J. Carr: „Applications of centre manifold theory"
[Appl. Math Sci 35, Springer, 1981.]

CHAPTER 1

INTRODUCTION TO CENTRE MANIFOLD THEORY

1.1. Introduction

In this chapter we state the main results of centre manifold theory for finite dimensional systems and give some simple examples to illustrate their application.

1.2. Motivation

To motivate the study of centre manifolds we first look at a simple example. Consider the system

$$\dot{x} = ax^3, \quad \dot{y} = -y + y^2, \tag{1.2.1}$$

where a is a constant. Since the equations are uncoupled we can easily show that the zero solution of (1.2.1) is asymptotically stable if and only if $a < 0$. Suppose now that

$$\begin{aligned} \dot{x} &= ax^3 + x^2y \\ \dot{y} &= -y + y^2 + xy - x^3. \end{aligned} \tag{1.2.2}$$

Since the equations are coupled we cannot immediately decide if the zero solution of (1.2.2) is asymptotically stable, but we might suspect that it is if $a < 0$. The key to understanding the relation of equation (1.2.2) to equation (1.2.1) is an abstraction of the idea of uncoupled equations.

A curve, say $y = h(x)$, defined for $|x|$ small, is said to be an invariant manifold for the system of differential equations

$$\dot{x} = f(x,y), \quad \dot{y} = g(x,y), \tag{1.2.3}$$

if the solution $(x(t),y(t))$ of (1.2.3) through $(x_0,h(x_0))$ lies on the curve $y = h(x)$ for small t, i.e., $y(t) = h(x(t))$. Thus, for equation (1.2.1), $y = 0$ is an invariant manifold. Note that in deciding upon the stability of the zero solution of (1.2.1), the only important equation is $\dot{x} = ax^3$, that is, we only need study a first order equation on a particular invariant manifold.

The theory that we develop tells us that equation (1.2.3) has an invariant manifold $y = h(x)$, $|x|$ small, with $h(x) = O(x^2)$ as $x \to 0$. Furthermore, the asymptotic stability of the zero solution of (1.2.2) can be proved by studying a first order equation. This equation is given by

$$\dot{u} = au^3 + u^2h(u) = au^3 + O(u^4), \tag{1.2.4}$$

and we see that the zero solution of (1.2.4) is asymptotically stable if $a < 0$ and unstable if $a > 0$. This tells us that the zero solution of (1.2.2) is asymptotically stable if $a < 0$ and unstable if $a > 0$ as we expected.

We are also able to use this method to obtain estimates for the rate of decay of solutions of (1.2.2) in the case $a < 0$. For example, if $(x(t),y(t))$ is a solution of (1.2.2) with $(x(0),y(0))$ small, we prove that there is a solution $u(t)$ of (1.2.4) such that $x(t) = u(t)(1+o(1))$, $y(t) = h(u(t))(1+o(1))$ as $t \to \infty$.

1.3. Centre Manifolds

We first recall the definition of an invariant manifold for the equation

$$\dot{x} = N(x) \tag{1.3.1}$$

where $x \in \mathbb{R}^n$. A set $S \subset \mathbb{R}^n$ is said to be a *local invariant manifold* for (1.3.1) if for $x_0 \in S$, the solution $x(t)$ of (1.3.1) with $x(0) = x_0$ is in S for $|t| < T$ where $T > 0$. If we can always choose $T = \infty$, then we say that S is an *invariant manifold*.

Consider the system

$$\begin{aligned} \dot{x} &= Ax + f(x,y) \\ \dot{y} &= By + g(x,y) \end{aligned} \tag{1.3.2}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ and A and B are constant matrices such that all the eigenvalues of A have zero real parts while all the eigenvalues of B have negative real parts. The functions f and g are C^2 with $f(0,0) = 0$, $f'(0,0) = 0$, $g(0,0) = 0$, $g'(0,0) = 0$. (Here, f' is the Jacobian matrix of f.)

If f and g are identically zero then (1.3.2) has two obvious invariant manifolds, namely $x = 0$ and $y = 0$. The invariant manifold $x = 0$ is called the stable manifold, since if we restrict initial data to $x = 0$, all solutions tend to zero. The invariant manifold $y = 0$ is called the centre manifold.

In general, if $y = h(x)$ is an invariant manifold for (1.3.2) and h is smooth, then it is called a *centre manifold* if $h(0) = 0$, $h'(0) = 0$. We use the term centre manifold in place of local centre manifold if the meaning is clear.

If f and g are identically zero, then all solutions of (1.3.2) tend exponentially fast, as $t \to \infty$, to solutions of

$$\dot{x} = Ax. \tag{1.3.3}$$

That is, the equation on the centre manifold determines the asymptotic behavior of solutions of the full equation modulo exponentially decaying terms. We now give the analogue of these results when f and g are non-zero. These results are proved in Chapter 2.

<u>Theorem 1</u>. There exists a centre manifold for (1.3.2), $y = h(x)$, $|x| < \delta$, where h is C^2.

The flow on the centre manifold is governed by the n-dimensional system

$$\dot{u} = Au + f(u,h(u)) \tag{1.3.4}$$

which generalizes the corresponding problem (1.3.3) for the linear case. The next theorem tells us that (1.3.4) contains all the necessary information needed to determine the asymptotic behavior of small solutions of (1.3.2).

<u>Theorem 2</u>. (a) Suppose that the zero solution of (1.3.4) is stable (asymptotically stable) (unstable). Then the zero solution of (1.3.2) is stable (asymptotically stable) (unstable).

(b) Suppose that the zero solution of (1.3.4) is stable. Let $(x(t),y(t))$ be a solution of (1.3.2) with $(x(0),y(0))$ sufficiently small. Then there exists a solution $u(t)$ of (1.3.4) such that as $t \to \infty$,

$$\begin{aligned} x(t) &= u(t) + O(e^{-\gamma t}) \\ y(t) &= h(u(t)) + O(e^{-\gamma t}) \end{aligned} \tag{1.3.5}$$

where $\gamma > 0$ is a constant.

If we substitute $y(t) = h(x(t))$ into the second equation in (1.3.2) we obtain

$$h'(x)[Ax + f(x,h(x))] = Bh(x) + g(x,h(x)). \tag{1.3.6}$$

Equation (1.3.6) together with the conditions $h(0) = 0$, $h'(0) = 0$ is the system to be solved for the centre manifold. This is impossible, in general, since it is equivalent to solving (1.3.2). The next result, however, shows that, in principle, the centre manifold can be approximated to any degree of accuracy.

For functions $\phi: \mathbb{R}^n \to \mathbb{R}^m$ which are C^1 in a neighborhood of the origin define

$$(M\phi)(x) = \phi'(x)[Ax + f(x,\phi(x))] - B\phi(x) - g(x,\phi(x)).$$

Note that by (1.3.6), $(Mh)(x) = 0$.

<u>Theorem 3</u>. Let ϕ be a C^1 mapping of a neighborhood of the origin in $\mathbb{R}^n$ into $\mathbb{R}^m$ with $\phi(0) = 0$ and $\phi'(0) = 0$. Suppose that as $x \to 0$, $(M\phi)(x) = O(|x|^q)$ where $q > 1$. Then as $x \to 0$, $|h(x) - \phi(x)| = O(|x|^q)$.

1.4. <u>Examples</u>

We now consider a few simple examples to illustrate the use of the above results.

