# Chapter 6. Asymptotic begavior of semigroups

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# Spectral bound and growth bound

Growth of solutions to (ACP) is determined by the growth of the semigroup.

Is the growth of semigroups determined by the spectrum of its generator?

**Definition**

*Let $(A, D(A))$ be the generator of a $C_0$-semigroup $T$.*

- *The spectral bound of $A$ is*

$$s(A) = \sup\{\Re\lambda : \lambda \in \sigma(A)\}.$$

- *The growth bound of $T$ is*

$$\omega(T) = \inf\{\omega \in \mathbb{R} : \exists M \geq 1 \; \|T(t)\| \leq Me^{\omega t} \; \forall t \geq 0\}.$$

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## Remark

*1. By Proposition 4 (Chapter 1) we have $s(A) \leq \omega(T)$.*

*2. For a matrix A we have $s(A) = \omega(e^{tA})$. In fact,*

$$\|e^{tA}\| \leq Me^{ct}(1+t)^{n-1} \leq \tilde{M}e^{(c+\varepsilon)t} \quad \forall t \geq 0$$

*where $c = \max\{\Re\lambda : \lambda \in \sigma(A)\}$ and n is the size of the biggest Jordan cell of A.*

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*Two counterexamples can be found in [EN], Chapter IV.3.*

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1. $\sigma(T) \setminus \{0\} = e^{t\sigma(A)}$ (spectral mapping theorem)
2. $\|T(t)\| \sim r(T(t))$

$$\|T(t)\| \overset{(1)}{=} r(T(t)) = \sup_{\lambda \in \sigma(T(t))} |\lambda| = \sup_{e^{\lambda t} \in \sigma(T(t))} |e^{\lambda t}| \overset{(2)}{=} \sup_{\lambda \in \sigma(A)} |e^{\lambda t}|$$

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## Proposition 1

*Let $T$ be a $C_0$-semigroup. Then $\omega(T) = \frac{1}{t} \ln r(T(t))$ for each $t \geq 0$*

Proof: see [EN], Proposition IV.2.2

# Spectral inclusion

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# decomposition of spectrum

## Definition

*Let $(A, D(A))$ be a closed densely defined operator and $\lambda \in \sigma(A)$. We say that $\lambda$ belongs to*

- *the point spectrum $P\sigma(A)$ if $\lambda - A$ is not injective,*
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## Remark

*1. Clearly, $P\sigma(A) \subset A\sigma(A)$.*
*2. It can be shown that $R\sigma(A) = P\sigma(A')$, where $A'$ is the adjoint of $A$.*
*3. $\lambda \in A\sigma(A)$ if and only if there exists an approximate eigenvector $(x_n)_{n=1}^{\infty} \subset D(A)$ such that $\|x_n\| = 1$ and $\lim_{n\to\infty}(\lambda - A)x_n = 0$.*

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# Spectral mapping theorem for point and residual spectrum

**Theorem 3**

*Let $(A, D(A))$ be the generator of a $C_0$-semigroup $T$. Then $P\sigma(T(t)) \setminus \{0\} = e^{tP\sigma(A)}$ and $R\sigma(T(t)) \setminus \{0\} = e^{tR\sigma(A)}$.*

Proof can be found in [EN], Theorem IV.3.7.

**Proposition 4**

*Let $\mu \in \sigma(T(t))$ and $\Lambda = \{\lambda \in \sigma(A) : \; e^{\lambda t} = \mu\}$. Then*

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# Spectral mapping theorem for regular semigroups

## Theorem 5

*Let $(A, D(A))$ be the generator of a $C_0$-semigroup $T$ which is norm continuous for $t > t_0$. Then $\sigma(T(t)) \setminus \{0\} = e^{t\sigma(A)}$.*

Proof can be found in [EN], Theorem IV.3.10

## Corollary 6

*Let $(A, D(A))$ be the generator of an analytic semigroup. Then $\sigma(T(t)) \setminus \{0\} = e^{t\sigma(A)}$.*