1. Dynamical Systems

Definition. By *dynamical system* (d.s.) we mean a couple (φ, Ω), where $\Omega \subset \mathbb{R}^n$ is open and $\varphi = \varphi(t,x) : \mathbb{R} \times \Omega \to \Omega$ is a map, satisfying

(i) $\varphi(0,x) = x$ for $\forall x \in \Omega$

(ii) $\varphi(s, \varphi(t,x)) = \varphi(s+t, x)$ for $\forall s,\ t \in \mathbb{R},\ x \in \Omega$

(iii) $(t,x) \mapsto \varphi(t,x)$ is continuous.

While Ω can be any topological space, we will consider mostly open domains in $\mathbb{R}^n$ and smooth $\varphi(t,x)$.

Example. If $\Omega \subset \mathbb{R}^n$ is open and $f = f(x) : \Omega \to \mathbb{R}^n$ of class C^1, then $\varphi(t,x_0) := x(t)$, where $x = x(t)$ is the (unique) maximal solution to

$$x' = f(x), \qquad x(0) = x_0 \tag{1}$$

is a 'local' dynamical system with $\varphi \in C^1$. This is a canonical example in the sense that any smooth d.s. arises as a solution operator to the equation (1).

Definition. Let (φ, Ω) be a dynamical system. A set $M \subset \Omega$ is called

- *positively invariant*, if $\varphi(t,x) \in M$ for $\forall t \geq 0,\ x \in M$
- *negatively invariant*, if $\varphi(t,x) \in M$ for $\forall t \leq 0,\ x \in M$
- *(fully) invariant*, if $\varphi(t,x) \in M$ for $\forall t \in \mathbb{R},\ x \in M$

Given a point $x_0 \in M$ we further define

- *positive (semi-)orbit* $\gamma^+(x_0) = \{\varphi(t,x_0);\ t \geq 0\}$
- *negative (semi-)orbit* $\gamma^-(x_0) = \{\varphi(t,x_0);\ t \leq 0\}$
- *(full) orbit* $\gamma(x_0) = \{\varphi(t,x_0);\ t \in \mathbb{R}\}$

Observe that positive (resp. negative resp. full) orbit is positively (resp. negatively resp. fully) invariant. The set M is positively (resp. negatively resp. fully) invariant, iff for any $x_0 \in M$, the orbit $\gamma^+(x_0)$ (resp. $\gamma^-(x_0)$ resp. $\gamma(x_0)$) is a subset of M.

Definition. Let (φ, Ω) be a dynamical system. We define the *ω-limit set* of a point $x_0 \in \Omega$ as

$$\omega(x_0) = \{y \in \Omega;\ \exists t_n \to +\infty \text{ s.t. } \varphi(t_n, x_0) \to y\}$$

Analogously, we define the *α-limit set* of x_0 as

$$\alpha(x_0) = \{y \in \Omega;\ \exists t_n \to -\infty \text{ s.t. } \varphi(t_n, x_0) \to y\}$$

Lemma 1. *Let (φ, Ω) be a dynamical system and $x_0 \in \Omega$. Then*

$$\omega(x_0) = \bigcap_{\tau > 0} \overline{\gamma^+(\varphi(\tau, x_0))}.$$

Remark. Recall that the set M is called *connected*, provided there *do not exist* open, disjoint sets $\mathcal{G}$, $\mathcal{H}$ such that $M \subset \mathcal{G} \cup \mathcal{H}$, while $M \cap \mathcal{G} \neq \emptyset$, $M \cap \mathcal{H} \neq \emptyset$.

Furthermore, any interval $I \subset \mathbb{R}$ is connected (in fact a subset of $\mathbb{R}$ is connected iff it is an interval), and a continuous image of a connected set is again connected.

Theorem 2 (Properties of $\omega(x_0)$.)**.** *Let (φ, Ω) be a dynamical system. Then*

1. *$\omega(x_0)$ is closed, fully invariant*

2. If $\gamma^+(x_0)$ relatively compact in Ω, then $\omega(x_0)$ is non-empty, compact, and connected.

Theorem 3. *Let (φ, Ω) be a dynamical system, let $K \subset \Omega$ be compact. Then*

$$\operatorname{dist}(\varphi(t, x_0), K) \to 0 \qquad \text{for } t \to +\infty\,, \tag{*}$$

if and only if $\emptyset \neq \omega(x_0) \subset K$.

In particular, $\omega(x_0) = \{z\}$ iff $\varphi(t, x_0) \to z$ for $t \to +\infty$.

Definition. Dynamical systems (φ, Ω) and (ψ, Θ) are called *topologically conjugate*, if there exists a homeomorphism $h : \Omega \to \Theta$ such that $h(\varphi(t, x)) = \psi(t, h(x))$ for all $t \in \mathbb{R}$, $x \in \Omega$.

Remark. Equivalently, for all t

$$\varphi(t, \cdot) = h_{-1} \circ \psi(t, \cdot) \circ h \quad \text{ in } \Omega.$$

Remark. Topological conjugacy preserves the key properties of dynamical systems: stationary points and their stability, periodic orbits, ω-limit sets, ...