<u>Example 1</u>. Consider the system

$$\begin{aligned} \dot{x} &= xy + ax^3 + by^2x \\ \dot{y} &= -y + cx^2 + dx^2y. \end{aligned} \tag{1.4.1}$$

By Theorem 1, equation (1.4.1) has a centre manifold $y = h(x)$. To approximate h we set

$$(M\phi)(x) = \phi'(x)[x\phi(x) + ax^3 + bx\phi^2(x)] + \phi(x) - cx^2 - dx^2\phi(x).$$

If $\phi(x) = O(x^2)$ then $(M\phi)(x) = \phi(x) - cx^2 + O(x^4)$. Hence, if $\phi(x) = cx^2$, $(M\phi)(x) = O(x^4)$, so by Theorem 3, $h(x) = cx^2 + O(x^4)$. By Theorem 2, the equation which determines the stability of the zero solution of (1.4.1) is

$$\dot{u} = uh(u) + au^3 + buh^2(u) = (a+c)u^3 + O(u^5).$$

Thus the zero solution of (1.4.1) is asymptotically stable if $a + c < 0$ and unstable if $a + c > 0$. If $a + c = 0$ then we have to obtain a better approximation to h.

Suppose that $a + c = 0$. Let $\phi(x) = cx^2 + \psi(x)$ where $\psi(x) = O(x^4)$. Then $(M\phi)(x) = \psi(x) - cdx^4 + O(x^6)$. Thus, if $\phi(x) = cx^2 + cdx^4$ then $(M\phi)(x) = O(x^6)$ so by Theorem 3, $h(x) = cx^2 + cdx^4 + O(x^6)$. The equation that governs the stability of the zero solution of (1.4.1) is

$$\dot{u} = uh(u) + au^3 + buh^2(u) = (cd+bc^2)u^5 + O(u^7).$$

Hence, if $a + c = 0$, then the zero solution of (1.4.1) is asymptotically stable if $cd + bc^2 < 0$ and unstable if $cd + bc^2 > 0$. If $cd + bc^2 = 0$ then we have to obtain a better approximation to h (see Exercise 1).

Exercise 1. Suppose that $a + c = cd + bc^2 = 0$ in Example 1. Show that the equation which governs the stability of the zero solution of (1.4.1) is $\dot{u} = -cd^2u^7 + O(u^9)$.

Exercise 2. Show that the zero solution of (1.2.2) is asymptotically stable if $a \le 0$ and unstable if $a > 0$.

Exercise 3. Suppose that in equation (1.3.2), $n = 1$ so that $A = 0$. Suppose also that $f(x,y) = ax^p + O(|x|^{p+1} + |y|^q)$ where $2q \ge p + 1$ and a is non-zero. Show that the zero solution of (1.3.2) is asymptotically stable if $a < 0$ and p is odd, and unstable otherwise.

Example 2. Consider the system

$$\begin{aligned}\dot{x} &= \varepsilon x - x^3 + xy \\ \dot{y} &= -y + y^2 - x^2\end{aligned} \qquad (1.4.2)$$

where ε is a real parameter. The object is to study small solutions of (1.4.2) for small $|\varepsilon|$.

The linearized problem corresponding to (1.4.2) has eigenvalues -1 and ε. This means that the results given in Section 3 do not apply directly. However, we can write (1.4.2) in the equivalent form

$$\begin{aligned}\dot{x} &= \varepsilon x - x^3 + xy \\ \dot{y} &= -y + y^2 - x^2 \\ \dot{\varepsilon} &= 0.\end{aligned} \qquad (1.4.3)$$

When considered as an equation on $\mathbb{R}^3$ the εx term in (1.4.3) is nonlinear. Thus the linearized problem corresponding to (1.4.3) has eigenvalues $-1,0,0$. The theory given in Section 3 now applies so that by Theorem 1, (1.4.3) has a two dimensional centre manifold $y = h(x,\varepsilon)$, $|x| < \delta_1$, $|\varepsilon| < \delta_2$. To find an approximation to h set

$$(M\phi)(x,\varepsilon) = \phi_x(x,\varepsilon)[\varepsilon x - x^3 + x\phi(x,\varepsilon)] + \phi(x,\varepsilon) + x^2 - \phi^2(x,\varepsilon).$$

Then, if $\phi(x,\varepsilon) = -x^2$, $(M\phi)(x,\varepsilon) = O(C(x,\varepsilon))$ where C is a homogeneous cubic in x and ε. By Theorem 3,

$h(x,\varepsilon) = -x^2 + O(C(x,\varepsilon))$. Note also that $h(0,\varepsilon) = 0$ (see Section 2.6). By Theorem 2 the equation which governs small solutions of (1.4.3) is

$$\begin{aligned} \dot{u} &= \varepsilon u - 2u^3 + O(|u|C(u,\varepsilon)) \\ \dot{\varepsilon} &= 0. \end{aligned} \qquad (1.4.4)$$

The zero solution $(u,\varepsilon) = (0,0)$ of (1.4.4) is stable so the representation of solutions given by Theorem 2 applies here. For $-\delta_2 < \varepsilon < 0$ the solution $u = 0$ of the first equation in (1.4.4) is asymptotically stable and so by Theorem 2 the zero solution of (1.4.2) is asymptotically stable.

For $0 < \varepsilon < \delta_2$, solutions of the first equation in (1.4.4) consist of two orbits connecting the origin to two small fixed points. Hence, for $0 < \varepsilon < \delta_2$ the stable manifold of the origin for (1.4.2) forms a separatrix, the unstable manifold consisting of two stable orbits connecting the origin to the fixed points.

<u>Exercise 4</u>. Study the behavior of all small solutions of $\ddot{w} + \dot{w} + \varepsilon w + w^3 = 0$ for small ε.

<u>Example 3</u>. Consider the equations

$$\begin{aligned} \dot{y} &= -y + (y+c)z \\ \varepsilon\dot{z} &= y - (y+1)z \end{aligned} \qquad (1.4.5)$$

where $\varepsilon > 0$ is small and $0 < c < 1$. The above equations arise from a model of the kinetics of enzyme reactions [33]. If $\varepsilon = 0$, then (1.4.5) degenerates into one algebraic equation and one differential equation. Solving the algebraic equation we obtain

$$z = \frac{y}{y+1} \qquad (1.4.6)$$

and substituting this into the first equation in (1.4.5) leads to the equation

$$\dot{y} = \frac{-\lambda y}{1+y} \qquad (1.4.7)$$

where $\lambda = 1 - c$.

Using singular perturbation techniques, it was shown in [33] that for ε sufficiently small, under certain conditions, solutions of (1.4.5) are close to solutions of the degenerate system (1.4.6), (1.4.7). We shall show how centre manifolds can be used to obtain a similar result.

Let $t = \varepsilon\tau$. We denote differentiation with respect to t by $\dot{}$ and differentiation with respect to τ by $'$. Equation (1.4.5) can be rewritten in the equivalent form

$$\begin{aligned} y' &= \varepsilon f(y,w) \\ w' &= -w + y^2 - yw + \varepsilon f(y,w) \\ \varepsilon' &= 0 \end{aligned} \qquad (1.4.8)$$

where $f(y,w) = -y + (y+c)(y-w)$ and $w = y - z$. By Theorem 1, (1.4.8) has a centre manifold $w = h(y,\varepsilon)$. To find an approximation to h set

$$(M\phi)(y,\varepsilon) = \varepsilon\phi_y(y,\varepsilon)f(y,\phi) + \phi(y,\varepsilon) - y^2 + y\phi(y,\varepsilon) - \varepsilon f(y,\phi).$$

If $\phi(y,\varepsilon) = y^2 - \lambda\varepsilon y$ then $(M\phi)(y,\varepsilon) = O(|y|^3 + |\varepsilon|^3)$ so that by Theorem 3,

$$h(y,\varepsilon) = y^2 - \lambda\varepsilon y + O(|y|^3 + |\varepsilon|^3).$$

By Theorem 2, the equation which determines the asymptotic behavior of small solutions of (1.4.8) is

$$u' = \varepsilon f(u,h(u,\varepsilon))$$

or in terms of the original time scale

$$\dot{u} = f(u,h(u,\varepsilon)) = -\lambda(u-u^2) + O(|\varepsilon u| + |u|^3). \tag{1.4.9}$$

Again, by Theorem 2, if ε is sufficiently small and $y(0)$, $z(0)$ are sufficiently small, then there is a solution $u(t)$ of (1.4.9) such that for some $\gamma > 0$,

$$\begin{aligned} y(t) &= u(t) + O(e^{-\gamma t/\varepsilon}) \\ z(t) &= y(t) - h(y(t),\varepsilon) + O(e^{-\gamma t/\varepsilon}). \end{aligned} \tag{1.4.10}$$

Note that equation (1.4.7) is an approximation to the equation on the centre manifold. Also, from (1.4.10), $z(t) \approx y(t) - y^2(t)$, which shows that (1.4.6) is approximately true.

