Ordinary differential equations 2

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1 Dynamical systems

1.1 Basic properties

Definition (Dynamical system)**.** *Dynamical system is a couple* (Ω, φ)*, where* $\Omega \subset \mathbb{R}^n$ *and* $\varphi : \mathbb{R} \times \Omega \to \Omega$ *is continuous and satisfies*

(i) $\varphi(0, x) = x$ *for all* $x \in \Omega$

(ii) $\varphi(t, \varphi(s, x)) = \varphi(t + s, x)$ *for all* t*,* $s \in \mathbb{R}$*,* $x \in \Omega$*.*

Definition (Orbit, positive and negative semiorbit)**.** *Let* (Ω, φ) *be a dynamical system. The orbit of* x *is the set* $\gamma(x) := \{\varphi(t, x) : \ t \in \mathbb{R}\}$*. The positive semiorbit, resp. negative semiorbit of* x *is the set* $\gamma_+(x) := \{\varphi(t, x) : \ t \geq 0\}$*, resp.* $\gamma_-(x) := \{\varphi(t, x) : \ t \leq 0\}$*.*

Definition (Invariant set, positively and negatively invariant set)**.** *Let* (Ω, φ) *be a dynamical system. A set* $A \subset \Omega$ *is invariant if for each* $x \in A$ *we have* $\gamma(x) \subset A$*. A set* $A \subset \Omega$ *is positively, resp. negatively invariant if for each* $x \in A$ *we have* $\gamma_+(x) \subset A$*, resp.* $\gamma_-(x) \subset A$*.*

Definition (ω-limit set, α-limit set)**.** *Let* (Ω, φ) *be a dynamical system and* $x_0 \in \Omega$*. The* ω*-limit set of* x_0 *is*

$$\omega(x_0) := \{x \in \Omega : \ \exists t_n \nearrow +\infty \ s.t. \ \lim_{n\to\infty} \varphi(t_n, x_0) = x\}.$$

The α*-limit set of* x_0 *is*

$$\alpha(x_0) := \{x \in \Omega : \ \exists t_n \searrow -\infty \ s.t. \ \lim_{n\to\infty} \varphi(t_n, x_0) = x\}.$$

Theorem 1.1. *Let* (Ω, φ) *be a dynamical system and* $x_0 \in \Omega$*.*

(i) Then $\omega(x_0)$ *is closed and invariant.*

(ii) If $\gamma_+(x_0)$ *is relatively compact, then* $\omega(x_0)$ *is nonempty, compact and connected.*

Lemma 1.2. *Let* (Ω, φ) *be a dynamical system and* $x_0 \in \Omega$. *Then*

$$\omega(x_0) = \bigcap_{\tau \geq 0} \overline{\gamma_+(\varphi(\tau, x_0))}.$$

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Definition (equilibrium). *A point* x_0 *is an equilibrium (stationary point) of the equation* $x' = f(x)$ *if* $f(x_0) = 0$. *A point* x_0 *is an equilibrium (stationary point) of* (Ω, φ) *if* $\varphi(t, x_0) = x_0$ *for all* $t \in \mathbb{R}$.

Definition (Topologically conjugate systems). *Dynamical systems* (Ω, φ), (Θ, ψ) *are topologically conjugate if there exists a homeomorphism* $h : \Omega \to \Theta$ *such that* $h(\varphi(t, x)) = \psi(t, h(x))$ *for all* $x \in \Omega$ *and* $t \in \mathbb{R}$.

Theorem 1.3 (Rectification theorem). *Let* $\Omega \subset \mathbb{R}^n$ *be open,* $f \in C^r(\Omega, \mathbb{R}^n)$, $r \geq 1$ *and* $f(x_0) \neq 0$. *Then there exist a neighborhood* V *of* x_0, *a neighborhood* W *of* 0 *in* $\mathbb{R}^n$ *and a homeomorphism* $g : V \to W$ *satisfying:* $t \mapsto x(t)$ *is a solution to* $x' = f(x)$ *if and only if* $t \mapsto g(x(t))$ *is a solution to* $y' = (1, 0, 0, \ldots, 0)^T$.

