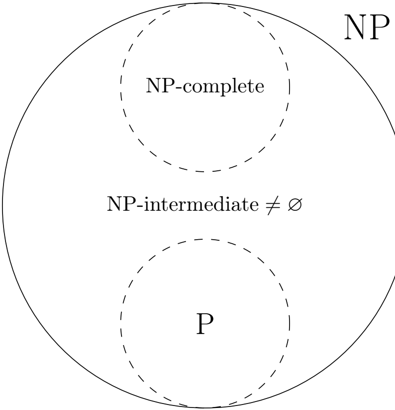
Remark

CSPs are usually thought of as digraph homomorphism problems. The target H is fixed, G is the input. Notation: $\mathrm{CSP}(H)$.



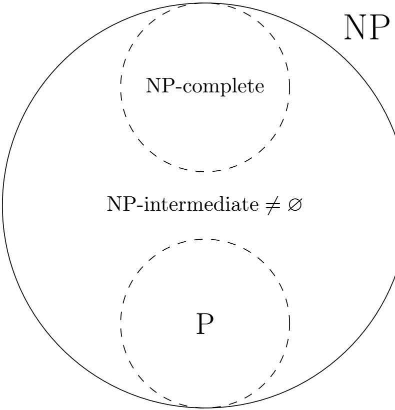


NP-intermediate Class and Dichotomy



Theorem (Ladner'75)

If $\mathrm{P} \neq \mathrm{NP}$ then there is a problem $L \in \mathrm{NP}$ such that $L \notin \mathrm{P} \sqcup \mathrm{NP}$-complete. The class of such problems is called NP-intermediate.



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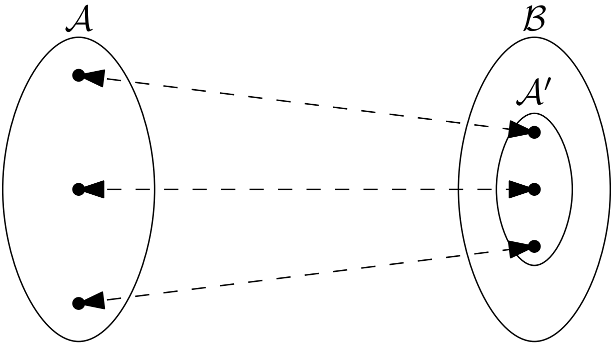
Dichotomy Question

Take a complexity class $\mathcal{C} \subset \mathrm{NP}$. Are there problems $L \in \mathcal{C}$ such that $L \in \mathrm{NP}$-intermediate? If there are no such L then we say that $\mathcal{C}$ has a dichotomy.

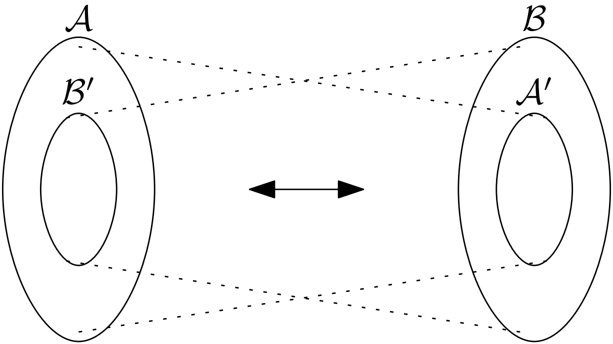
P-time Equivalence



Observation

If for any problem $P_{\mathcal{A}}$ of a class $\mathcal{A}$ there is a problem $P_{\mathcal{B}}$ of $\mathcal{B}$ such that $P_{\mathcal{A}}$ and $P_{\mathcal{B}}$ are P-time equivalent, then the existence of a dichotomy for $\mathcal{B}$ implies the existence for $\mathcal{A}$, denoted as $\mathcal{B} \to \mathcal{A}$.

P-time Equivalence



Definition

$\mathcal{A}$ is P-time equivalent to $\mathcal{B}$ if for any problem $P_{\mathcal{A}}$ of $\mathcal{A}$ there is a P-time equivalent problem $P_{\mathcal{B}}$ of $\mathcal{B}$ and for any $P_{\mathcal{B}} \in \mathcal{B}$ there is a P-time equivalent problem $P_{\mathcal{A}} \in \mathcal{A}$. It is denoted as $\mathcal{A} \leftrightarrow \mathcal{B}$.