The above results are not satisfactory since we have to assume that the initial data is small. In Chapter 2, we show how we can deal with more general initial data. Here we briefly indicate the procedure involved there. If $y_0 \neq -1$, then

$$(y,w,\varepsilon) = (y_0, y_0^2(1+y_0)^{-1}, 0)$$

is a curve of equilibrium points for (1.4.8). Thus, we expect that there is an invariant manifold $w = h(y,\varepsilon)$ for (1.4.8) defined for ε small and $0 \leq y \leq m$ $(m = O(1))$, and with $h(y,\varepsilon)$ close to the curve

$$w = y^2(1+y)^{-1}. \tag{1.4.11}$$

For initial data close to the curve given by (1.4.11), the stability properties of (1.4.8) are the same as the stability properties of the reduced equation

$$\dot{u} = f(u,h(u,\varepsilon)).$$

1.5. Bifurcation Theory

Consider the system of ordinary differential equations

$$\begin{aligned} \dot{w} &= F(w,\varepsilon) \\ F(0,\varepsilon) &\equiv 0 \end{aligned} \tag{1.5.1}$$

where $w \in \mathbb{R}^{n+m}$ and ε is a p-dimensional parameter. We say that $\varepsilon = 0$ is a bifurcation point for (1.5.1) if the qualitative nature of the flow changes at $\varepsilon = 0$; that is, if in any neighborhood of $\varepsilon = 0$ there exist points ε_1 and ε_2 such that the local phase portraits of (1.5.1) for $\varepsilon = \varepsilon_1$ and $\varepsilon = \varepsilon_2$ are not topologically equivalent.

Suppose that the linearization of (1.5.1) about $w = 0$ is

$$\dot{w} = C(\varepsilon)w. \tag{1.5.2}$$

If the eigenvalues of $C(0)$ all have non-zero real parts then, for small $|\varepsilon|$, small solutions of (1.5.1) behave like solutions of (1.5.2) so that $\varepsilon = 0$ is not a bifurcation point. Thus, from the point of view of local bifurcation theory the only interesting situation is when $C(0)$ has eigenvalues with zero real parts.

Suppose that $C(0)$ has n eigenvalues with zero real parts and m eigenvalues whose real parts are negative. We are assuming that $C(0)$ does not have any positive eigenvalues since we are interested in the bifurcation of stable phenomena.

Because of our hypothesis about the eigenvalues of $C(0)$ we can rewrite (1.5.1) as

$$\begin{aligned} \dot{x} &= Ax + f(x,y,\varepsilon) \\ \dot{y} &= By + g(x,y,\varepsilon) \\ \dot{\varepsilon} &= 0 \end{aligned} \tag{1.5.3}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, A is an $n \times n$ matrix whose eigenvalues all have zero real parts, B is an $m \times m$ matrix whose eigenvalues all have negative real parts, and f and g vanish together with each of their derivatives at $(x,y,\varepsilon) = (0,0,0)$.

By Theorem 1, (1.5.3) has a centre manifold $y = h(x,\varepsilon)$, $|x| < \delta_1$, $|\varepsilon| < \delta_2$. By Theorem 2 the behavior of small solutions of (1.5.3) is governed by the equation

$$\begin{aligned} \dot{u} &= Au + f(u,h(u,\varepsilon),\varepsilon) \\ \dot{\varepsilon} &= 0. \end{aligned} \tag{1.5.4}$$

In applications n is frequently 1 or 2 so this is a very useful reduction. The reduction to a lower dimensional problem is analogous to the use of the Liapunov-Schmidt procedure in the analysis of static problems.

Our use of centre manifold theory in bifurcation problems follows that of Ruelle and Takens [57] and of Marsden and McCracken [51]. For the relationship between centre manifold theory and other perturbation techniques such as amplitude expansions see [14].

We emphasize that the above analysis is local. In general, given a parameter dependent differential equation it is difficult to classify all the possible phase portraits. For an example of how complicated such an analysis can be, see [66] where a model of the dynamic behavior of a continuous stirred tank reactor is studied. The model consists of a

parameter dependent second order system of ordinary differential equations. The authors show that there are 35 possible phase portraits!

1.6. Comments on the Literature

Theorems 1-3 are the simplest such results in centre manifold theory and we briefly mention some of the possible generalizations.

(1) The assumption that the eigenvalues of the linearized problem all have non-positive real parts is not necessary.

(2) The equations need not be autonomous.

(3) In certain circumstances we can replace 'equilibrium point' by 'invariant set'.

(4) Similar results can be obtained for certain classes of infinite-dimensional evolution equations, such as partial differential equations.

There is a vast literature on invariant manifold theory [1,8,22,23,27,28,30,32,34,35,42,44,45,48,51]. For applications of invariant manifold theory to bifurcation theory see [1,14, 17,18,19,24,31,34,36,37,38,47,48,49,51,57,65]. For a simple discussion of stable and unstable manifolds see [22, Chapter 13] or [27, Chapter 3].

In Chapter 2 we prove Theorems 1-3. Our proofs of Theorems 1 and 2 are modeled on Kelly [44,45]. Theorem 3 is a special case of a result of Henry [34] and our proof follows his. The method of approximating the centre manifold in Theorem 3 was essentially used by Hausrath [32] in his work on stability in critical cases for neutral functional differential equations. Throughout Chapter 2 we use methods that

CHAPTER 2

PROOFS OF THEOREMS

2.1. Introduction

In this chapter we give proofs of the three main theorems stated in Chapter 1. The proofs are essentially applications of the contraction mapping principle. The procedure used for defining the mappings is rather involved, so we first give a simple example to help clarify the technique. The proofs that we give can easily be extended to the corresponding infinite dimensional case; indeed essentially all we have to do is to replace the norm $|\cdot|$ in finite dimensional space by the norm $\|\cdot\|$ in a Banach space.

2.2. A Simple Example

We consider a simple example to illustrate the method that we use to prove the existence of centre manifolds.

Consider the system

$$\dot{x}_1 = x_2,\ \dot{x}_2 = 0,\ \dot{y} = -y + g(x_1,x_2), \tag{2.2.1}$$

where g is smooth and $g(x_1,x_2) = O(x_1^2+x_2^2)$ as $(x_1,x_2) \to (0,0)$. We prove that (2.2.1) has a local centre manifold.

Let $\psi: \mathbb{R}^2 \to \mathbb{R}$ be a C^∞ function with compact support such that $\psi(x_1,x_2) = 1$ for (x_1,x_2) in a neighborhood of the origin. Define G by $G(x_1,x_2) = \psi(x_1,x_2)g(x_1,x_2)$. We prove that the system of equations

$$\dot{x}_1 = x_2,\ \dot{x}_2 = 0,\ \dot{y} = -y + G(x_1,x_2), \tag{2.2.2}$$

has a centre manifold $y = h(x_1,x_2)$, $(x_1,x_2) \in \mathbb{R}^2$. Since $G(x_1,x_2) = g(x_1,x_2)$ in a neighborhood of the origin, this proves that $y = h(x_1,x_2)$, $x_1^2 + x_2^2 < \delta$ for some δ, is a local centre manifold for (2.2.1).

