<u>Theorem 4.8</u>: Let the assumptions of Thm 4.1 hold and let $\psi \in C^1(\mathbb{R}^n, \mathbb{R}^m)$ s.t. $\psi(0)=0$, $\nabla\psi(0)=0$ and $[\Pi\psi](x) = \mathcal{O}(|x|^q)$ as $x \to 0$ for a $q>1$. Then $|\psi(x) - \Phi(x)| = \mathcal{O}(|x|^q)$ as $|x| \to 0$ where Φ is the center manifold from Thm 4.1.

proof: $\mathcal{X} = \{\varphi : \mathbb{R}^n \to \mathbb{R}^m ;\ \varphi(0)=0,\ \|\varphi\| \le b,\ \mathrm{Lip}\,\varphi \le \ell\}$, $T\Phi \doteq \Phi$

$$T\varphi(\eta_0) = \int_{-\infty}^{0} e^{-Bs} g(p(s), \varphi(p(s)))\, ds$$

Define $\mathcal{Y} := \{\varphi \in \mathcal{X} : |\varphi(x)| \le K|x|^q \ \forall x \in \mathbb{R}^n\}$, K is a large num to be specified later.

Define $\vartheta(x) := \begin{cases} \psi(x) & \text{on } \mathcal{U}(0,\sigma) \\ 0 & \text{on } \mathcal{U}(0,2\sigma)^c \end{cases}$ with small gradient for $|x| \in (\sigma, 2\sigma)$

$N(x) := [\Pi\vartheta](x)$

Then ~~$\vartheta \in \mathcal{Y}$ (for appropriate K)~~ $|N(x)| \le c_1|x|^q \quad \forall x \in \mathbb{R}^n$

Define $S : \mathcal{Y} \to C(\mathbb{R}^n, \mathbb{R}^m)$ by $S(\varphi) = T(\varphi + \vartheta) - \vartheta$

If $S(\mathcal{Y}) \subset \mathcal{Y}$, obviously $\mathrm{Lip}_S = \mathrm{Lip}_T < 1$ $\rightsquigarrow$ S contraction

then S has a fixed point $\tilde\varphi$,

so $T(\tilde\varphi + \vartheta) - \vartheta = \tilde\varphi \Rightarrow T(\tilde\varphi + \vartheta) = \tilde\varphi + \vartheta \Rightarrow \tilde\varphi + \vartheta = \Phi$

and $\vartheta - \Phi = \tilde\varphi \in \mathcal{Y} \Rightarrow |\vartheta - \Phi| = \mathcal{O}(|x|^q) \Rightarrow |\psi - \Phi| = \mathcal{O}(|x|^q)$.

It remains to show $S(\mathcal{Y}) \subset \mathcal{Y}$.

$$[S(\varphi)](\eta_0) = [T(\varphi + \vartheta)](\eta_0) - \vartheta(\eta_0)$$

$$[T(\varphi+\vartheta)](\eta_0) = \int_{-\infty}^{0} e^{-Bs} g(p(s), (\varphi+\vartheta)(p(s)))\, ds$$

$$\vartheta(\eta_0) = \left[e^{-Bs}\vartheta(p(s))\right]_{-\infty}^{0} = \int_{-\infty}^{0} e^{-Bs}\cdot(-B)\vartheta(s) + e^{-Bs}\nabla\vartheta(p(s))\cdot\left[Ap(s) + f(p(s), (\varphi+\vartheta)(p(s))\right] ds$$

$$[S(\varphi)](\eta_0) = \int_{-\infty}^{0} e^{-Bs}\,\Pi\vartheta(p(s)) + e^{-Bs}\Big[g(p(s), \varphi+\vartheta(p(s))) - g(p(s), \vartheta(s)) - \nabla\vartheta(p(s))\big[f(\quad) - f(\quad \vartheta)\big]\Big]$$

$$= \int_{-\infty}^{0} e^{-Bs}\, Q(p(s), \varphi(p(s)))\, ds$$

$$Q(x,y) = N(x) + g(x, y + \vartheta(x)) - g(x, \vartheta(x)) - \nabla\vartheta(x)\left[f(x, y+\vartheta(x)) - f(x, \vartheta(x))\right]$$

$$|Q(x,y)| \le c_1|x|^q + \sigma|y| + \sigma^2|y|$$

Moreover, $|p(s)| \le |\eta_0|\, e^{(-\frac{\beta}{2q} + (\ell+1)\sigma)s} \quad \forall s \le 0$

$$|\varphi(p(s))| \le K|\eta_0|^q\, e^{(-\frac{\beta}{2} + q\sigma(\ell+1))s}$$

$$|Q(p(s), \varphi(p(s)))| \le c_1|\eta_0|^q e^{\cdots s} + (\sigma+\sigma^2)\cdot K|\eta_0|^q e^{\cdots s}$$

$$= |\eta_0|^q e^{(-\frac{\beta}{2} + q\sigma(\ell+1))s}\left(c_1 + (\sigma+\sigma^2)K\right)$$

$$|[S\varphi](\eta_0)| \le |\eta_0|^q \underbrace{\frac{c_1 + (\sigma+\sigma^2)K}{\frac{\beta}{2} + q\sigma(\ell+1)} c_0}_{< K} \quad \text{if } K \text{ large}, \sigma \text{ small}$$

It remains to show $S\varphi \in \mathcal{X}$. similar to Thm 4.1 $\square$