State of the Art

$\mathrm{NP} = \mathsf{ESO}$

SNP with $\neq$

MonotoneSNP MMSNP$_{\neq}$ MonadicSNP

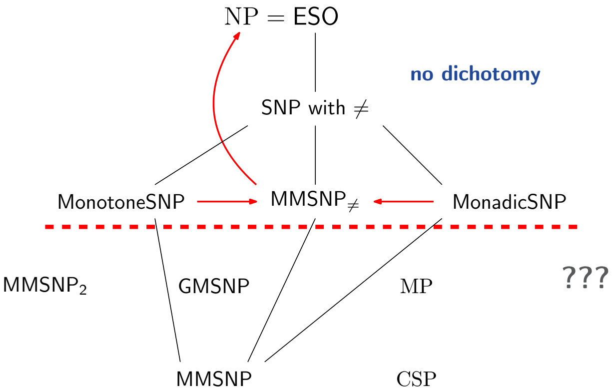
MMSNP$_2$ GMSNP MP ???

MMSNP CSP

Theorem (Fagin'74)

The set of properties expressible by the existential second-order logic (ESO) is exactly NP.

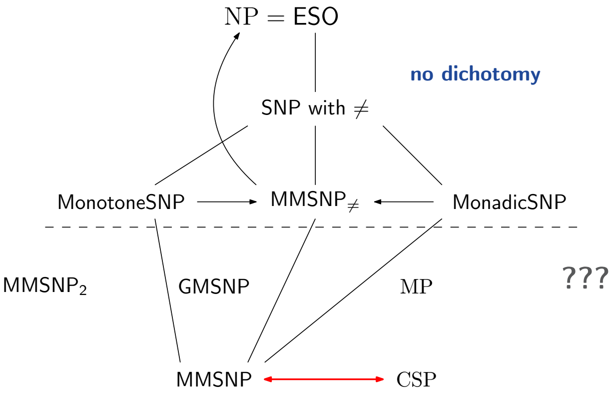
State of the Art



Theorem (Feder, Vardi'98)

MMSNP without one of its properties is P-time equivalent to NP.

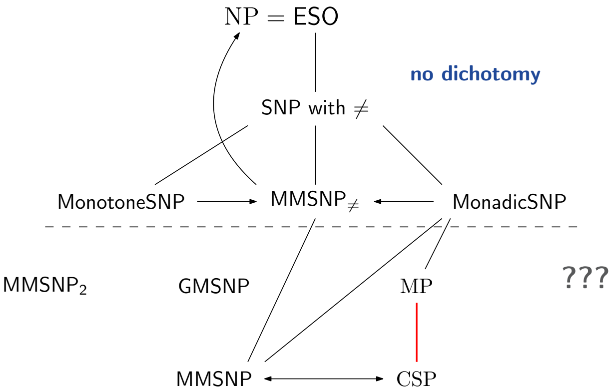
State of the Art



Theorem (Feder, Vardi'98)

MMSNP is P-time equivalent to CSP under randomized P-time reductions.

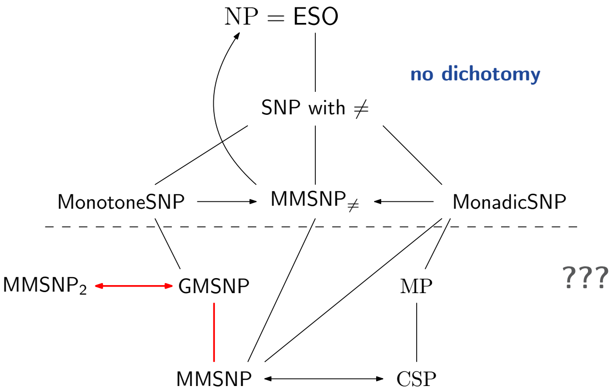
State of the Art



Fact

The class MP (Hell et al.) includes CSP and is included in MonadicSNP.