The solution of the first two equations in (2.2.2) is $x_1(t) = z_1 + z_2t$, $x_2(t) = z_2$, where $x_i(0) = z_i$. If $y(t) = h(x_1(t),x_2(t))$ is a solution of the third equation in (2.2.2) then

$$\frac{d}{dt}h(z_1+z_2t,z_2) = -h(z_1+z_2t,z_2) + G(z_1+z_2t,z_2). \tag{2.2.3}$$

To determine a centre manifold for (2.2.2) we must single out a special solution of (2.2.3). Since $G(x_1,x_2)$ is small for all x_1 and x_2, solutions of the third equation in (2.2.2) behave like solutions of the linearized equation $\dot{y} = -y$. The general solution of (2.2.2) therefore contains a term like e^{-t}. As $t \to \infty$, this component approaches the origin perpendicular to the z_1,z_2 plane. Since the centre manifold is tangent to the z_1,z_2 plane at the origin we must eliminate the e^{-t} component, that is we must eliminate the component that approaches the origin along the stable manifold as $t \to \infty$. To do this we solve (2.2.3) together with the condition

$$\lim_{t\to-\infty} h(z_1+z_2t,z_2)e^{t} = 0. \tag{2.2.4}$$

Integrating (2.2.3) between $-\infty$ and 0 and using (2.2.4) we obtain

$$h(z_1,z_2) = \int_{-\infty}^{0} e^s G(z_1+z_2 s, z_2)ds.$$

By construction, $y = h(z_1,z_2)$ is an invariant manifold for (2.2.2). Using the fact that G has compact support and that $G(x_1,x_2)$ has a second order zero at the origin it follows that $h(z_1,z_2)$ has a second order zero at the origin; that is, h is a centre manifold.

2.3. Existence of Centre Manifolds

In this section we prove that the system

$$\begin{aligned} \dot{x} &= Ax + f(x,y) \\ \dot{y} &= By + g(x,y) \end{aligned} \tag{2.3.1}$$

has a centre manifold. As before $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, the eigenvalues of A have zero real parts, the eigenvalues of B have negative real parts and f and g are C^2 functions which vanish together with their derivatives at the origin.

Theorem 1. Equation (2.3.1) has a local centre manifold $y = h(x)$, $|x| < \delta$, where h is C^2.

Proof: As in the example given in the previous section, we prove the existence of a centre manifold for a modified equation. Let $\psi\colon \mathbb{R}^n \to [0,1]$ be a C^∞ function with $\psi(x) = 1$ when $|x| \le 1$ and $\psi(x) = 0$ when $|x| \ge 2$. For $\varepsilon > 0$ define F and G by

$$F(x,y) = f(x\psi(\tfrac{x}{\varepsilon}),y), \quad G(x,y) = g(x\psi(\tfrac{x}{\varepsilon}),y).$$

The reason that the cut-off function ψ is only a function of

x is that the proof of the existence of a centre manifold generalizes in an obvious way to infinite dimensional problems.

We prove that the system

$$\begin{aligned} \dot{x} &= Ax + F(x,y) \\ \dot{y} &= By + G(x,y) \end{aligned} \tag{2.3.2}$$

has a centre manifold $y = h(x)$, $x \in \mathbb{R}^n$, for small enough ε. Since F and G agree with f and g in a neighborhood of the origin, this proves the existence of a local centre manifold for (2.3.1).

For $p > 0$ and $p_1 > 0$ let X be the set of Lipschitz functions $h\colon \mathbb{R}^n \to \mathbb{R}^m$ with Lipschitz constant p_1, $|h(x)| \le p$ for $x \in \mathbb{R}^n$ and $h(0) = 0$. With the supremum norm $\|\cdot\|$, X is a complete space.

For $h \in X$ and $x_0 \in \mathbb{R}^n$, let $x(t,x_0,h)$ be the solution of

$$\dot{x} = Ax + F(x,h(x)), \quad x(0,x_0,h) = x_0. \tag{2.3.3}$$

The bounds on F and h ensure that the solution of (2.3.3) exists for all t. We now define a new function Th by

$$(Th)(x_0) = \int_{-\infty}^{0} e^{-Bs} G(x(s,x_0,h),h(x(s,x_0,h)))ds. \tag{2.3.4}$$

If h is a fixed point of (2.3.4) then h is a centre manifold for (2.3.2). We prove that for p, p_1, and ε small enough, T is a contraction on X.

Using the definitions of F and G, there is a continuous function $k(\varepsilon)$ with $k(0) = 0$ such that

$$\begin{aligned} |F(x,y)| + |G(x,y)| &\le \varepsilon k(\varepsilon), \\ |F(x,y) - F(x',y')| &\le k(\varepsilon)[|x-x'| + |y-y'|], \\ |G(x,y) - G(x',y')| &\le k(\varepsilon)[|x-x'| + |y-y'|], \end{aligned} \tag{2.3.5}$$

for all $x, x' \in \mathbb{R}^n$ and all $y, y' \in \mathbb{R}^m$ with $|y|, |y'| < \varepsilon$.

Since the eigenvalues of B all have negative real parts, there exist positive constants β, C such that for $s \le 0$ and $y \in \mathbb{R}^m$,

$$|e^{-Bs}y| \le Ce^{\beta s}|y|. \tag{2.3.6}$$

Since the eigenvalues of A all have zero real parts, for each $r > 0$ there is a constant $M(r)$ such that for $x \in \mathbb{R}^n$ and $s \in \mathbb{R}$,

$$|e^{As}x| \le M(r)e^{r|s|}|x|. \tag{2.3.7}$$

Note that in general, $M(r) \to \infty$ as $r \to 0$.

If $p < \varepsilon$, then we can use (2.3.5) to estimate terms involving $G(x(s,x_0,h), h(x(s,x_0,h)))$ and similar terms. We shall suppose that $p < \varepsilon$ from now on.

If $x_0 \in \mathbb{R}^n$, then using (2.3.6) and the estimates on G and h, we have from (2.3.4) that

$$|Th(x_0)| \le C\beta^{-1}\varepsilon k(\varepsilon). \tag{2.3.8}$$

Now let $x_0, x_1 \in \mathbb{R}^n$. Using (2.3.7) and the estimates on F and h, we have from (2.3.3) that for $r > 0$ and $t \le 0$,

$$\begin{aligned} |x(t,x_0,h) - x(t,x_1,h)| &\le M(r)e^{-rt}|x_0-x_1| \\ &\quad + (1+p_1)M(r)k(\varepsilon)\int_t^0 e^{r(s-t)}|x(s,x_0,h) - x(s,x_1,h)|ds. \end{aligned}$$

By Gronwall's inequality, for $t \le 0$,

$$|x(t,x_0,h) - x(t,x_1,h)| \le M(r)|x_1-x_0|e^{-\gamma t}, \tag{2.3.9}$$

where $\gamma = r + (1+p_1)M(r)k(\varepsilon)$. Using (2.3.9) and the bounds on G and h, we obtain from (2.3.4)

$$|Th(x_0) - Th(x_1)| \le C(M(r)+p_1)k(\varepsilon)(\beta-\gamma)^{-1}|x_0-x_1| \tag{2.3.10}$$

if ε and r are small enough so that $\beta > \gamma$.

