Initial remarks

- Regularity of semigroup corresponds to regularity of solutions to DE
  - Initial value \( x_0 \in D(A) \), then solution is differentiable for all \( t \geq 0 \)
  - Initial value \( x_0 \not\in D(A) \), then solution is not differentiable for \( t = 0 \).
    It may and may no be differentiable for some \( t > 0 \).
- From PDE’s: for every \( x_0 \in X \) the solution to heat equation is smooth for all \( t > 0 \) … high regularity of the heat semigroup.
- From PDE’s: solution to transport equation preserves regularity of the initial value … low regularity of the shift semigroup.
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Types of regularity

We distinguish several types of regularity, the following implications hold

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\text{NORM CONTINUOUS} \iff \text{DIFFERENTIABLE} \iff \text{ANALYTIC}
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\textit{Norm continuous} means that the mapping \( t \mapsto S(t) \) is continuous in the operator topology for all \( t > 0 \) (strictly!).

\textit{Differentiable} means that the mapping \( t \mapsto S(t)x \) is differentiable for all \( x \in X \) and all \( t > 0 \) (strictly!).

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Analytic semigroups

Definition

A $C_0$-semigroup $T$ is called analytic, if there exists $\theta > 0$ s.t. $T$ has an analytic extension $\tilde{T} : \Sigma_\theta \to \mathcal{L}(X)$ satisfying

1. $\tilde{T}(z + w) = \tilde{T}(z)\tilde{T}(w)$ for all $z, w \in \Sigma_\theta$
2. $z \mapsto T(z)$ is analytic in $\Sigma_\theta \setminus \{0\}$
3. $\lim_{\Sigma_{\theta'} \ni z \to 0} T(z)x = x$ for all $x \in X$, $\theta' \in (0, \theta)$.

An analytic semigroup is called bounded analytic semigroup if it is bounded on each $\Sigma_{\theta'}$, $\theta' \in (0, \theta)$. 
Sectorial operators

**Definition**

An operator \((A, D(A))\) is called sectorial if there exists \(\delta \in (0, \frac{\pi}{2}]\) such that

1. \(\Sigma_{\frac{\pi}{2}+\delta} \setminus \{0\} \subset \rho(A)\)
2. **for every** \(\varepsilon \in (0, \delta)\) there exists \(M_{\varepsilon}\), \(\|R(\lambda, A)\| \leq \frac{M_{\varepsilon}}{|\lambda|}\) **for all** \(\lambda \in \Sigma_{\frac{\pi}{2}+\delta-\varepsilon} \setminus \{0\}\)
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Remark 1

The resolvent estimate holds with \(|\lambda|\) instead of \(\Re \lambda\).
Generator of an analytic semigroup

Theorem 1

Let \((A, D(A))\) be a \(\theta\)-sectorial operator. Then it generates a bounded analytic semigroup on \(\Sigma_\theta\) given by

\[
T(z) = \frac{1}{2\pi i} \int_{\gamma} e^{\mu z} R(\mu, A) d\mu
\]

where \(\gamma\) is as follows with \(\theta > \theta' > |\text{arg } z|\).
Characterization of analytic semigroups

Theorem 2

Let \((A, D(A))\) is the generator of a \(C_0\)-semigroup \(T\). Then the following assertions are equivalent:

- \(A\) is \(\theta\)-sectorial
- \(T\) is a bounded analytic semigroup on \(\Sigma_\theta\).
- \(T(t)X \subset D(A)\) and \(\{\|tAT(t)\| : t \in (0, 1]\}\) is bounded.
- \(e^{\pm i\theta'} A\) generate bounded \(C_0\)-semigroups \(T(e^{\pm i\theta'} t)\) for all \(\theta' \in (0, \theta)\).

parts of the proof in HW4
Proposition 3

A multiplicative operator $A_m$ on $L^2(\Omega)$ is $\theta$-sectorial if and only if the essential range of $m$ is contained in $\mathbb{C} \setminus \Sigma_{\frac{\pi}{2}}^{\pi/2} + \theta \cup \{0\}$.

Corollary 4

Every self-adjoint dissipative operator on a Hilbert space generates a bounded analytic semigroup on $\Sigma_{\frac{\pi}{2}}$.

Example

Dirichlet Laplacian on $L^2(\Omega)$, $\Omega$ bounded domain in $\mathbb{R}^n$ is self-adjoint and dissipative. Therefore, the heat semigroup is analytic and solutions to the corresponding heat equation are analytic for $t > 0$...?
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