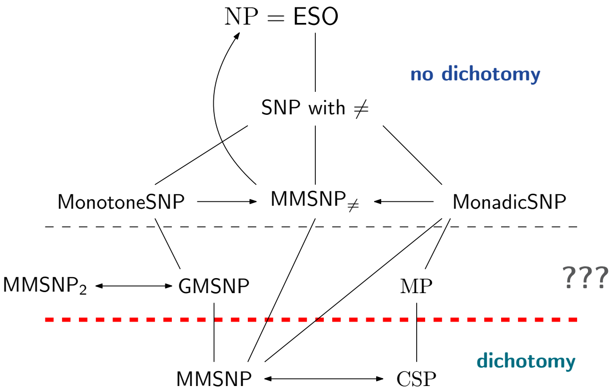
State of the Art



Theorem (Bienvenu, Ten Cate, Lutz, Wolter'14)

GMSNP strictly includes MMSNP and is P-time equivalent to MMSNP_2 introduced by Madelaine.

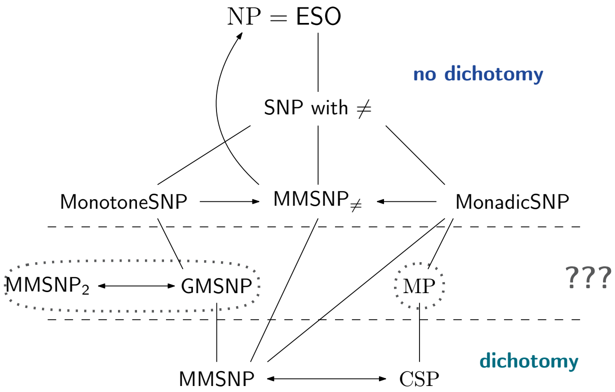
State of the Art



Algebraic Tractability Theorem (Bulatov, Zhuk'18)

Either a structure A satisfies a certain algebraic property and $\mathrm{CSP}(A) \in \mathrm{P}$ or it doesn't satisfy and $\mathrm{CSP}(A)$ is NP-complete.

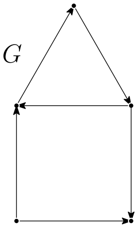
Open Questions

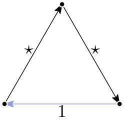


- Does the class MMSNP_2 have dichotomy?
- Does the class of Matrix Partition problems have dichotomy?

Matrix Partition

Matrix Partition





$$\mathsf{M} = \begin{pmatrix} 0 & \star & 0 \\ 0 & 0 & \star \\ 1 & 0 & 0 \end{pmatrix}$$

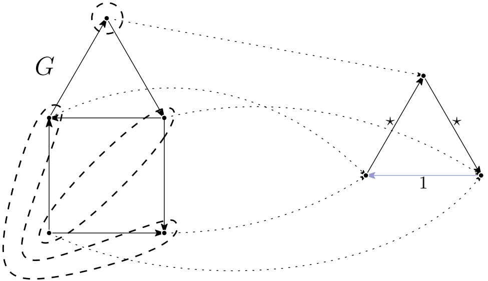
Definition

G admits M-partition if there is a partition $V(G) = P_1 \sqcup \ldots \sqcup P_n$ such that

- if $\mathsf{M}(i,j) = 0$ then, $\forall v_i \in P_i, v_j \in P_j$, $v_i v_j$ isn't an edge;
- if $\mathsf{M}(i,j) = 1$ then, $\forall v_i \in P_i, v_j \in P_j$, $v_i v_j$ is an edge.

The class of all such problems is called MP.

Matrix Partition



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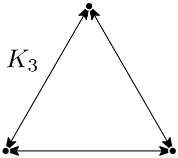
Every digraph homomorphism problem (CSP) can be represented as an M-partition problem.



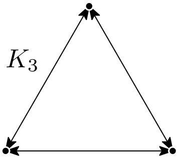
$$\mathsf{M}_3 = \begin{pmatrix} 0 & \star & \star \\ \star & 0 & \star \\ \star & \star & 0 \end{pmatrix}$$

Example

Let K_3 and M_3 be as to the left. Then $\mathrm{CSP}(K_3)$ and $\mathrm{MP}(\mathsf{M}_3)$ is the same problem.