Similarly, if $h_1, h_2 \in X$ and $x_0 \in \mathbb{R}^n$, we obtain

$$|Th_1(x_0)-Th_2(x_0)| \le Ck(\varepsilon)[\beta^{-1}+(1+p_1)M(r)k(\varepsilon)r^{-1}(\beta-\gamma)^{-1}] \cdot \|h_1-h_2\|. \tag{2.3.11}$$

By a suitable choice of p, p_1, ε and r, we see from (2.3.8), (2.3.10) and (2.3.11) that T is a contraction on X. This proves the existence of a Lipschitz centre manifold for (2.3.2). To prove that h is C^1 we show that T is a contraction on a subset of X consisting of Lipschitz differentiable functions. The details are similar to the proof given above so we omit the details. To prove that h is C^2 we imitate the proof of Theorem 4.2 on page 333 of [22].

2.4. Reduction Principle

The flow on the centre manifold is governed by the n-dimensional system

$$\dot{u} = Au + f(u,h(u)). \tag{2.4.1}$$

In this section we prove a theorem which enables us to relate the asymptotic behavior of small solutions of (2.3.1) to solutions of (2.4.1).

We first prove a lemma which describes the stability properties of the centre manifold.

Lemma 1. Let $(x(t),y(t))$ be a solution of (2.3.2) with $|(x(0),y(0))|$ sufficiently small. Then there exist positive C_1 and μ such that

$$|y(t) - h(x(t))| \leq C_1 e^{-\mu t}|y(0) - h(x(0))|$$

for all $t \geq 0$.

Proof: Let $(x(t),y(t))$ be a solution of (2.3.2) with $(x(0),y(0))$ sufficiently small. Let $z(t) = y(t) - h(x(t))$, then by an easy computation

$$\dot{z} = Bz + N(x,z) \tag{2.4.2}$$

where

$$N(x,z) = h'(x)[F(x,h(x)) - F(x,z+h(x))] + G(x,z + h(x)) - G(x,h(x)).$$

Using the definitions of F and G and the bounds on h, there is a continuous function $\delta(\varepsilon)$ with $\delta(0) = 0$ such that $|N(x,z)| < \delta(\varepsilon)|z|$ if $|z| < \varepsilon$. Using (2.3.6) we obtain, from (2.4.2),

$$|z(t)| \leq Ce^{-\beta t}|z(0)| + C\delta(\varepsilon)\int_0^t e^{-\beta(t-s)}|z(s)|ds$$

and the result follows from Gronwall's inequality.

Before giving the main result in this section we make some remarks about the matrix A. Since the eigenvalues of A all have zero real parts, by a change of basis we can put A in the form $A = A_1 + A_2$ where A_2 is nilpotent and

$$|e^{A_1 t}x| = |x|. \tag{2.4.3}$$

Since A_2 is nilpotent, we can choose the basis such that

$$|A_2 x| \leq (\beta/4)|x|, \tag{2.4.4}$$

where β is defined by (2.3.6).

We assume for the rest of this section that a basis has been chosen so that (2.4.3) and (2.4.4) hold.

Theorem 2. (a) Suppose that the zero solution of (2.4.1) is stable (asymptotically stable) (unstable). Then the zero solution of (2.3.1) is stable (asymptotically stable) (unstable).

(b) Suppose that the zero solution of (2.3.1) is stable. Let $(x(t),y(t))$ be a solution of (2.3.1) with $(x(0),y(0))$ sufficiently small. Then there exists a solution $u(t)$ of (2.4.1) such that as $t \to \infty$,

$$\begin{aligned} x(t) &= u(t) + O(e^{-\gamma t}) \\ y(t) &= h(u(t)) + O(e^{-\gamma t}) \end{aligned} \tag{2.4.5}$$

where $\gamma > 0$ is a constant depending only on B.

Proof: If the zero solution of (2.4.1) is unstable then by invariance, the zero solution of (2.3.1) is unstable. From now on we assume that the zero solution of (2.3.1) is stable. We prove that (2.4.5) holds where $(x(t),y(t))$ is a solution of (2.3.2) with $|(x(0),y(0))|$ sufficiently small. Since F and G are equal to f and g in a neighborhood of the origin this proves Theorem 2. We divide the proof into two steps.

I. Let $u_0 \in \mathbb{R}^n$ and $z_0 \in \mathbb{R}^m$ with $|(u_0,z_0)|$ sufficiently small. Let $u(t)$ be the solution of (2.4.1) with $u(0) = u_0$. We prove that there exists a solution $(x(t),y(t))$ of (2.3.2) with $y(0) - h(x(0)) = z_0$ and $x(t) - u(t)$,

$y(t) - h(u(t))$ exponentially small as $t \to \infty$.

II. By Step I we can define a mapping S from a neighborhood of the origin in $\mathbb{R}^{n+m}$ into $\mathbb{R}^{n+m}$ by $S(u_0,z_0) = (x_0,z_0)$ where $x_0 = x(0)$. For $|(x_0,z_0)|$ sufficiently small, we prove that (x_0,z_0) is in the range of S.

I. Let $(x(t),y(t))$ be a solution of (2.3.2) and $u(t)$ a solution of (2.4.1). Note that if $u(0)$ is sufficiently small,

$$\dot{u} = Au + F(u,h(u)) \tag{2.4.6}$$

since solutions of (2.4.1) are stable. Let $z(t) = y(t) - h(x(t))$, $\phi(t) = x(t) - u(t)$. Then by an easy computation

$$\dot{z} = Bz + N(\phi+u,z) \tag{2.4.7}$$

$$\dot{\phi} = A\phi + R(\phi,z) \tag{2.4.8}$$

where N is defined in the proof of Lemma 1 and

$$R(\phi,z) = F(u+\phi,z+h(u+\phi)) - F(u,h(u)).$$

We now formulate (2.4.7), (2.4.8) as a fixed point problem. For $a > 0$, $K > 0$, let X be the set of continuous functions $\phi\colon [0,\infty) \to \mathbb{R}^n$ with $|\phi(t)e^{at}| \leq K$ for all $t \geq 0$. If we define $\|\phi\| = \sup\{|\phi(t)e^{at}|: t \geq 0\}$, then X is a complete space. Let (u_0,z_0) be sufficiently small and let $u(t)$ be the solution of (2.4.6) with $u(0) = u_0$. Given $\phi \in X$ let $z(t)$ be the solution of (2.4.7) with $z(0) = z_0$. Define $T\phi$ by

$$(T\phi)(t) = -\int_t^\infty e^{A_1(t-s)}[A_2\phi(s) + R(\phi(s),z(s))]ds. \tag{2.4.9}$$

We solve (2.4.9) by means of the contraction mapping principle. If ϕ is a fixed point of T, then retracing our steps we find that $x(t) = u(t) + \phi(t)$, $y(t) = z(t) + h(x(t))$ is a solution of (2.3.2). We can take a to be as close to β as we please at the cost of increasing K and shrinking the neighborhood on which the result is valid. For simplicity however, we take $K = 1$ and $2a = \beta$ where β is defined by (2.3.6).

Using the bounds on F,G,h and the fact that $N(\phi,0) = 0$, there is a continuous function $k(\varepsilon)$ with $k(0) = 0$ such that if $\phi_1,\phi_2 \in \mathbb{R}^n$ and $z_1,z_2 \in \mathbb{R}^m$ with $|z_i| < \varepsilon$, then

$$\begin{aligned} |N(\phi_1,z_1) - N(\phi_2,z_2)| &\leq k(\varepsilon)[|z_1||\phi_1-\phi_2| + |z_1-z_2|] \\ |R(\phi_1,z_1) - R(\phi_2,z_2)| &\leq k(\varepsilon)[|z_1-z_2| + |\phi_1-\phi_2|]. \end{aligned} \tag{2.4.10}$$

From (2.4.7),

$$|z(t)| \leq C|z_0|e^{-\beta t} + Ck(\varepsilon)\int_0^t e^{-\beta(t-s)}|z(s)|ds$$

where we have used (2.3.6) and (2.4.10). By Gronwall's inequality

$$|z(t)| \leq C|z_0|e^{-\beta_1 t} \tag{2.4.11}$$

where $\beta_1 = \beta - Ck(\varepsilon)$. From (2.4.9), if ε is sufficiently small,

$$|T\phi(t)| \leq \frac{e^{-at}}{2} + k(\varepsilon)\int_t^\infty (e^{-as} + C|z_0|e^{-\beta_1 s})ds \leq e^{-at}$$

where we have used (2.4.3), (2.4.4), (2.4.10) and (2.4.11). Hence T maps X into X.