Definition (hyperbolic equilibrium). *An equilibrium* x_0 *of* $x' = f(x)$ *is called hyperbolic if* $\sigma(\nabla f(x_0)) \cap i\mathbb{R} = \emptyset$, *i.e.* $\nabla f(x_0)$ *has no eigenvalues on the imaginary axis (*$\sigma(A)$ *denotes the spectrum of a matrix* A*).*

Theorem (Hartman–Grobman). *Let* x_0 *be a hyperbolic equilibrium of* $x' = f(x)$ *and denote* $A = \nabla f(x_0)$. *Then there exist a neighborhood* V *of* x_0, *a neighborhood* W *of* 0 *in* $\mathbb{R}^n$ *and a homeomorphism* $g : V \to W$ *satisfying:* $t \mapsto x(t)$ *is a solution to* $x' = f(x)$ *if and only if* $t \mapsto g(x(t))$ *is a solution to* $y' = Ay$.

1.2 LaSalle's invariance principle

Definition (orbital derivative). *Consider the equation* $x' = f(x)$ *with* $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ *and let* $V \in C^1(\Omega)$. *The orbital derivative of* V *in a point* x *is* $\dot{V}(x) := \nabla V(x) \cdot f(x)$.

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Theorem 1.4. *Let* $\Omega \subset \mathbb{R}^n$ *be open,* $f : \Omega \to \mathbb{R}^n$ *be Lipschitz continuous and* φ *be the solving function of* $x' = f(x)$. *Let* $V \in C^1(\Omega)$ *be bounded from below and* $l \in \mathbb{R}$ *be such that* $\Omega_l := \{x \in \Omega : \ V(x) \leq l\}$ *is bounded and* $\dot{V} \leq 0$ *on* Ω_l. *Denote* $S := \{x \in \Omega : \ \dot{V}(x) = 0\}$ *and* $M := \{x \in S : \ \gamma(x) \subset S\}$. *Then* $\omega(x_0) \subset M$ *for all* $x_0 \in \Omega_l$.

1.3 Poincaré–Bendixson Theory

Let Ω be a connected subset of $\mathbb{R}^2$, $f \in C^1(\Omega)$ and φ be the solving function of

$$x' = f(x). \tag{AE}$$

Assume that φ is defined at least on $[0, +\infty)$ for every $x \in \Omega$.

Theorem 1.5 (Poincaré–Bendixson). *Let $p \in \Omega$ and $\gamma_+(p)$ be relatively compact. If $\omega(p)$ does not contain any stationary points, then $\omega(p)$ is an orbit of a nontrivial periodic solution.*

Definition (Jordan curve). *A curve in $\mathbb{R}^2$ is Jordan if there exists its continuous parametrization $\psi : [0,1] \to \mathbb{R}^2$ which is injective on $[0,1)$ and satisfies $\psi(0) = \psi(1)$.*

Theorem (Jordan). *Let $\gamma \subset \mathbb{R}^2$ be a Jordan curve. Then there exist unique Ω_1, $\Omega_2 \subset \mathbb{R}^2$ such that Ω_1 is bounded, Ω_2 is not bounded, Ω_1, Ω_2 and γ are pairwise disjoint and $\mathbb{R}^2 = \Omega_1 \cup \gamma \cup \Omega_2$. We denote $\Omega_1 =: \operatorname{Int}\gamma$ and $\Omega_2 =: \operatorname{Ext}\gamma$.*

Definition (Transversal). *A transversal to φ is an open line segment $\Sigma \subset \Omega$ which is not parallel to f in any point, i.e. $\Sigma = \{a + tb : \ t \in (0,1)\}$ for some fixed a, $b \in \mathbb{R}^2$ and for every $x \in \Sigma$ the vectors $f(x)$ and b are linearly independent.*

Definition (Flow-box). *A set $U \subset \Omega$ is called a flow-box if the dynamical system (U, φ) is topologically conjugate to (V, ψ), where $V = \{(x_1, x_2) \in \mathbb{R}^n : \ |x_1| < \varepsilon_1, \ |x_2| < \varepsilon_2\}$ for some ε_1, $\varepsilon_2 > 0$ and $\psi(t, (x_1, x_2)) = (x_1 + t, x_2)$.*

Lemma 1.6. *Let Σ be a transversal to φ and $p \in \Sigma$. Then there exists a flow-box U containing p such that for every $y \in U$ the set $\Sigma \cap \gamma_U(y)$ contains exactly one point. By $\gamma_U(y)$ we denote the orbit of y in the (local) dynamical system (U, φ).*

Lemma 1.7. *Let Σ be a transversal to φ, $p \in \Sigma$. Then intersections of $\gamma_+(p)$ and Σ form a monotone sequence. In particular, if $t_1 < t_2 < t_3$ be such that $\varphi(t_j, p) \in \Sigma$, $j = 1, 2, 3$, then either $\varphi(t_1, p) = \varphi(t_2, p) = \varphi(t_3, p)$ or $\varphi(t_2, p)$ lies strictly between $\varphi(t_1, p)$ and $\varphi(t_3, p)$.*