Theorem 4 (Rectification lemma)**.** *Let $f(x)$ be C^1 in a neighborhood of $x_0 \in \mathbb{R}^n$, let $f(x_0) \neq 0$. Then there exist*

- *a neighborhood $\mathcal{V}$ of x_0,*
- *a neighborhood $\mathcal{W}$ of $0 \in \mathbb{R}^n$ and*
- *a diffeomorphism $g : \mathcal{V} \to \mathcal{W}$*

such that $x(t)$ is a solution to (1) in $\mathcal{V}$ iff $y(t) = g(x(t))$ is a solution to

$$y' = (1, 0, 0, \ldots, 0)^T \tag{2}$$

in $\mathcal{W}$.

Remark. In terms of the previous definition the Rectification lemma says: d.s. given by (1) and (2) are topologically conjugate (in fact C^1-conjugate) on respective neighborhoods.

Remark. Rectification lemma says that close to non-stationary points there is no interesting dynamics. The following (and considerably more difficult) theorem implies that close to stationary hyperbolic points, there is no nonlinear dynamics.

Recall that a stationary point x_0 to equation (1) is called *hyperbolic*, if $\operatorname{Re}\lambda \neq 0$ for any λ from the spectrum of $A = \nabla f(x_0)$.

Theorem 5 (Hartman-Grobman.)**.** *Let $f(x)$ be C^1 on some neighborhood of x_0, where x_0 is a hyperbolic stationary point to (1). Let $A = \nabla f(x_0)$. Then there exist a neighborhood $\mathcal{V}$ of x_0 and a neighborhood $\mathcal{W}$ of $0 \in \mathbb{R}^n$ such that the d.s. given by (1) and $y' = Ay$ are topologically conjugate on respective neighborhoods.*

La Salle's invariance principle

Recall that given a C^1 function $V : \Omega \to \mathbb{R}$ we define the *orbital derivative* – w.r.t. solutions of (1) – as

$$\dot{V}_f(x) = \nabla V(x) \cdot f(x) = \sum_{j=1}^{n} \frac{\partial V}{\partial x_j}(x) f_j(x)$$

By chain rule for any $x = x(t)$ a solution of (1) in Ω one has

$$\frac{d}{dt} V(x(t)) = \dot{V}_f(x(t)).$$

Example. Consider the mathematical pendulum with friction $x'' + q(x') + \sin x = 0$. Here $x = x(t)$ is the displacement angle, and $q = q(y)$ friction, depending on the velocity $y = x'$. It is natural to assume $q(0) = 0$ and $q(y)y > 0$ for $y \neq 0$. In such a case the equilibrium $(x, y) = (0, 0)$ is stable, using the Lyapunov function $V = y^2/2 + 1 - \cos x$.

But is it even asymptotically stable? If $q'(0) > 0$, this follows by the linearization argument. But the more delicate (in fact, non-hyperbolic) case when $q'(0) = 0$ requires a more subtle argument, which is contained in the following abstract theorem.

Theorem 6 (La Salle)**.** *Let (φ, Ω) be the d.s. given by (1). Let $V : \Omega \to \mathbb{R}$ be a C^1 function bounded from below, and let $\ell \in \mathbb{R}$ be such that the set $\Omega_\ell = \{x \in \Omega;\ V(x) < \ell\}$ is bounded. Assume finally that $\dot{V}_f(x) \leq 0$ in Ω_ℓ.*

Denote

$$R = \{x \in \Omega_\ell;\ \dot{V}_f = 0\}$$
$$M = \{y \in R;\ \gamma(y) \subset R\}$$

Then for any $x_0 \in \Omega_\ell$ one has $\emptyset \neq \omega(x_0) \subset M$.

Remark. M is the largest fully invariant subset of R. In a typical application, M reduces to a single point which (in view of Theorem 13.2) is thus asymptotically stable (in fact it attracts all of Ω_ℓ).

Poincarè-Bendixson theory

- existence and non-existence of periodic solutions in $\mathbb{R}^2$.
- it is essential that we are in two dimensions only.

Standing assumptions. Throughout this chapter,

- $\Omega \subset \mathbb{R}^2$ is a domain (i.e. open, connected set),
- $f(x) : \Omega \to \mathbb{R}^2$ is C^1 and
- $\varphi = \varphi(t, x)$ is the d.s. given by (1).

Theorem 7 (Poincarè-Bendixson.)**.** *Let $p \in \Omega$ be such that $\gamma^+(p)$ is relatively compact in Ω, let furthermore $\omega(p)$ contains no stationary point. Then $\omega(p) = \Gamma$, where Γ is a (non-trivial) periodic orbit.*

Reminder

We say that γ is a *curve*, if $\gamma = \psi([a, b])$, where ψ is injective, continuous. It is a *Jordan curve*, provided that ψ is continuous, injective on $[a, b)$ and $\psi(a) = \psi(b)$. Finally, γ is a *(line) segment*, provided that ψ can be taken affine, i.e. $\psi(t) = at + b$ for some vectors $a \neq 0$ and b.

Remark. Orbit (periodic orbit) is a curve (Jordan curve).