Now let $\phi_1,\phi_2 \in X$ and let z_1,z_2 be the corresponding solutions of (2.4.7) with $z_i(0) = z_0$. We first estimate $w(t) = z_1(t) - z_2(t)$. From (2.4.7) and (2.4.10),

$$|w(t)| \le Ck(\varepsilon)\int_0^t e^{-\beta(t-s)}[|z_1(s)||\phi_1(s)-\phi_2(s)| + |w(s)|]ds.$$

Using (2.4.11),

$$|w(t)| \le C_1 k(\varepsilon)\|\phi_1-\phi_2\|e^{-\beta t} + Ck(\varepsilon)\int_0^t e^{-\beta(t-s)}|w(s)|ds$$

where C_1 is a constant, so that by Gronwall's inequality

$$|w(t)| = |z_1(t)-z_2(t)| \le C_1 k(\varepsilon)\|\phi_1-\phi_2\| e^{-\beta_1 t}. \quad (2.4.12)$$

Using (2.4.4) and (2.4.12), for ε sufficiently small,

$$|T\phi_1(t)-T\phi_2(t)| \le \frac{1}{2}\|\phi_1-\phi_2\| + k(\varepsilon)\int_t^\infty (|\phi_1(s) - \phi_2(s)| + |z_1(s) - z_2(s)|)ds < \alpha\|\phi_1-\phi_2\|$$

where $\alpha < 1$.

The above analysis proves that for each (u_0,z_0) sufficiently small, T has a unique fixed point. If U is a neighborhood of the origin in $\mathbb{R}^{n+m}$ then it is easy to repeat the above analysis to show that $T: X \times U \to X$ is a continuous uniform contraction. This proves that the fixed point depends continuously on u_0 and z_0.

II. Define S by $S(u_0,z_0) = (x_0,z_0)$ where $x_0 = u_0 + \phi(0)$. Since ϕ depends continuously on u_0 and z_0, S is continuous. We prove that S is one-to-one, so that by the Invariance of Domain Theorem (see [11] or [60]) S is an open mapping. Since $S(0,0) = 0$, this proves that the range of S is a full neighborhood of the origin in $\mathbb{R}^{n+m}$.

Proving that S is one-to-one is clearly equivalent to proving that if $u_0 + \phi_0(0) = u_1 + \phi_1(0)$ then $u_0 = u_1$ and $\phi_0(0) = \phi_1(0)$. If $u_0 + \phi_0(0) = u_1 + \phi_1(0)$ then the initial values for x and y are the same, so that by

uniqueness of solution of (2.3.2), $u_0(t) + \phi_0(t) = u_1(t) + \phi_1(t)$ for all $t \ge 0$, where $u_i(t)$ is the solution of (2.4.6) with $u_i(0) = u_i$. Hence, for $t \ge 0$,

$$u_0(t) - u_1(t) = \phi_1(t) - \phi_0(t). \quad (2.4.13)$$

Since the real parts of the eigenvalues of A are all zero, $\lim_{t\to\infty}|u_1(t) - u_0(t)|e^{\varepsilon t} = \infty$ for any $\varepsilon > 0$ unless $u_1(0) = u_0(0)$. Also, $|\phi_i(t)| \le e^{-at}$ for all $t \ge 0$. It now follows from (2.4.13) that S is one-to-one and this completes the proof of the theorem.

2.5. Approximation of the Centre Manifold

For functions $\phi: \mathbb{R}^n \to \mathbb{R}^m$ which are C^1 in a neighborhood of the origin define

$$(M\phi)(x) = \phi'(x)[Ax + f(x,\phi(x))] - B\phi(x) - g(x,\phi(x)).$$

Theorem 3. Suppose that $\phi(0) = 0$, $\phi'(0) = 0$ and that $(M\phi)(x) = O(|x|^q)$ as $x \to 0$ where $q > 1$. Then as $x \to 0$,

$$|h(x) - \phi(x)| = O(|x|^q).$$

Proof: Let $\theta: \mathbb{R}^n \to \mathbb{R}^m$ be a continuously differentiable function with compact support such that $\theta(x) = \phi(x)$ for $|x|$ small. Set

$$N(x) = \theta'(x)[Ax + F(x,\theta(x))] - B\theta(x) - G(x,\theta(x)), \quad (2.5.1)$$

where F and G are defined in Theorem 1. Note that $N(x) = O(|x|^q)$ as $x \to 0$.

In Theorem 1, we proved that h was the fixed point of a contraction mapping $T: X \to X$. Define a mapping S by $Sz = T(z+\theta) - \theta$; the domain of S being a closed subset

$Y \subset X$. Since T is a contraction mapping on X, S is a contraction mapping on Y. For $K > 0$ let

$$Y = \{z \in X: |z(x)| \le K|x|^q \text{ for all } x \in \mathbb{R}^n\}.$$

If we can find a K such that S maps Y into Y then we will have proved the theorem.

We first find an alternative formulation of the map S. For $z \in Y$ let $x(t,x_0)$ be the solution of

$$\dot{x} = Ax + F(x,z(x) + \theta(x)), \quad x(0,x_0) = x(0). \tag{2.5.2}$$

From (2.3.4)

$$(T(z+\theta))(x_0) = \int_{-\infty}^{0} e^{-Bs}G(x(s,x_0),z(x(s,x_0)) + \theta(x(s,x_0)))ds.$$

Now

$$-\theta(x_0) = -\int_{-\infty}^{0} \frac{d}{ds}[e^{-Bs}\theta(x(s,x_0))]ds$$
$$= \int_{-\infty}^{0} e^{-Bs}[B\theta(x(s,x_0)) - \frac{d}{ds}\theta(x(s,x_0))]ds.$$

Writing x for $x(s,x_0)$ etc., from (2.5.1) and (2.5.2)

$$B\theta(x) - \frac{d}{ds}\theta(x) = B\theta(x) - \theta'(x)[Ax + F(x,z(x) + \theta(x))]$$
$$= -N(x) - G(x,\theta(x)) + \theta'(x)[F(x,\theta) - F(x,z(x) + \theta(x))].$$

Using $Sz = T(z+\theta) - \theta$ and the above calculations

$$(Sz)(x_0) = \int_{-\infty}^{0} e^{-Bs}Q(x(s,x_0),\ z(x(s,x_0)))ds \tag{2.5.3}$$

where $x(s,x_0)$ is the solution of (2.5.2) and

$$Q(x,z) = G(x,\theta+z) - G(x,\theta) - N(x) + \theta'(x)[F(x,\theta) - F(x,\theta+z)]. \tag{2.5.4}$$

We now show that S maps Y into Y for some $K > 0$. By choosing θ suitably, we may assume that $|\theta(x)| \le \varepsilon$ for all $x \in \mathbb{R}^n$. Since $N(x) = O(|x|^q)$ as $x \to 0$,

$$|N(x)| \le C_1|x|^q, \quad x \in \mathbb{R}^n \tag{2.5.5}$$

where C_1 is a constant. Now

$$|Q(x,z)| \le |Q(x,0)| + |Q(x,z) - Q(x,0)|$$
$$= |N(x)| + |Q(x,z) - Q(x,0)|. \tag{2.5.6}$$