Matrix Partition and CSP

Every digraph homomorphism problem (CSP) can be represented as an M-partition problem.



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Conclusion

CSP is a subclass of MP.

Questions about MP apart from dichotomy

1. To show that any generalization of MP to an arbitrary relational signature is P-time equivalent to MP on digraphs.

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Questions about MP apart from dichotomy

1. To show that any generalization of MP to an arbitrary relational signature is P-time equivalent to MP on digraphs.
2. To find a logic that is equivalent to MP. Similarly, as MMSNP $\leftrightarrow \mathrm{CSP}$.
3. To determine when a MP problem has finitely many minimal obstructions.

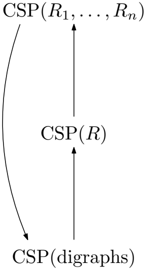
Definition

A directed graph G is called a minimal obstruction of a problem $\mathrm{MP}(\mathsf{M})$ if $G \notin \mathrm{MP}(\mathsf{M})$ and, for any induced subgraph $G' \subsetneq G$, $G' \in \mathrm{MP}(\mathsf{M})$.

Theorem (Atserias'08)

$\mathrm{CSP}(H)$ is definable in first-order logic iff it has finitely many minimal obstructions.

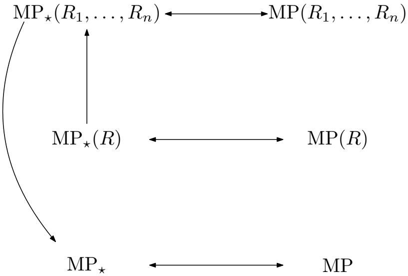
CSP Inclusion into Digraphs



Theorem (Feder, Vardy'98) (Bulin et al.'15)

CSP over an arbitrary finite signature is P-time equivalent to CSP on digraphs.

Results for the MP case



The difference between $\mathrm{MP}_\star$ and MP is the presence of $\star$-graphs in the input.

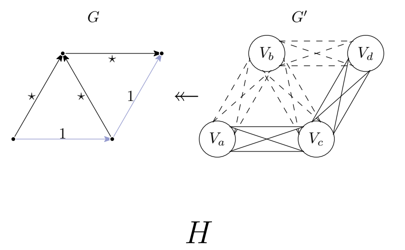
Theorem (B., Kanté '21)

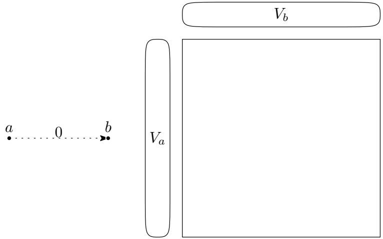
For any finite signature σ, $\mathrm{MP}(\sigma) \leftrightarrow \mathrm{MP}_\star(\sigma)$.

Theorem (B., Kanté '21)

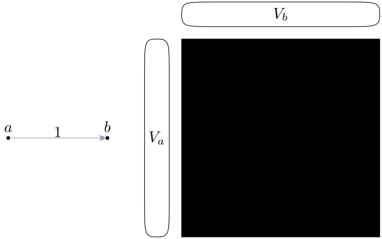
For any finite signature σ, there exists a signature $\tilde{\sigma} = \{R\}$ such that $\mathrm{MP}(\tilde{\sigma}) \rightarrow \mathrm{MP}(\sigma)$.