Jordan theorem. If $\gamma \subset \mathbb{R}^2$ is a Jordan curve, then $\mathbb{R}^2 \setminus \gamma$ consists precisely of two domains, of which one is bounded and simply connected ("the interior") and the other is unbounded ("the exterior").

Definition. An open segment Σ is called *transversal*, provided that $f(p) \cdot n \neq 0$ for any $p \in \Sigma$, where n is the normal vector to Σ.

Geometrically: solutions of (1) traverse Σ with a non-zero speed (and in particular, in the same direction) at all points. Clearly every non-stationary point lies on some transversal.

Lemma 8. *Let $\Sigma \subset \Omega$ be transversal, $y \in \Sigma$. Then there exist two neighborhoods $\mathcal{U} \supset \tilde{\mathcal{U}}$ of y and $\Delta > 0$ such that for any $x_0 \in \tilde{\mathcal{U}}$ we have* *(i)* $\varphi(t, x_0) \in \mathcal{U}$ *for all* $|t| < \Delta$ *and*

(ii) there is a unique $|t_0| < \Delta/2$ *such that* $\varphi(t_0, x_0) \in \Sigma \cap \tilde{\mathcal{U}}$

Lemma 9. *Let* $\Sigma \subset \Omega$ *be a transversal, let* $p \in \Omega$*. Then the intersections of* $\gamma^+(p)$ *with* Σ *form a monotone sequence. More precisely: if* $t_1 < t_2 < t_3$ *are such that* $\varphi(t_i, p) \in \Sigma$*,* $i = 1, 2, 3$*, then either (i)* $\varphi(t_1, p) = \varphi(t_2, p) = \varphi(t_3, p)$*, or (ii) the point* $\varphi(t_2, p)$ *lies strictly between* $\varphi(t_1, p)$ *and* $\varphi(t_3, p)$*.*

Lemma 10. *Let* $\Sigma \subset \Omega$ *be a transversal, let* $p \in \Omega$*. Then* $\omega(p) \cap \Sigma$ *consists of at most one point.*

Corollary. *Let* $\Sigma \subset \Omega$ *be a transversal, let* $\Gamma \subset \Omega$ *be a periodic orbit. Then* $\Gamma \cap \Sigma$ *consists of at most one point.*

Theorem 11 (Bendixson-Dulac)**.** *Let* $\Omega \subset \mathbb{R}^2$ *be simply connected and let there exist a* C^1 *function* $B(x) : \Omega \to \mathbb{R}$ *such that* $\operatorname{div}(Bf)(x) > 0$ *a.e. in* Ω*. Then* (1) *has no (non-trivial) periodic orbit in* Ω*.*

2. Carathéodory theory

In this chapter I, J denote arbitrary intervals.

Definition. Function $x : I \to \mathbb{R}^n$ is called *absolutely continuous*, denoted $x \in AC(I)$, provided that for any $\varepsilon > 0$ there is $\delta > 0$ such that for arbitrary *disjoint* intervals $(a_i, b_i) \subset I$ one has

$$\sum_i |a_i - b_i| < \delta \qquad \Longrightarrow \qquad \sum_i |x(a_i) - x(b_i)| < \varepsilon$$

Function x is called *locally absolutely continuous*, denoted $x \in AC_{\text{loc}}(I)$, provided that $x \in AC(J)$ for any compact $J \subset I$.

Proposition 12. *Let $x \in AC(I)$. Then a finite derivative x' exists a.e. in I, $x' \in L^1(I)$ and $x(t_2) - x(t_1) = \int_{t_1}^{t_2} x'(s)\,ds$ for all t_1, $t_2 \in I$.*

Proposition 13. *Let $h \in L^1(I)$, and $t_0 \in I$ be fixed. Then the function $x(t) := \int_{t_0}^{t} h(s)\,ds$ belongs to $AC(I)$; furthermore $x' = h$ a.e. in I.*

Notation.

- $\Omega \subset \mathbb{R}^{n+1}$ is an open set of points $(t, x) \in \mathbb{R} \times \mathbb{R}^n$,
- $U(x_0, \delta)$ is an open ball in $\mathbb{R}^n$,
- $Q(t_0, x_0; \delta, \Delta)$ is a cylinder $U(t_0, \delta) \times U(x_0, \Delta)$ in $\mathbb{R}^{n+1}$.
- for $x : I \to \mathbb{R}^n$ we denote $\operatorname{graph} x = \{(t, x(t)); t \in I\}$.

Definition. We say that the function $f(t, x) : \Omega \to \mathbb{R}^n$ satisfies *Carathéodory conditions*, writing $f \in \mathcal{C}ar(\Omega)$, if for all $(t_0, x_0) \in \Omega$ there exists a cylinder $Q(t_0, x_0; \delta, \Delta) \subset \Omega$ and a function $m \in L^1(U(t_0, \delta))$ such that

(i) for any $x \in U(x_0, \Delta)$ the function $f(\cdot, x)$ is measurable in $U(t_0, \delta)$,

(ii) for almost every $t \in U(t_0, \delta)$ the function $f(t, \cdot)$ is continuous in $U(x_0, \Delta)$, and

(iii) $|f(t, x)| \le m(t)$ for a.e. t for all x in $Q(t_0, x_0; \delta, \Delta)$.

