Homework 1 - Multiplicative semigroups

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Let $\Omega \subset \mathbb{R}^n$ be measurable, $m : \Omega \to \mathbb{C}$ be measureable and such that

$$\operatorname{esssup}\{\operatorname{Re} m(x) : \; x \in \Omega\} = K < +\infty$$

(i.e. $\operatorname{Re} m(x) \leq K$ a.e. in Ω and it is not true if we replace K by any $K' < K$).
Alternatively (easier version). Consider m continuous and $K = \sup\{\Re m(x) : \; x \in \Omega\}$.
Denote $(A_m, D(A_m))$ the multiplicative operator on $L^p(\Omega, \mathbb{C})$, $1 \leq p < +\infty$ associated with m, i.e.

$$(A_m f)(x) := m(x)f(x), \quad D(A_m) := \{f \in L^p(\Omega, \mathbb{C}); \; mf \in L^p(\Omega, \mathbb{C})\}$$

and for $t \geq 0$ let $S_m(t)$ be the linear operator defined by

$$(S_m(t)f)(x) := e^{tm(x)} f(x) \quad \text{for } f \in L^p(\Omega, \mathbb{C}).$$

1. Show that $t \mapsto S_m(t)$ is a C_0-semigroup.

2. Show that $(A_m, D(A_m))$ is its generator. [You can use e.g. the Lebesque dominated convergence and the fact that convergence in L^p-norm implies a.e. convergence of a subsequence.]

3. Determine so called *growth bound* of the semigroup S_m, i.e.

$$\omega_0(S_m) := \inf\{\omega \in \mathbb{R}; \; \exists M \geq 1, \; \|S_m(t)\| \leq M e^{\omega t}, \; \forall t \geq 0\}.$$

4. Optional. Find spectrum of A_m, i.e. the set of all $\lambda \in \mathbb{C}$ s.t. $\lambda - A_m$ is not a bijection from $D(A_m)$ onto $L^p(\Omega, \mathbb{C})$. Find the spectrum of $S_m(t)$. What is the relation between the spectrum of the generator, spectrum of the semigroup and the growth bound of the semigroup?