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Lemma 1.8. *Let Σ be a transversal to φ and $p \in \Sigma$. Then $\omega(p) \cap \Sigma$ contains at most one point.*

Theorem 1.9 (Bendixson–Dulac criterion). *Let Ω be open and simply connected.*

(i) If $\operatorname{div} f > 0$ a.e. in Ω, then (AE) *has no nontrivial periodic solutions.*

(ii) If there exists $B \in C^1(\Omega)$ such that $\operatorname{div}(B \cdot f) > 0$ a.e. in Ω, then (AE) *has no nontrivial periodic solutions.*

2 Carathéodory Theory

Throughout this chapter, we assume that I is an interval, $\Omega \subset \mathbb{R}^{n+1}$ be an open set with points (t, x), $t \in \mathbb{R}$, $x \in \mathbb{R}^n$. We denote $U(x_0, \Delta)$ the ball in $\mathbb{R}^n$ centered in x_0 with radius Δ. By $Q(t_0, x_0)$ or $Q(t_0, x_0, \delta, \Delta)$ we denote a cylinder $U(t_0, \delta) \times U(x_0, \Delta) \subset \mathbb{R}^n$. The graph of a function $x : I \to \mathbb{R}^n$ is $\operatorname{graph} x := \{(t, x(t)) : \ t \in I\} \subset \mathbb{R}^{n+1}$. In this chapter, we consider a diferential equation

$$x' = f(t, x) \tag{DE}$$

Definition (AC function)**.** *A function $x : I \to \mathbb{R}^n$ is called* absolutely continuous *($x \in AC(I)$) if for every $\varepsilon > 0$ there exists $\delta > 0$ satisfying: for any finite sequence of pairwise disjoint intervals (a_i, b_i), $i = 1, \dots n$ it holds that*

$$\sum_{i=1}^{n} |b_i - a_i| < \delta \quad \Rightarrow \quad \sum_{i=1}^{n} |f(b_i) - f(a_i)| < \varepsilon.$$

We say that x is locally absolutely continuous on I *($x \in AC_{loc}(I)$) if $x \in AC(J)$ for every compact interval $J \subset I$.*

Proposition 2.1. *Let $x \in AC(I)$, then $x'(t)$ exists for almost every $t \in I$, $x' \in L^1_{loc}(I)$ and $x(t) - x(s) = \int_s^t x'(r)dr$ for every s, $t \in I$.*

Proposition 2.2. *Let $h \in L^1(I)$, $c \in I$ and define $x(t) := \int_c^t h(r)dr$. Then $x \in AC(I)$ and $x' = h$ almost everywhere on I.*

Definition (Carathéodory conditions)**.** *We say that $f : \Omega \to \mathbb{R}^n$ satisfies* Carathéodory conditions *($f \in CAR(\Omega)$) if for every $(t_0, x_0) \in \Omega$ there exist a cylinder $Q(t_0, x_0, \delta, \Delta)$ and a function $m \in U(t_0, \delta)$ such that*

- *(i) $f(\cdot, x)$ is measurable on $U(t_0, \delta)$ for every $x \in U(x_0, \Delta)$*
- *(ii) $f(t, \cdot)$ is continuous on $U(x_0, \Delta)$ for a.e. $t \in U(t_0, \delta)$*
- *(iii) $|f(t, x)| \le m(t)$ for a.e. $t \in U(t_0, \delta)$ for every $x \in U(x_0, \Delta)$.*

Definition (AC solution)**.** *Let $f \in CAR(\Omega)$. We say that $x : I \to \mathbb{R}^n$ is* an absolutely continuous solution *to $x' = f(t, x)$ if $x \in AC(I)$,* $\operatorname{graph} x \subset \Omega$ *and $x'(t) = f(t, x(t))$ for a.e. $t \in I$.*

Lemma 2.3. *Let $f \in CAR(\Omega)$, $x : I \to \mathbb{R}^n$ continuous and* $\operatorname{graph} x \subset \Omega$*. Then $f(\cdot, x(\cdot)) \in L^1_{loc}(I)$.*