We can estimate $|Q(x,z) - Q(x,0)|$ in terms of the Lipschitz constants of F and G. Using (2.3.5), there is a continuous function $k(\varepsilon)$ with $k(0) = 0$, such that

$$|Q(x,z) - Q(x,0)| \le k(\varepsilon)|z| \tag{2.5.7}$$

for $|z| \le \varepsilon$. Using (2.5.5), (2.5.6), (2.5.7), for $z \in Y$ and $x \in \mathbb{R}^n$, we have that

$$|Q(x,z)| \le C_1|x|^q + k(\varepsilon)|z(x)|$$
$$\le (C_1 + Kk(\varepsilon))|x|^q. \tag{2.5.8}$$

Using the same calculations as in the proof of Theorem 1, if $x(t,x_0)$ is the solution of (2.5.2), then for each $r > 0$, there is a constant $M(r)$ such that

$$|x(t,x_0)| \le M(r)|x_0|e^{-\gamma t}, \quad t \le 0 \tag{2.5.9}$$

where $\gamma = r + 2M(r)k(\varepsilon)$.

Using (2.3.6), (2.5.8) and (2.5.9), if $z \in Y$,

$$|(Sz)(x_0)| \le C(C_1+Kk(\varepsilon))(M(r))^q(\beta-q\gamma)^{-1}|x_0|^q = C_2|x_0|^q$$

provided ε and r are small enough so that $\beta - q\gamma > 0$.

By choosing K large enough and ε small enough, we have that $C_2 \leq K$ and this completes the proof of the theorem.

2.6. Properties of Centre Manifolds

(1) In general (2.3.1) does not have a unique centre manifold. For example, the system $\dot{x} = -x^3$, $\dot{y} = -y$, has the two parameter family of centre manifolds $y = h(x,c_1,c_2)$ where

$$h(x,c_1,c_2) = \begin{cases} c_1 \exp(-\frac{1}{2}x^{-2}), & x > 0 \\ 0 & x = 0 \\ c_2 \exp(-\frac{1}{2}x^{-2}), & x < 0. \end{cases}$$

However, if h and h_1 are two centre manifolds for (2.3.1), then by Theorem 3, $h(x) - h_1(x) = O(|x|^q)$ as $x \to 0$ for all $q > 1$.

(2) If f and g are C^k, $(k \geq 2)$, then h is C^k [44]. If f and g are analytic, then in general (2.3.1) does not have an analytic centre manifold, for example, it is easy to show that the system

$$\dot{x} = -x^3, \quad \dot{y} = -y + x^2 \qquad (2.6.1)$$

does not have an analytic centre manifold (see exercise (1)).

(3) Centre manifolds need not be unique but there are some points which must always be on any centre manifold. For example, suppose that (x_0,y_0) is a small equilibrium point of (2.3.1) and let $y = h(x)$ be any centre manifold for (2.3.1). Then by Lemma 1 we must have $y_0 = h(x_0)$. Similarly, if Γ is a small periodic orbit of (2.3.1), then Γ must lie on all centre manifolds.

(4) Suppose that $(x(t),y(t))$ is a solution of (2.3.1) which remains in a neighborhood of the origin for all $t \geq 0$. An examination of the proof of Theorem 2, shows that there is a solution $u(t)$ of (2.4.1) such that the representation (2.4.5) holds.

(5) In many problems the initial data is not arbitrary, for example, some of the components might always be nonnegative. Suppose $S \subset \mathbb{R}^{n+m}$ with $0 \in S$ and that (2.3.1) defines a local dynamical system on S. It is easy to check, that with the obvious modifications, Theorem 2 is valid when (2.3.1) is studied on S.

Exercise 1. Consider

$$\dot{x} = -x^3, \quad \dot{y} = -y + x^2. \qquad (2.6.1)$$

Suppose that (2.6.1) has a centre manifold $y = h(x)$, where h is analytic at $x = 0$. Then

$$h(x) = \sum_{n=2}^{\infty} a_n x^n$$

for small x. Show that $a_{2n+1} = 0$ for all n and that $a_{n+2} = na_n$ for $n = 2,4,\ldots$, with $a_2 = 1$. Deduce that (2.6.1) does not have an analytic centre manifold.

Exercise 2 (Modification of an example due to S. J. van Strien [63]). If f and g are C^∞ functions, then for each r, (2.3.1) has a C^r centre manifold. However, the size of the neighborhood on which the centre manifold is defined depends on r. The following example shows that in general (2.3.1) does not have a C^∞ centre manifold, even if f and g are analytic.

Consider

$$\dot{x} = -\varepsilon x - x^3, \quad \dot{y} = -y + x^2, \quad \dot{\varepsilon} = 0. \tag{2.6.2}$$

Suppose that (2.6.2) has a C^∞ centre manifold $y = h(x,\varepsilon)$ for $|x| < \delta$, $|\varepsilon| < \delta$. Choose $n > \delta^{-1}$. Then since $h(x,(2n)^{-1})$ is C^∞ in x, there exist constants $a_1, a_2, \ldots, a_{2n}$ such that

$$h(x,(2n)^{-1}) = \sum_{i=1}^{2n} a_i x^i + O(x^{2n+1})$$

for $|x|$ small enough. Show that $a_i = 0$ for odd i and that if $n > 1$,

$$(1 = (2i)(2n)^{-1})a_{2i} = (2i-2)a_{2i-2}, \quad i = 2,\ldots,n \qquad a_2 \neq 0. \tag{2.6.3}$$

Obtain a contradiction from (2.6.3) and deduce that (2.6.2) does not have a C^∞ centre manifold.

Exercise 3. Suppose that the nonlinearities in (2.3.1) are odd, that is $f(x,y) = -f(-x,-y)$, $g(x,y) = -g(-x,-y)$. Prove that (2.3.1) has a centre manifold $y = h(x)$ with $h(x) = -h(-x)$. [The example $\dot{x} = -x^3$, $\dot{y} = -y$, shows that if h is any centre manifold for (2.3.1) then $h(x) \neq -h(-x)$ in general.]

2.7. Global Invariant Manifolds for Singular Perturbation Problems

To motivate the results in this section we reconsider Example 3 in Chapter 1. In that example we applied centre manifold theory to a system of the form

$$\begin{aligned} y' &= \varepsilon f(y,w) \\ w' &= -w + y^2 - yw + \varepsilon f(y,w) \\ \varepsilon' &= 0 \end{aligned} \tag{2.7.1}$$

where $f(0,0) = 0$. Because of the local nature of our results on centre manifolds, we only obtained a result concerning small initial data. Let $v = -w(1+y) + y^2$, $s = s(\tau)$, where $s'(\tau) = 1 + y(\tau)$; then we obtain a system of the form

$$\begin{aligned} y' &= \varepsilon g_1(y,v) \\ v' &= -v + \varepsilon g_2(y,v) \\ \varepsilon' &= 0 \end{aligned} \tag{2.7.2}$$

where $g_i(0,0) = 0$, $i = 1,2$. Note that if $y \neq -1$, then $(y,0,0)$ is always an equilibrium point for (2.7.2) so we expect that (2.7.2) has an invariant manifold $v = h(y,\varepsilon)$ defined for $-1 < y < m$, say, and ε sufficiently small.