The phrase "for almost every t for all …" means: there is a zero measure set N such that for all $t \in N$ and all …

Definition. Let $f \in \mathcal{C}ar(\Omega)$. Function $x : I \to \mathbb{R}^n$ is called a *Carathéodory solution* to

$$x' = f(t, x) \tag{1}$$

in Ω, provided that $\operatorname{graph} x \subset \Omega$, $x \in AC_{\text{loc}}(I)$ and one has $x'(t) = f(t, x(t))$ for a.e. $t \in I$.

Lemma 14. *Let $f \in \mathcal{C}ar(\Omega)$, $x : I \to \mathbb{R}^n$ be continuous and* $\operatorname{graph} x \subset \Omega$. *Then the function $t \mapsto f(t, x(t))$ belongs to $L^1_{\text{loc}}(I)$.*

Lemma 15. *Let $f \in \mathcal{C}ar(\Omega)$, $x : I \to \mathbb{R}^n$ be a continuous function, and* $\operatorname{graph} x \subset \Omega$. *Then x is a Carathéodory solution to* (1) *if and only if*

$$x(t_2) - x(t_1) = \int_{t_1}^{t_2} f(s, x(s))\,ds \tag{2}$$

for all t_1, $t_2 \in I$.

Remark. Based on the above integral formulation, one can develope the theory of AC (Carathéodory) solutions, in an analogy to the C^1 (classical) theory: local existence and uniqueness, maximal solutions, continuous dependence on the initial condition … We will only prove a certain variant of (a generalized) Picard's theorem, which will include even global existence of solutions together with a continuous dependence on the (initial) data.

Theorem 16 (Generalized Banach contraction theorem)**.** *Let Λ, X be metric spaces, with X being complete and non-empty. Let $\Phi : \Lambda \times X \to X$ be continuous w.r.t. $\lambda \in \Lambda$ for any fixed $x \in X$. Let (the key assumption of uniform contraction) there exist $\kappa \in (0,1)$ such that*

$$\|\Phi(\lambda, x) - \Phi(\lambda, y)\|_X \leq \kappa \|x - y\|_X \quad \forall\, \lambda \in \Lambda,\ x,\ y \in X.$$

Then

(i) for any $\lambda \in \Lambda$ there is a unique $x(\lambda) \in X$ such that $\Phi(\lambda, x(\lambda)) = x(\lambda)$,

(ii) the map $\lambda \mapsto x(\lambda)$ is continuous, and

(iii) $\|y - x(\lambda)\|_X \leq (1-\kappa)^{-1} \|y - \Phi(\lambda, y)\|_X$ for $\forall \lambda \in \Lambda$, $y \in X$.

Theorem 17 (Generalized Picard theorem)**.** *Let $I = [0, T]$ be an interval, Π a metric space and $f : I \times \mathbb{R}^n \times \Pi \to \mathbb{R}^n$ satisfy the following:*

1. *$f(\cdot, \cdot, p) \in \mathcal{C}ar(I \times \mathbb{R}^n)$ for all $p \in \Pi$ fixed*
2. *there exists $\ell \in L^1(I)$ such that $|f(t,x,p) - f(t,y,p)| \leq \ell(t)|x - y|$ for a.e. $t \in I$ for all x, $y \in \mathbb{R}^n$, $p \in \Pi$*
3. *the map $p \mapsto \int_0^t f(s, x(s), p)\, ds$ is continuous from Π to $C(I)$, for arbitrary fixed $t \in I$ and $x \in C(I)$*

Then

(i) for any $x_0 \in \mathbb{R}^n$ and $p_0 \in \Pi$ there exists a unique Caratheodory solution $x \in AC(I)$ of $x' = f(t, x, p_0)$, $x(0) = x_0$ and

(ii) this solution depends continuously on x_0 and p_0

Remark. By continuous dependence we mean: If $x_{0n} \to x_0$ and $p_{0n} \to p_0$ then $x_n \rightrightarrows x$ in I, where x_n resp. x are the solutions corresponding to x_{0n}, p_{0n} and x_0, p_0, respectively.

Remark. Second assumption of the above theorem can be called a *generalized Lipschitz continuity* of $f(t, x, p)$ w.r.t. x.

Example. Consider linear equation

$$x' = A(t)x + b(t) \tag{3}$$

where $A(t) : [0, T] \to \mathbb{R}^{n \times n}$, $b(t) : [0, T] \to \mathbb{R}^n$ are L^1 functions. Clearly the assumptions of Theorem 17 hold (take $\ell(t) = \|A(t)\|$). The right-hand side $b(t)$ is considered as a parameter in $\Pi = L^1(0, T)$. We obtain existence of a global unique solution $x \in AC(I)$ which depends continuously on x_0 and $b(\cdot)$.

Bifurcation theory

Definition (Bifurcation)**.** A point (x_0, μ_0) is called *regular point* of the equation

$$x' = f(x, \mu) \tag{4}$$

provided there exist $\delta > 0$ and $\mathcal{U}$ a neighborhood of x_0 such that for all $|\mu - \mu_0| < \delta$ are the dynamical systems of (4) topologically conjugate in $\mathcal{U}$.