Lemma 2.4. *Let $f \in CAR(\Omega)$, $x : I \to \mathbb{R}^n$ continuous and* $\operatorname{graph} x \subset \Omega$*. Then x is an AC solution to* (DE) *if and only if for all s, $t \in I$ it holds that*

$$x(t) - x(s) = \int_s^t f(r, x(r))dr.$$

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Theorem 2.5 (Generalized Banach Contraction Theorem)**.** *Let Λ, X be metric spaces, X nonempty and complete. Let $\Phi : \Lambda \times X \to X$ is continuous w.r.t. λ for each fixed $x \in X$. Let there exists $\kappa \in (0,1)$ such that*

$$\|\Phi(\lambda, x) - \Phi(\lambda, y)\|_X \le \kappa \|x - y\|_X \qquad \forall\ \lambda \in \Lambda,\ x, y \in X.$$

Then

(i) for every $\lambda \in \Lambda$ there exists a unique $x(\lambda)$ such that $\Phi(\lambda, x(\lambda)) = x(\lambda)$.

(ii) the mapping $\lambda \mapsto x(\lambda)$ is continuous.

(iii) $\|y - x(\lambda)\| \le (1-\kappa)^{-1} \|y - \Phi(\lambda, y)\|$ for all $\lambda \in \Lambda$, $y \in X$.

Theorem 2.6 (Generalized Picard Theorem)**.** *Let $I = [0, T]$ be a bounded interval and $f \in CAR(I \times \mathbb{R}^n)$. Let us assume that there exists $l \in L^1(I)$ such that*

$$|f(t, x, p) - f(t, y, p)| \le l(t)|x - y| \quad \text{for a.e. } t \in I \text{ for all } x,\ y \in \mathbb{R}^n.$$

Then for every $x_0 \in \mathbb{R}^n$ there exists a unique AC solution $x \in AC(I)$ of (DE) *with $x(0) = x_0$ and the solution depends continuously on the initial value in the following sense. If $x_{0n} \to x_0$, then $x_n \rightrightarrows x$ uniformly on I (where x, resp. x_n are the solutions corresponding to x_0, resp. x_{0n}).*

6th Lecture

3 Bifurcations

3.1 Basic properties

In this chapter we study autonomous differential equations with a parameter $\mu \in \mathbb{R}$.

$$\dot{x} = f(x, \mu) \tag{AR$_\mu$}$$

We assume $\Omega \subset \mathbb{R}^n$ to be an open set, $f \in C^1(\Omega \times \mathbb{R})$ or more smooth.

Definition (bifurcation)**.** *We say that $(x_0, \mu) \in \Omega \times \mathbb{R}$ is a point of bifurcation if in any neighborhood of μ there exist μ_1, μ_2 such that the dynamical systems $\dot{x} = f(x, \mu_1)$ and $\dot{x} = f(x, \mu_2)$ are not topologically conjugate on any neighborhoods of x_0. We say that $\mu \in \mathbb{R}$ is a point of bifurcation if in any neighborhood of μ there exist μ_1, μ_2 such that the dynamical systems $\dot{x} = f(x, \mu_1)$ and $\dot{x} = f(x, \mu_2)$ are not topologically conjugate.*

Proposition 3.1. *If* $f(x_0, \mu_0) \neq 0$*, then* (x_0, μ_0) *is not a point of bifurcation.*

Theorem 3.2. *If* x_0 *is a hyperbolic equilibrium for* $\dot{x} = f(x, \mu_0)$*, then* (x_0, μ_0) *is not a point of bifurcation. In particular, for every* μ *close enough to* μ_0 *the system* $\dot{x} = f(x, \mu)$ *has a unique hyperbolic equilibrium* x_μ *near* x_0 *and the dimensions of stable and unstable manifolds do not depend on* μ*.*

Corollary 3.3. *If* (x_0, μ_0) *is a point of bifurcation, then* x_0 *is a nonhyperbolic equilibrium.*

3.2 Bifurcations on $\mathbb{R}$

Theorem 3.4. *Let* $f \in C^2(\Omega \times \mathbb{R})$*,* $0 \in \Omega$*,* $f(0,0) = 0$*,* $f_x(0,0) = 0$*,* $f_\mu(0,0) \neq 0$ *and* $f_{xx}(0,0) \neq 0$*. Then* $(0,0)$ *is a point of bifurcation. In particular, it is a saddle-node bifurcation, i.e., there are no equilibria for* $\mu < 0$ *and two equilibria for* $\mu > 0$ *in a neighborhood of* 0 *or vice versa.*