MP $\leftrightarrow$ MP$_\star$

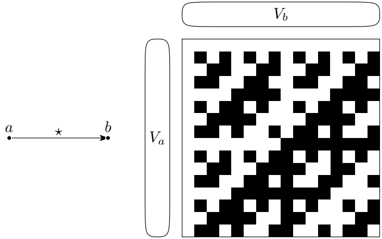




All-zero matrix

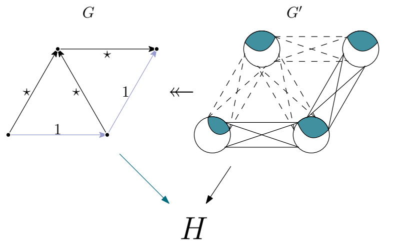


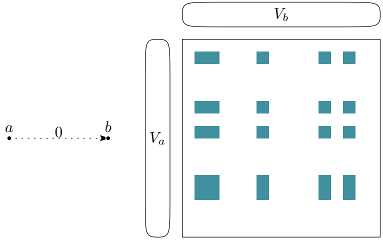
All-one matrix



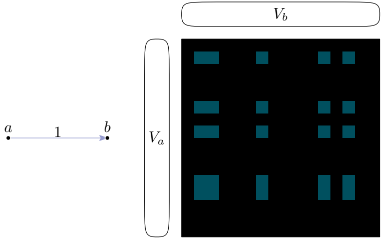
Hadamard matrix

MP $\leftrightarrow$ MP$_\star$





All-zero matrix



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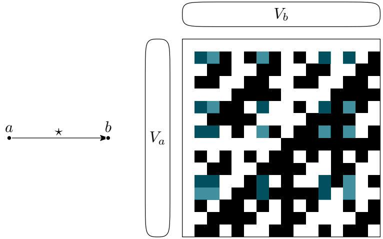


Hadamard matrix

Lemma

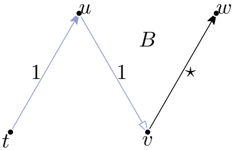
$\mathrm{MP}_\star(A)$ reduces in P-time to $\mathrm{MP}_\star(\tilde{A})$.

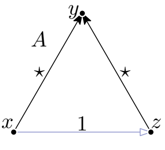
Proof

$R_1(t,u) = 1$

$R_1(v,w) = \star$

$R_2(u,v) = 1$





$R_1(x,y) = \star$

$R_1(x,z) = \star$

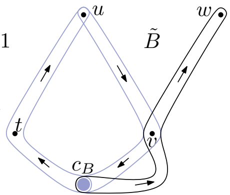
$R_2(x,z) = 1$

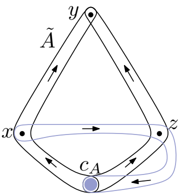
$R(c_B, c_B, c_B) = 1$

$R(c_B, t, u) = 1$

$R(c_B, v, w) = \star$

$R(u, v, c_B) = 1$





$R(c_A, c_A, c_A) = 1$

$R(c_A, x, y) = \star$

$R(c_A, z, y) = \star$

$R(x, z, c_A) = 1$

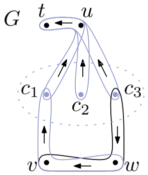
$\mathrm{MP}_\star(R) \to \mathrm{MP}_\star(R_1, \ldots, R_n)$

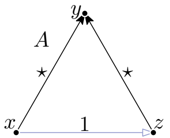
Lemma

$\mathrm{MP}_\star(\tilde{A})$ reduces in P-time to $\mathrm{MP}_\star(A)$.

Proof

$$R(w, v, c_1) = 1$$
$$R(c_3, w, v) = \star$$
$$\forall i\colon R(c_i, u, t) = 1$$
$$\forall i, j, k\colon R(c_i, c_j, c_k) = 1$$





$$R_1(x, y) = \star$$
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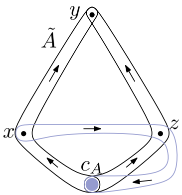

$$R(c_A, c_A, c_A) = 1$$
$$R(c_A, x, y) = \star$$
$$R(c_A, z, y) = \star$$
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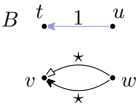

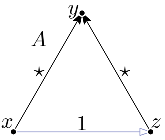

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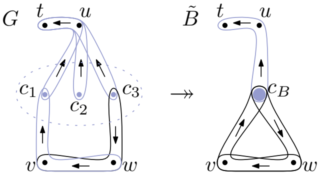

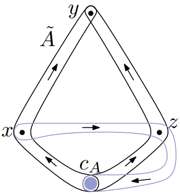

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