A point (x_0, μ_0) is called a *point of bifurcation* if it is not a regular point.

Remark. Here $\mu \in \mathbb{R}$ is called a bifurcation parameter. Typically "bifurcation theorem" describes the behavior near the bifurcation point in a more precise way (e.g. the curve(s) of stationary points and their stability).

Remark.

- A non-stationary point of (4) is always regular (by Theorem 4).
- A hyperbolic stationary point is also regular (Hale and Kocak: Dynamics and Bifurcations, Thm 8.15, in 2D; Arnold: Ordinary differential equations, Â§22, general case).

Hence, a necessary condition for bifurcation is presence of a non-hyperbolic stationary point, i.e. $f(x_0) = 0$ and $\nabla f(x_0)$ has an eigenvalue with zero real part.

Bifurcations in 1D

Lemma 18 (Division lemma)**.** *Let $U \subset \mathbb{R}^2$ be a neighborhood of $(0,0)$ and $h \in C^k(U)$, $k \geq 1$. Let $h(0,\mu) = 0$ for $(0,\mu) \in U$. Then there exists $V \subset \mathbb{R}^2$ a neighborhood of $(0,0)$ and $H \in C^{k-1}(V)$ such that $h(x,\mu) = xH(x,\mu)$ on V. Moreover, one has*

- $H(0,0) = \partial_x h(0,0)$,
- $\partial_x H(0,0) = \frac{1}{2}\partial^2_{xx} h(0,0)$,
- $\partial_\mu H(0,0) = \partial^2_{x\mu} h(0,0)$,
- $\partial^2_{xx} H(0,0) = \frac{1}{3}\partial^3_{xxx} h(0,0)$.

Theorem 19 (Saddle-node in 1d)**.** *Let f be C^2 in a neighborhood of $(0,0) \in \mathbb{R}^2$. Let*

- $f(0,0) = 0$, $\partial_x f(0,0) = 0$,
- $\partial_\mu f(0,0) \neq 0$,
- $\partial^2_{xx} f(0,0) \neq 0$.

Then $(0,0)$ is a point of bifurcation of the equation (4). In particular, there are no equilibria for $\mu < 0$ and two equilibria, one asymptotically stable and one unstable, for $\mu > 0$ in a neighborhood of 0, or vice versa.

Theorem 20 (Transcritical in 1d)**.** *Let f be C^2 in a neighborhood of $(0,0) \in \mathbb{R}^2$. Let*

- $f(0,0) = 0$, $\partial_x f(0,0) = 0$,
- $f(0,\mu) = 0$ *(hence also $\partial_\mu f(0,0) = 0$) for μ close to 0,*
- $\partial^2_{\mu x} f(0,0) \neq 0$,
- $\partial^2_{xx} f(0,0) \neq 0$.

Then $(0,0)$ is a point of bifurcation. In particular, for every $\mu \in (-\delta,\delta) \setminus \{0\}$ there exist exactly two equilibria in $(-\varepsilon,\varepsilon)$: $x_0 = 0$ and $x_1 \neq 0$. Moreover, x_0 is stable for $\mu < 0$ and unstable for $\mu > 0$, or vice versa.

Theorem 21 (Pitchfork in 1d)**.** *Let f be C^2 in a neighborhood of $(0,0) \in \mathbb{R}^2$. Let*

- $f(0,0) = 0$, $\partial_x f(0,0) = 0$,
- $f(0,\mu) = 0$ *(hence also* $\partial_\mu f(0,0) = 0$*) for* μ *close to* 0,
- $\partial^2_{\mu x} f(0,0) \neq 0$,
- $\partial^2_{xx} f(0,0) = 0$,
- $\partial^3_{xxx} f(0,0) \neq 0$.

Then $(0,0)$ is a point of bifurcation. In particular, for $\mu < 0$ there is a unique equilibrium $x_0 = 0$ in a neighborhood of zero and for $\mu > 0$ there are exactly three equilibria $x_1 < x_0 = 0 < x_2$ in a neighborhood of 0 or vice versa. Moreover, x_0 is stable for $\mu < 0$ and unstable for $\mu > 0$ or vice versa.

Hopf bifurcation in 2D

Consider the system in a neighborhood of $(0,0,0)$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A_\mu \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f(x,y,\mu) \\ g(x,y,\mu) \end{pmatrix}, \tag{5}$$

where

- μ is a bifurcation parameter
- A_μ is a 2×2 matrix dependent μ and
- $f, g \in C^3$, $f = g = 0$, $\nabla_{xy} f = \nabla_{xy} g = 0$ in $(0,0,\mu)$.

(f, g contain higher order terms)

Theorem 22 (Hopf)**.** *Let*

$$\sigma(A_\mu) = \{\alpha(\mu) \pm i\omega(\mu)\},$$

where α, $\omega \in C^2$ on a neighborhood of 0 be such that

- $\alpha(0) = 0$,
- $\alpha'(0) \neq 0$,
- $\omega(0) > 0$.

Then there exist δ, $\Delta > 0$ and a function $\phi \in C^1((0,\delta), (-\Delta, \Delta))$ such that for every $a \in (0,\delta)$ there exists a nontrivial periodic solution to (5) with $\mu = \phi(a)$ going through the point $(x,y) = (a,0)$.