Lemma 3.5. *Let* $h \in C^k$*,* $k \geq 2$ *on a neighborhood of* $(0,0)$ *and* $h(0,\lambda) = 0$ *on a neighborhood of 0. Then there exists* $H \in C^{k-1}$ *on a neighborhood of* $(0,0)$ *such that* $h(x,\lambda) = xH(x,\lambda)$ *and, moreover, it holds that* $H(0,\lambda) = h_x(0,\lambda)$*,* $H_x(0,0) = \frac{1}{2}h_{xx}(0,0)$*,* $H_\lambda(0,0) = h_{x\lambda}(0,0)$*, and (if* $k \geq 3$*)* $H_{xx}(0,0) = \frac{1}{3}h_{xxx}(0,0)$*.*

Theorem 3.6. *Let* $f \in C^2(\Omega \times \mathbb{R})$*,* $0 \in \Omega$*,* $f(0,\mu) = 0$ *for all* $\mu \in \mathbb{R}$*,* $f_x(0,0) = 0$*,* $f_{\mu,x}(0,0) \neq 0$ *and* $f_{xx}(0,0) \neq 0$*. Then* $(0,0)$ *is a point of bifurcation. In particular, it is a transcritical bifurcation, i.e., for every* $\mu \in (-\delta,\delta) \setminus \{0\}$ *there exist exactly two equilibria in* $(-\varepsilon,\varepsilon)$*:* $x_0 = 0$ *and* $x_1 \neq 0$*. Moreover,* x_0 *is stable for* $\mu < 0$ *and unstable for* $\mu > 0$ *or vice versa.*

Theorem 3.7. *Let* $f \in C^3(\Omega \times \mathbb{R})$*,* $0 \in \Omega$*,* $f(0,\mu) = 0$ *for all* $\mu \in \mathbb{R}$*,* $f_x(0,0) = 0$*,* $f_{\mu,x}(0,0) \neq 0$ *and* $f_{xx}(0,0) = 0$*,* $f_{xxx}(0,0) \neq 0$*. Then* $(0,0)$ *is a point of bifurcation. In particular, it is a pitchfork bifurcation, i.e., for* $\mu < 0$ *there is a unique equilibrium* $x_0 = 0$ *in a neighborhood of zero and for* $\mu > 0$ *there are exactly three equilibria* $x_1 < x_0 = 0 < x_2$ *in a neighborhood of* 0 *or vice versa. Moreover,* x_0 *is stable for* $\mu < 0$ *and unstable for* $\mu > 0$ *or vice versa.*

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3.3 Hopf bifurcation in $\mathbb{R}^2$

We consider the following system in a neighborhood of $(0,0,0)$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A_\mu \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f(x,y,\mu) \\ g(x,y,\mu) \end{pmatrix}, \tag{1}$$

where A_μ is a 2×2 matrix dependent on a parameter μ and f, g contain higher order terms, i.e., $f = g = 0$, $\nabla_{xy} f = \nabla_{xy} g = 0$ in $(0,0,\mu)$.

Theorem 3.8 (Hopf)**.** *Let* $\sigma(A_\mu) = \{\alpha(\mu) \pm i\omega(\mu)\}$*, where* $\alpha,\ \omega \in C^2$ *on a neighborhood of 0 and it holds that* $\alpha(0) = 0,\ \alpha'(0) \neq 0,\ \omega(0) \neq 0$*. Then there exist* $\delta,\ \Delta > 0$ *and a function* $\varphi \in C^1((0,\delta),(-\Delta,\Delta))$ *such that for every* $a \in (0,\delta)$ *there exists a nontrivial periodic solution to* (1) *with* $\mu = \varphi(a)$ *going through the point* $(x,y) = (a,0)$*.*

Theorem 3.9 (Hopf 2)**.** *Let the assumptions of Theorem 3.8 hold and moreover*

$$A_0 = \begin{pmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{pmatrix}.$$

Then the system is near $(0,0,0)$ *topologically conjugate to*

$$r' = d\mu r + ar^3, \qquad \varphi' = 1,$$

where $d = \alpha'(0)$ *and* $16a$ *is equal to*

$$\left(f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy} + \frac{1}{\omega_0}[f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - f_{xx}g_{xx} + f_{yy}g_{yy}]\right)(0,0,0).$$

8th Lecture

4 Center manifold

For the linear equation

$$X' = AX$$

with a matrix $A \in \mathbb{R}^{n\times n}$ we have stable, unstable and center subspaces defined as

$$\begin{aligned} V_s &:= \{x \in \mathbb{R}^n :\ \exists\, C,\beta > 0\ \forall t \geq 0\ \|e^{tA}x\| \leq Ce^{-\beta t}\}, \\ V_u &:= \{x \in \mathbb{R}^n :\ \exists\, C,\beta > 0\ \forall t \leq 0\ \|e^{tA}x\| \leq Ce^{\beta t}\}, \\ V_c &:= \{x \in \mathbb{R}^n :\ \exists\, C > 0, n \in \mathbb{N} \forall t \in \mathbb{R}\ \|e^{tA}x\| \leq C(1+|x|)^n\}. \end{aligned}$$

It holds that $\mathbb{R}^n = V_s \oplus V_u \oplus V_c$.
Consider a nonlinear equation

$$X' = F(X) \tag{2}$$

with $F \in C^1(\mathbb{R}^N,\mathbb{R}^N)$ and $F(0) = 0$.