Theorem 4. Consider the system

$$\begin{aligned} x' &= Ax + \varepsilon f(x,y,\varepsilon) \\ y' &= By + \varepsilon g(x,y,\varepsilon) \\ \varepsilon' &= 0 \end{aligned} \tag{2.7.3}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ and A,B are as in Theorem 1. Suppose also that f,g are C^2 with $f(0,0,0) = 0$, $g(0,0,0) = 0$. Let $m > 0$. Then there is a $\delta > 0$ such that (2.7.3) has an invariant manifold $y = h(x,\varepsilon)$, $|x| < m$, $|\varepsilon| < \delta$, with $|h(x,\varepsilon)| < C|\varepsilon|$, where C is a constant which depends on m,A,B,f and g.

Proof: Let $\psi: \mathbb{R}^n \to [0,1]$ be a C^∞ function with $\psi(x) = 1$ if $|x| \leq m$ and $\psi(x) = 0$ if $|x| \geq m + 1$. Define F and G by

$$F(x,y,\varepsilon) = \varepsilon f(x\psi(x),y,\varepsilon), \quad G(x,y,\varepsilon) = \varepsilon g(x\psi(x),y,\varepsilon).$$

We can then prove that the system

$$\begin{aligned} x' &= Ax + F(x,y,\varepsilon) \\ y' &= By + G(x,y,\varepsilon) \end{aligned} \qquad (2.7.4)$$

has an invariant manifold $y = h(x,\varepsilon)$, $x \in \mathbb{R}^n$, for $|\varepsilon|$ sufficiently small. The proof is essentially the same as that given in the proof of Theorem 1 so we omit the details.

Remark. If $x = (x_1,x_2,\ldots,x_n)$ then we can similarly prove the existence of $h(x,\varepsilon)$ for $\underline{m}_i < x_i < \overline{m}_i$.

The flow on the invariant manifold is given by the equation

$$u' = Au + \varepsilon f(u,h(u,\varepsilon)). \qquad (2.7.5)$$

With the obvious modifications it is easy to show that the stability of solutions of (2.7.3) is determined by equation (2.7.5) and that the representation of solutions given in (2.4.5) holds.

Finally, we state an approximation result.

Theorem 5. Let $\phi: \mathbb{R}^{n+1} \to \mathbb{R}^m$ satisfy $\phi(0,0) = 0$ and $|(M\phi)(x,\varepsilon)| \le C\varepsilon^p$ for $|x| \le m$ where p is a positive integer, C is a constant and

$$(M\phi)(x,\varepsilon) = D_x\phi(x,\varepsilon)[Ax + \varepsilon f(x,\phi(x,\varepsilon))] - B\phi(x,\varepsilon) - \varepsilon g(x,\phi(x,\varepsilon)).$$

Then, for $|x| \le m$,

$$|h(x,\varepsilon) - \phi(x,\varepsilon)| \le C_1\varepsilon^p$$

for some constant C_1.

Theorem 5 is proved in exactly the same way as Theorem 3 so we omit the proof.

For further information on the application of centre manifold theory to singular perturbation problems see Fenichel [24] and Henry [34].

2.8. Centre Manifold Theorems for Maps

In this section we briefly indicate some results on centre manifolds for maps. We first indicate how the study of maps arises naturally in studying periodic solutions of differential equations.

Consider the following equation in $\mathbb{R}^n$

$$\dot{x} = f(x,\lambda) \qquad (2.8.1)$$

where f is smooth and λ is a real parameter. Suppose that for $\lambda = \lambda_0$, (2.8.1) has a periodic solution γ with period T. One way to study solutions of (2.8.1) near γ for $|\lambda - \lambda_0|$ small is to consider the Poincaré map $P(\lambda)$. To define $P(\lambda)$ let y be some point on γ, let U be a local cross section of γ through y and let U_1 be an open neighborhood of y in U. Then $P(\lambda): U_1 \to U$ is defined by $P(\lambda)(z) = x(s)$, where $x(t)$ is the solution of (2.8.1) with $x(0) = z$ and $s > 0$ is the first time $x(t)$ hits U. (See [51] for the details).

If $P(\lambda)$ has a fixed point then (2.8.1) has a periodic orbit with period close to T. If $P(\lambda)$ has a periodic point of order n, $(P(\lambda))^k z \neq z$ for $1 \le k \le n-1$ and $P^n(\lambda)z = z$) then (2.8.1) has a periodic solution with period close to nT. If $P(\lambda)$ preserves orientation and there is a closed curve which is invariant under $P(\lambda)$ then there exists an invariant torus for (2.8.1).

If none of the eigenvalues of the linearized map $P'(\lambda_0)$ lie on the unit circle then it can be shown that $P(\lambda)$ has essentially the same behavior as $P(\lambda_0)$ for $|\lambda - \lambda_0|$ small. Hence in this case, for $|\lambda - \lambda_0|$ small, solutions of (2.8.1) near γ have the same behavior as when $\lambda = \lambda_0$. If some of the eigenvalues of $P'(\lambda_0)$ lie on the unit circle then there is the possibility of bifurcations taking place. In this case centre manifold theory reduces the dimension of the problem. As in ordinary differential equations we only discuss the stable case, that is, none of the eigenvalues of the linearized problem lie outside the unit circle.

Let $T: \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ have the following form:

$$T(x,y) = (Ax + f(x,y),\ By + g(x,y)) \tag{2.8.2}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, A and B are squıre matrices such that each eigenvalue of A has modulus 1 and each eigenvalue of B has modulus less than 1, f and g are C^2 and f,g and their first order derivatives are zero at the origin.

Theorem 6. There exists a centre manifold $h: \mathbb{R}^n \to \mathbb{R}^m$ for T. More precisely, for some $\varepsilon > 0$ there exists a C^2 function $h: \mathbb{R}^n \to \mathbb{R}^m$ with $h(0) = 0$, $h'(0) = 0$ such that $|x| < \varepsilon$ and $(x_1,y_1) = T(x,h(x))$ implies $y_1 = h(x_1)$.

In order to determine h we have to solve the equation

$$(x_1,h(x_1)) = T(x,h(x)).$$

By (2.8.2) this is equivalent to

$$h(Ax + f(x,h(x))) = Bh(x) + g(x,h(x)).$$

For functions $\phi: \mathbb{R}^n \to \mathbb{R}^m$ define $M\phi$ by

$$(M\phi)(x) = \phi(Ax + f(x,\phi(x))) - B\phi(x) - g(x,\phi(x))$$

so that $Mh = 0$.

Theorem 7. Let $\phi: \mathbb{R}^n \to \mathbb{R}^m$ be a C^1 map with $\phi(0) = 0$, $\phi'(0) = 0$ and $(M\phi)(x) = O(|x|^q)$ as $x \to 0$ for some $q > 1$. Then $h(x) = \phi(x) + O(|x|^q)$ as $x \to 0$.

We now study the difference equation

$$\begin{aligned} x_{r+1} &= Ax_r + f(x_r,y_r) \\ y_{r+1} &= By_r + g(x_r,y_r). \end{aligned} \tag{2.8.3}$$

As in the ordinary differential equation case, the asymptotic behavior of small solutions of (2.8.3) is determined by the flow on the centre manifold which is given by

$$u_{r+1} = Au_r + f(u_r,h(u_r)). \tag{2.8.4}$$

Theorem 8. (a) Suppose that the zero solution of (2.8.4) is stable (asymptotically stable) (unstable). Then the zero solution of (2.8.3) is stable (asymptotically stable) (unstable).

(b) Suppose that the zero solution of (2.8.3) is stable. Let (x_r,y_r) be a solution of (2.8.3) with (x_1,y_1) sufficiently small. Then there is a solution u_r of (2.8.4) such that $|x_r - u_r| \le K\beta^r$ and $|y_r - h(u_r)| \le K\beta^r$ for all r where K and β are positive constants with $\beta < 1$.

The proof of Theorem 6 and the stability claim of Theorem 8 can be found in [30,40,51]. The rest of the assertions are proved in the same way as the ordinary differential