Theorem 23 (Hopf — normal form)**.** *Let the assumptions of Theorem 22 hold and moreover*

$$A_0 = \begin{pmatrix} 0 & -\omega(0) \\ \omega(0) & 0 \end{pmatrix}.$$

Then the system is near $(0,0,0)$ topologically conjugate to

$$r' = d\mu r + ar^3, \qquad \phi' = 1,$$

where $d = \alpha'(0)$ and $16a$ is equal to

$$\begin{gathered} f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy} \\ + \tfrac{1}{\omega(0)}[f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - f_{xx}g_{xx} + f_{yy}g_{yy}] \end{gathered}$$

evaluated in $(0,0,0)$.

Invariant manifolds

For the linear equation
$$X' = AX$$
with a matrix $A \in \mathbb{R}^{n\times n}$ we have **stable, unstable and center subspaces** defined as

$$\begin{aligned}
V_s &:= \{x \in \mathbb{R}^n : \ \exists\, C, \beta > 0\ \forall t \ge 0\ \|e^{tA}x\| \le Ce^{-\beta t}\},\\
V_u &:= \{x \in \mathbb{R}^n : \ \exists\, C, \beta > 0\ \forall t \le 0\ \|e^{tA}x\| \le Ce^{\beta t}\},\\
V_c &:= \{x \in \mathbb{R}^n : \ \exists\, C > 0, n \in \mathbb{N} \forall t \in \mathbb{R}\ \|e^{tA}x\| \le C(1+|x|)^n\}.
\end{aligned}$$

Moreover,
$$\mathbb{R}^n = V_s \oplus V_u \oplus V_c.$$

Consider a nonlinear equation
$$X' = F(X) \tag{6}$$
with $F \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ and $F(0) = 0$.

Definition (Stable, unstable manifold)**.** Let ϕ be the solving function to (6). We define the *stable manifold* $\tilde{V}_s$ and *unstable manifold* $\tilde{V}_u$ in $0 \in \mathbb{R}^N$ by

$$\begin{aligned}
V_s &:= \{x \in \mathbb{R}^N : \ \exists\, C, \beta > 0\ \forall t \ge 0\ \|\phi(t,x)\| \le Ce^{-\beta t}\},\\
V_u &:= \{x \in \mathbb{R}^N : \ \exists\, C, \beta > 0\ \forall t \le 0\ \|\phi(t,x)\| \le Ce^{\beta t}\}.
\end{aligned}$$

Definition (Center manifold)**.** Let V_c be the center subspace of $X' = \nabla F(0)X$. A *center manifold* $\tilde{V}_c$ for (6) in $0 \in \mathbb{R}^N$ is any invariant manifold, that is tangent to V_c in 0 and has the same dimension as V_c.

Remark. Stable and unstable manifolds do exist but it is not clear, whether they are manifolds.

The center manifold is a manifold by definition. But it is not clear whether it exists and is unique.

Existence of center manifold

General assumptions. Consider a system of equations

$$\begin{aligned}
x' &= Ax + f(x,y),\\
y' &= By + g(x,y),
\end{aligned} \tag{S}$$

such that

- $A \in \mathbb{R}^{n\times n}$, $x^T Ax \ge -\varepsilon|x|^2$,
- $B \in \mathbb{R}^{m\times m}$, $y^t By \le -\beta|y|^2$, $\|e^{tB}\| \le c_0 e^{-\beta t}$
- for some $\beta > \varepsilon > 0$, $c_0 > 0$ and all $t \ge 0$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$.
- $f(0,0) = g(0,0) = 0$, $\nabla f(0,0) = \nabla g(0,0) = 0$,
- $|f|$, $|g| < \rho$, $|\nabla f|$, $|\nabla g| < \sigma$ on $\mathbb{R}^{n+m}$ for some σ, $\rho > 0$.

Define
$$\mathcal{X}_{b,L} := \{\Phi \in \mathrm{Lip}(\mathbb{R}^n, \mathbb{R}^m) : \ \|\Phi\| \le b,\ Lip_\Phi \le L,\ \Phi(0) = 0\}.$$

Theorem 24. *Let ε, β, c_0, L, $b > 0$ are given, $\varepsilon < \beta$. If σ, ρ are small enough, then there exists a unique $\Phi \in \mathcal{X}_{b,L}$ satisfying*

$$\begin{gathered}
(x(t), y(t)) \textit{ solves } (\mathrm{S}) \quad \& \quad y(0) = \Phi(x(0))\\
\Downarrow\\
y(t) = \Phi(x(t))\ \forall\, t \ge 0.
\end{gathered} \tag{INV}$$

Moreover, this Φ satisfies $\nabla\Phi(0) = 0$.

Application 1.
If $\Re\sigma(A) > 0$, $\Re\sigma(B) < 0$, then graph Φ is the unstable manifold.

Application 2.
If $\Re\sigma(\tilde{A}) < 0$, $\Re\sigma(\tilde{B}) > 0$ and we apply Theorem 24 with $A = -\tilde{B}$ and $B = -\tilde{A}$, then graph Φ is the stable manifold for the system with $\tilde{A}$, $\tilde{B}$.