Definition (Stable, unstable manifold)**.** *Let* φ *be the solving function to* (2)*. We define the stable manifold* $\tilde{V}_s$ *and unstable manifold* $\tilde{V}_u$ *in* $0 \in \mathbb{R}^N$ *by*

$$\begin{aligned} V_s &:= \{x \in \mathbb{R}^N :\ \exists\, C,\beta > 0\ \forall t \geq 0\ \|\varphi(t,x)\| \leq Ce^{-\beta t}\}, \\ V_u &:= \{x \in \mathbb{R}^N :\ \exists\, C,\beta > 0\ \forall t \leq 0\ \|\varphi(t,x)\| \leq Ce^{\beta t}\}, \end{aligned}$$

Definition (Center manifold)**.** *Let* V_c *be the center subspace of* $X' = \nabla F(0)X$*. A center manifold* $\tilde{V}_c$ *for* (2) *in* $0 \in \mathbb{R}^N$ *is any invariant manifold, that is tangent to* V_c *in* 0 *and has the same dimension as* V_c*.*

4.1 Existence of center manifold

General assumptions. We consider a system of equations

$$\begin{aligned} x' &= Ax + f(x,y), \\ y' &= By + g(x,y), \end{aligned} \tag{S}$$

such that $A \in \mathbb{R}^{n\times n}$, $x^T Ax \geq -\varepsilon|x|^2$, $B \in \mathbb{R}^{m\times m}$, $y^t By \leq -\beta|y|^2$, $\|e^{tB}\| \leq c_0 e^{-\beta t}$ for some $\beta > \varepsilon > 0$, $c_0 > 0$ and all $t \geq 0$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$. Functions f, g are such that $f(0,0) = g(0,0) = 0$, $\nabla f(0,0) = \nabla g(0,0) = 0$, and $|f|$, $|g| < \rho$, $|\nabla f|$, $|\nabla g| < \sigma$ on $\mathbb{R}^{n+m}$ for some σ, $\rho > 0$.
Define

$$\mathcal{X}_{b,L} := \{\Phi \in \mathrm{Lip}(\mathbb{R}^n, \mathbb{R}^m) : \ \|\Phi\| \leq b, \ Lip_\Phi \leq L, \ \Phi(0) = 0\}.$$

Theorem 4.1. *Let ε, β, c_0, L, $b > 0$ are given, $\varepsilon < \beta$. If σ, ρ are small enough, then there exists a unique $\Phi \in \mathcal{X}_{b,L}$ satisfying*

$$(x(t), y(t)) \text{ solves (S) \& } y(0) = \Phi(x(0)) \quad \Rightarrow \quad y(t) = \Phi(x(t)) \ \forall\, t \geq 0. \tag{INV}$$

Moreover, this Φ satisfies $\nabla\Phi(0) = 0$.

Application 1. If $\Re\sigma(A) > 0$, $\Re\sigma(B) < 0$, then $\operatorname{graph}\Phi$ is the unstable manifold.
Application 2. If $\Re\sigma(\tilde{A}) < 0$, $\Re\sigma(\tilde{B}) > 0$ and we apply Theorem 4.1 with $A = -\tilde{B}$ and $B = -\tilde{A}$, then $\operatorname{graph}\Phi$ is the stable manifold for the system with $\tilde{A}$, $\tilde{B}$.
Application 3. If $\Re\sigma(A) = 0$, $\Re\sigma(B) < 0$, then $\operatorname{graph}\Phi$ is a center manifold.
Let us consider so called reduced equation

$$p' = Ap + f(p, \Phi(p)). \tag{RE}$$

Lemma 4.2. *Let $\Phi \in \mathcal{X}_{b,L}$. Then (INV) is equivalent to*

$$p \text{ solves (RE)} \quad \Rightarrow \quad (p, \Phi(p)) \text{ solves (S)}. \tag{RED}$$

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Lemma 4.3. *Let $\gamma : (-\infty, 0] \to \mathbb{R}^n$ be bounded and continuous. Then there exists a unique solution to $y' = By + \gamma$, which is bounded on $(-\infty, 0]$. Moreover, this solution satisfies $y(0) = \int_{-\infty}^0 e^{-sB}\gamma(s)ds$.*

Lemma 4.4. *Let $\Phi \in \mathcal{X}_{b,L}$. Then (INV) is equivalent to*

$$p \text{ solves (RE) with } p(0) = p_0 \quad \Rightarrow \quad \Phi(p_0) = \int_{-\infty}^0 e^{-sB} g(p(s), \Phi(p(s)))ds. \tag{FPP}$$

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4.2 Tracking property and reduction of stability

In this section, we assume that $\Phi \in \mathcal{X}_{b,L}$ satisfies (INV) and $\mu > L$ is fixed. We denote

$$K = \{X = (x, y) \in \mathbb{R}^{n+m} : \ |y| \leq \mu|x|\}$$
$$V = \{X = (x, y) \in \mathbb{R}^{n+m} : \ |y| \geq \mu|x|\}$$

and

$$K(X_0) = \{X = (x, y) \in \mathbb{R}^{n+m} : \ X - X_0 \in K\}$$
$$V(X_0) = \{X = (x, y) \in \mathbb{R}^{n+m} : \ X - X_0 \in V\}$$

Lemma 4.5. *Let σ be small enough and let X_1, $X_2 : \mathbb{R} \to \mathbb{R}^{n+m}$, $X_1 = (x_1, y_1)$, $X_2 = (x_2, y_2)$ be two solutions of* (S)*.*

- *If $X_1(0) \in K(X_2(0))$, then $X_1(t) \in K(X_2(t))$ for all $t \geq 0$*
- *There exists $\gamma > 0$ such that: If $X_1(t) = V(X_2(t))$ for all $t \in I$, then*

$$|X_1(t) - X_2(t)| \leq e^{-\gamma(t-s)}|X_1(s) - X_2(s)| \quad \text{for all } s,\, t \in I,\ s < t.$$

Theorem 4.6 (Tracking property)**.** *Let σ be small enough. For every solution X of* (S) *there exists a solution p of* (RE) *such that $P = (p, \Phi(p))$ satisfies*

$$|X(t) - P(t)| \leq Ce^{-\gamma t}|X(0) - P(0)| \quad \text{for all } t \geq 0$$

with γ from Lemma 4.5. Moreover, $P(0)$ can be taken small if $X(0)$ is small.

Corollary 4.7 (Reduction of stability)**.** $(0, 0) \in \mathbb{R}^{n+m}$ *is (assymptotically) stable for* (S) *if and only if $0 \in \mathbb{R}^n$ is (assymptotically) stable for* (RE)*.*

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4.3 Approximation of center manifold

Let us denote for $\Psi \in C^1(\mathbb{R}^n, \mathbb{R}^m)$

$$[M\Psi](x) = \nabla\Psi(x)[Ax + f(x, \Psi(x))] - B\Psi(x) - g(x, \Psi(x)).$$

We know that $M\Psi \equiv 0$ if and only if Ψ satisfies (INV).

Theorem 4.8 (Approximation of center manifold)**.** *Let $q > 1$ and let $\Psi \in C^1(\mathbb{R}^n, \mathbb{R}^m)$ satisfies $\Psi(0) = 0$, $\nabla\Psi(0) = 0$ and $[M\Psi](x) = O(|x|^q)$ as $x \to 0$. Then $|\Psi(x) - \Phi(x)| = O(|x|^q)$ as $x \to 0$ for any $\Phi \in \mathcal{X}_{b,L}$ satisfying* (INV)*.*

5 Optimal control theory

5.1 Controllability

Let $\Omega \subset \mathbb{R}^n$ be open, $U \subset \mathbb{R}^m$, $f \in C^1(\Omega \times U, \mathbb{R}^n)$ and $x_0 \in \Omega$. *A controlled ordinary differential equation* is

$$x' = f(x,u), \qquad x(0) = x_0. \tag{CDE}$$

Let $0 < T \leq +\infty$. A set $\mathcal{U} \subset \{u : [0,T] \to U :\ u \text{ measureable}\}$ is called *a set of admissible controls*, any function $u \in \mathcal{U}$ is called *a control* and the solution $x : [0,T] \to \mathbb{R}^n$ of (CDE) with a given control u is called *response of the system.*

A controled linear equation is

$$x' = Ax + Bu, \tag{CLE}$$

where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$.