Application 3.
If $\Re\sigma(A) = 0$, $\Re\sigma(B) < 0$, then graph Φ is a center manifold. Let us consider so called reduced equation

$$p' = Ap + f(p, \Phi(p)). \tag{RE}$$

Lemma 25. *Let $\Phi \in \mathcal{X}_{b,L}$. Then* (INV) *is equivalent to*

$$p \textit{ solves } \text{(RE)} \quad \Rightarrow \quad (p, \Phi(p)) \textit{ solves } \text{(S)}. \tag{RED}$$

Lemma 26. *Let $\gamma : (-\infty, 0] \to \mathbb{R}^n$ be bounded and continuous. Then there exists a unique solution to $y' = By + \gamma$, which is bounded on $(-\infty, 0]$. Moreover, this solution satisfies $y(0) = \int_{-\infty}^{0} e^{-sB}\gamma(s)ds$.*

Lemma 27. *Let $\Phi \in \mathcal{X}_{b,L}$. Then* (INV) *is equivalent to*

$$\begin{gathered} p \textit{ solves } \text{(RE)} \textit{ with } p(0) = p_0 \\ \Downarrow \\ \Phi(p_0) = \int_{-\infty}^{0} e^{-sB} g(p(s), \Phi(p(s)))ds. \end{gathered} \tag{FPP}$$

Tracking property and reduction of stability

In this section, we assume that $\Phi \in \mathcal{X}_{b,L}$ satisfies (INV) and $\mu > L$ is fixed. We denote

$$K = \{(x, y) \in \mathbb{R}^{n+m} : \ |y| \le \mu|x|\},$$
$$V = \{(x, y) \in \mathbb{R}^{n+m} : \ |y| \ge \mu|x|\},$$

and for $X_0 \in \mathbb{R}^{n+m}$

$$K(X_0) = \{X \in \mathbb{R}^{n+m} : \ X - X_0 \in K\},$$
$$V(X_0) = \{X \in \mathbb{R}^{n+m} : \ X - X_0 \in V\}.$$

Lemma 28. *Let σ be small enough and let X_1, $X_2 : \mathbb{R} \to \mathbb{R}^{n+m}$, $X_1 = (x_1, y_1)$, $X_2 = (x_2, y_2)$ be two solutions of* (S)*.*

- *If $X_1(0) \in K(X_2(0))$, then $X_1(t) \in K(X_2(t))$ for all $t \ge 0$.*
- *There exist c, $\gamma > 0$ such that: If $X_1(t) \in V(X_2(t))$ for all $t \in I$, then*

  $$|X_1(t) - X_2(t)| \le ce^{-\gamma(t-s)}|X_1(s) - X_2(s)|$$

  for all s, $t \in I$, $s < t$.

Theorem 29 (Tracking property)**.** *Let σ be small enough. For every solution X of* (S) *there exists a solution p of* (RE) *such that $P = (p, \Phi(p))$ satisfies*

$$|X(t) - P(t)| \le Ce^{-\gamma t}|X(0) - P(0)| \quad \textit{for all } t \ge 0$$

with γ from Lemma 28. Moreover, $P(0)$ can be taken small if $X(0)$ is small.

Corollary 30 (Reduction of stability)**.** *$(0, 0) \in \mathbb{R}^{n+m}$ is (assymptotically) stable for* (S) *if and only if $0 \in \mathbb{R}^n$ is (assymptotically) stable for* (RE)*.*

Approximation of center manifold

Let us denote for $\Psi \in C^1(\mathbb{R}^n, \mathbb{R}^m)$

$$[M\Psi](x) = \nabla\Psi(x)[Ax + f(x, \Psi(x))] - B\Psi(x) - g(x, \Psi(x)).$$

Remark.

$$M\Psi \equiv 0 \qquad \Leftrightarrow \qquad \Psi \text{ satisfies (INV)}$$

Theorem 31 (Approximation of center manifold)**.** *Let the assumptions of Theorem 24 hold. Assume $q > 1$ and let $\Psi \in C^1(\mathbb{R}^n, \mathbb{R}^m)$ satisfies*

- $\Psi(0) = 0$,
- $\nabla\Psi(0) = 0$ *and*
- $[M\Psi](x) = O(|x|^q)$ *as* $x \to 0$.

Then

$$|\Psi(x) - \Phi(x)| = O(|x|^q) \qquad \textit{as } x \to 0$$

for every $\Phi \in \mathcal{X}_{b,L}$ *satisfying* (INV).

Optimal control

Given

- $\Omega \subset \mathbb{R}^n$ open,
- $U \subset \mathbb{R}^m$, usually $m < n$.
- $f \in C^1(\Omega \times U, \mathbb{R}^n)$ and $x_0 \in \Omega$
- $0 < T \leq +\infty$ and $\mathcal{U} \subset \{u : [0,T] \to U : \ u \text{ measurable}\}$

A controlled ordinary differential equation is

$$x' = f(x,u), \qquad x(0) = x_0. \tag{CDE}$$

- Set $\mathcal{U}$ is *set of admissible controls*,
- function $u \in \mathcal{U}$ is *(admissible) control*
- the solution $x : [0,T] \to \mathbb{R}^n$ of (CDE) with a given control u is *response of the system.*

Typical tasks to be addressed:

1. for which x_0, $t > 0$ is there $u(\cdot) \in \mathcal{U}$ such that $x(t) = 0$ (controllability)
2. analogous question, but with a minimal time $t > 0$ (time optimal control)
3. more generally: find $u(\cdot) \in \mathcal{U}$ such that the functional

$$P[u(\cdot)] = g(x(T)) + \int_0^T r(x(s), u(s))\, ds$$

has a maximal value.