Notation: $x_0 \underset{u}{\overset{t}{\to}} 0$ means "control u brings x_0 to 0 in time t", i.e. if we insert u into (CDE), then the solution x of (CDE) satisfies $x(t) = 0$.

Definition. *Let $t \in [0,T]$. The set $R(t) = \{x_0 \in \mathbb{R}^n :\ \exists u \in \mathcal{U},\ x_0 \underset{u}{\overset{t}{\to}} 0\}$ is called* the reachable set for time *t.*

Definition. *Kalman controllability matrix for* (CLE) *is* $\mathcal{K}(A|B) = (B, AB, A^2B, \dots, A^{n-1}B) \in \mathbb{R}^{n\times mn}$.

Theorem 5.1. *Consider* (CLE) *with $\mathcal{U} = L^1_{loc}([0,T], \mathbb{R}^m)$. Then $R(t) = \operatorname{Im}\mathcal{K}(A|B)$ for all $t > 0$.*

Corollary 5.2. *The following is equivalent for the system* (CLE) *with $\mathcal{U} = L^1_{loc}([0,T], \mathbb{R}^m)$.*

(i) (CLE) *is globally controllable (i.e. $R(t) = \mathbb{R}^n$) for some/every $t > 0$,*

(ii) (CLE) *is locally controllable (i.e. $0 \in R(t)^o$, where $R(t)^o$ is the interior of $R(t)$) for some/every $t > 0$,*

(iii) $\operatorname{rank}\mathcal{K}(A|B) = n$.

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Theorem 5.3. *Let U be any neighborhood of 0 and $\mathcal{U} = L^1_{loc}([0,T], U)$. Let $0 \in \Omega$, $f(0,0) = 0$, $A = \nabla_x f(0,0)$, and $B = \nabla_u f(0,0)$. If $\operatorname{rank}\mathcal{K}(A|B) = n$, then* (CDE) *is locally controllable for all $t > 0$.*

5.2 Time-optimal control and Bang-bang principle

In this section we consider (CLE) with $U = [-1, 1]^m$, $\mathcal{U} = L^1_{loc}([0, T], U)$.

Proposition 5.4. *The system* (CLE) *is locally controllable if and only if* $\operatorname{rank} \mathcal{K}(A|B) = n$.

Proposition 5.5. *For every $t > 0$, $R(t)$ is closed, convex and symmetric ($x \in R(t) \Rightarrow -x \in R(t)$). If $t_1 < t_2$ then $R(t_1) \subset R(t_2)$.*

Theorem 5.6. *Let* $\operatorname{rank} \mathcal{K}(A|B) = n$ *and* $\Re\lambda \leq 0$ *for all* $\lambda \in \sigma(A)$. *Then* (CLE) *is globally controllable.*

Definition. *An admissible control u is called* a bang-bang control *if $u_i(t) = \pm 1$ for all $t \in [0, T]$ and all $i = 1, 2, \dots, m$.*

Theorem 5.7. *For each $x_0 \in R(t)$ there exists a bang-bang control $\tilde{u}$ such that $x_0 \xrightarrow[\tilde{u}]{t} 0$.*

Theorem 5.8. *For each $x_0 \in \bigcup_{t \geq 0} R(t)$ there exists $\tilde{t} = \min\{t \geq 0 : x_0 \in R(t)\}$ and a bang-bang control $\tilde{u}$ such that $x_0 \xrightarrow[\tilde{u}]{\tilde{t}} 0$.*

5.3 Pontryagin maximum principle

In this section, we are looking for an admissible control u which maximizes the functional

$$P[u] = g(x(T)) + \int_0^T r(x(s), u(s)) ds,$$

where x is the solution to (CDE) (with the control u). Functions $g \in C^1(\mathbb{R}^n)$, $f \in C^1(\mathbb{R}^n \times U)$ and $r \in C(\mathbb{R}^n \times U)$ are given.

Theorem 5.9. *Let $u^* \in \mathcal{U}$ is a point of a local maximum of P and x^* is the corresponding system response. Then there exists a solution $P^* : [0, T] \to \mathbb{R}^n$ to the adjoint equation*

$$P^{*\prime} = -\nabla_x H(x^*, P^*, u^*), \qquad P^*(T) = (\nabla_x g)(x^*(T)) \tag{ADJ}$$

and the maximum principle

$$H(x^*(t), P^*(t), u^*(t)) = \max_{\eta \in U} H(x^*(t), P^*(t), \eta), \tag{MP}$$

holds, where $H(x, P, u) = P \cdot f(x, u) + r(x, u)$.