Controllability

A *controled linear equation* is

$$x' = Ax + Bu, \tag{CLE}$$

where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$.

Notation: $x_0 \xrightarrow[u]{t} 0$ means "control u brings x_0 to 0 in time t", i.e. if we insert u into (CDE), then the solution x of (CDE) satisfies $x(t) = 0$.

Definition. Let $t \in [0,T]$. The set

$$R(t) = \{x_0 \in \mathbb{R}^n : \ \exists u \in \mathcal{U}, \ x_0 \xrightarrow[u]{t} 0\}$$

is called *the reachable set* for time t.

Definition. *Kalman controllability matrix* for (CLE) is

$$\mathbb{K}(A|B) = (B, AB, A^2B, \dots, A^{n-1}B) \in \mathbb{R}^{n\times mn}.$$

Theorem 32. *Consider* (CLE) *with* $\mathcal{U} = L^1_{loc}([0,T], \mathbb{R}^m)$*. Then*

$$R(t) = \operatorname{Range} \mathbb{K}(A|B)$$

for all $t > 0$.

Corollary 33. *The following is equivalent for the system* (CLE) *with* $\mathcal{U} = L^1_{loc}([0,T], \mathbb{R}^m)$.

(i) (CLE) *is globally controllable (i.e.* $R(t) = \mathbb{R}^n$*) for some/every* $t > 0$,

(ii) (CLE) *is locally controllable (i.e.* $0 \in \operatorname{Int} R(t)$*) for some/every* $t > 0$,

(iii) $\operatorname{rank} \mathbb{K}(A|B) = n$.

Theorem 34. *Let* U *be any neighborhood of* 0 *and* $\mathcal{U} = L^1_{loc}([0,T], U)$*. Let* $0 \in \Omega$, $f(0,0) = 0$, $A = \nabla_x f(0,0)$*, and* $B = \nabla_u f(0,0)$*. If* $\operatorname{rank} \mathbb{K}(A|B) = n$*, then* (CDE) *is locally controllable for all* $t > 0$.

Time-optimal control and Bang-bang principle

In this section we consider (CLE) with

$$U = [-1,1]^m, \qquad \mathcal{U} = L^1_{loc}([0,T],U).$$

Proposition 35. *The system* (CLE) *is locally controllable if and only if* $\operatorname{rank}\mathbb{K}(A|B) = n$.

Proposition 36. *For every $t > 0$, $R(t)$ is closed, convex and symmetric ($x \in R(t) \Rightarrow -x \in R(t)$). If $t_1 < t_2$ then $R(t_1) \subset R(t_2)$.*

Theorem 37. *Let* $\operatorname{rank}\mathbb{K}(A|B) = n$ *and $\Re\lambda \leq 0$ for all $\lambda \in \sigma(A)$. Then* (CLE) *is globally controllable.*

Definition. An admissible control u is called *a bang-bang control* if $u_i(t) = \pm 1$ for all $t \in [0,T]$ and all $i = 1,2,\ldots,m$.

Theorem 38. *For each $x_0 \in R(t)$ there exists a bang-bang control $\tilde{u}$ such that $x_0 \xrightarrow[\tilde{u}]{t} 0$.*

Theorem 39. *For each $x_0 \in \bigcup_{t\geq 0} R(t)$ there exists*

$$\tilde{t} = \min\{t \geq 0 : x_0 \in R(t)\}$$

and a bang-bang control $\tilde{u}$ such that $x_0 \xrightarrow[\tilde{u}]{\tilde{t}} 0$.

Pontryagin maximum principle

In this section, we are looking for an admissible control u which maximizes the functional

$$P[u] = g(x(T)) + \int_0^T r(x(s),u(s))ds,$$

where x is the solution to (CDE) (with the control u). Functions $g \in C^1(\mathbb{R}^n)$, $f \in C^1(\mathbb{R}^n \times U)$ and $r \in C(\mathbb{R}^n \times U)$ are given.

Theorem 40. *Let $u^* \in \mathcal{U}$ be a point of a local maximum of P and x^* is the corresponding system response. Then there exists a solution $P^* : [0,T] \to \mathbb{R}^n$ to the adjoint equation*

$$P^{*\prime} = -\nabla_x H(x^*,P^*,u^*), \quad P^*(T) = (\nabla_x g)(x^*(T)) \tag{ADJ}$$

and the maximum principle

$$H(x^*(t),P^*(t),u^*(t)) = \max_{\eta\in U} H(x^*(t),P^*(t),\eta), \tag{MP}$$

holds, where $H(x,P,u) = P \cdot f(x,u) + r(x,u)$.