# ODE 2
## 21.1.2014, Exam A

**1.** Consider the system

$$\begin{aligned} x' &= 2x^2y - y^2, \\ y' &= -2y - 3x^2y - x^5. \end{aligned}$$

(a) Find an aproximation of a central manifold, which allows to determine the type of stability of the origin. **(6 points)**
(b) Decide (with proof), whether the origin is stable, asymptotically stable or unstable. **(4 points)**

**2.** Consider the system

$$\begin{aligned} x' &= y + xy^2(1 - x^2 - 2y^4), \\ y' &= -x + y(1 - x^2 - 2y^4). \end{aligned}$$

(a) Show that the origin is the only stationary point. **(3 points)**
(b) Show that the system has a periodic solution. **(7 points)**
Hint: In both parts investigate the distance of the solution from the origin.

**3.** Consider the following system with a controll $u$ and a parameter $p \in \mathbb{R}$

$$\begin{aligned} x' &= -x + 2y + z + u, \\ y' &= -y + pz - u, \\ z' &= x - y - 4z \end{aligned}$$

and the sets of admissible controlls $U_1 = L^1([0, +\infty))$, $U_2 = L^1([0, +\infty), [-1, 1])$.
(a) Find the reachable set (depending on $p$) for $U_1$. **(5 points)**
(b) Show global controllability for $U_2$ and $p = 0$. **(5 points)**

# ODE 2

## 3.2.2014, Exam B

**1.** Consider the following system with a controll $u$

$$\begin{aligned} x' &= x + 2\sin y + 3z + 3z\sin z + u, \\ y' &= -e^x y + 2z + u^2, \\ z' &= x + \sin y - u \end{aligned}$$

and the set of admissible controlls $U = \{u : [0, +\infty) \to [-5, 5], u \text{ measureable}\}$.
(a) Show that the system is locally controllable. **(6 points)**
(b) Show that the system is not controllable for large $x(0)$, $z(0)$. **(4 points)**

**2.** Consider the following system with a parameter $\mu \in \mathbb{R}$

$$\begin{aligned} x' &= \mu \sin x + y\cos y + x^3, \\ y' &= \mu \sin y - x\cos x. \end{aligned}$$

(a) Show that the system has periodic solutions for $\mu \in (0, \varepsilon)$ or $(-\varepsilon, 0)$. **(5 points)**
(b) Determine the type of stability of the periodic solutions a decide, whether they exist for $\mu > 0$ or $\mu < 0$. **(5 points)**
Verify the assumptions of the theorems you apply. Hint: $16a = f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy} + \frac{1}{\omega_0}[f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - f_{xx}g_{xx} + f_{yy}g_{yy}]$.

**3.** Consider the system

$$\begin{aligned} x' &= \sin y - x\sin^2 y, \\ y' &= -x. \end{aligned}$$

(a) Find (with proof) all asymptotically stable stationary points. **(6 points)**
(b) Find (with proof) all unstable stationary points. **(4 points)**
Hint: Investigate the behavior of $V(x, y) = x^2 - 2\cos y$ along solutions.

# ODE 2
# 10.2.2014, Exam C

**1.** Consider the following system with a parameter $\mu \in \mathbb{R}$

$$\begin{aligned} x' &= x + 2y + y^2 + \mu, \\ y' &= 2x - 2y. \end{aligned}$$

(a) Find a bifurcation point $(x_0, y_0, \mu_0)$. **(3 points)**
(b) Determine the type of bifurcation. Describe the behavior of the system (the number of stationary points and their stability) on a neighborhood of $(x_0, y_0)$ for $\mu = \mu_0 + \varepsilon$ and for $\mu = \mu_0 - \varepsilon$. **(7 points)**

**2.** Consider the equation $x' = 2x - 2u$ (DE) with a controll $u$, the initial condition $x(0) = 3$ (IC) and a functional

$$F(u) := \int_0^2 x(t) + u(t) + u^2(t)\ dt$$

and the set of admissible controlls $U := \{u : [0, 2] \to [-2, 2];\ u \text{ measurable}\}$.
(a) Find a controll $u \in U$, for which is $F(u)$ maximal. **(5 points)**
(b) Find $x$ corresponding to this controll $u$. **(5 points)**

**3.** Consider a system

$$\begin{aligned} x' &= y + xy^2 + xz^2, \\ y' &= -x - x^2y + yz^2, \\ z' &= -z + z^2 - xy. \end{aligned}$$

(a) Find the stable and central subspace of the linearized problem on a neighborhood of the origin. **(3 points)**
(b) Find the best approximation of the central manifold with a 2nd order polynomial and write an error estimate in the form $O(\dots)$. **(7 points)**

# ODE 2
# 10.9.2014, Exam D

**1.** Consider the following system with a controll $u$

$$\begin{aligned} x' &= \sin(x+y+z+u)+(x+y)^2, \\ y' &= e^{x+u}(y+z), \\ z' &= x+y+2z+u \end{aligned}$$

and the set of admissible controlls $U = \{u : [0,+\infty) \to [-5,5], u \text{ measureable}\}$.
(a) Show that the system is locally controllable. **(6 points)**
(b) Show that the system is not controllable for large $x(0)$, $y(0)$, $z(0)$. **(4 points)**

**2.** Consider the system

$$\begin{aligned} x' &= -x+3yz, \\ y' &= -2z+xyz, \\ z' &= 2y-3x^2y. \end{aligned}$$

(a) Find the stable and central manifolds of the linearized problem on a neighborhood of the origin. **(3 points)**
(b) Find the best approximation of the central manifold with a 2nd order polynomial and write an error estimate in the form $O(\dots)$. **(7 points)**

**3.** Consider the following system with a parameter $\mu \in \mathbb{R}$

$$\begin{aligned} x' &= \sin(xy)+y+\mu\sin x, \\ y' &= (1-\cos x)y-x+\mu y. \end{aligned}$$

(a) Show that the system has periodic solutions for $\mu \in (0,\varepsilon)$ or $(-\varepsilon,0)$. **(5 points)**
(b) Determine the type of stability of the periodic solutions and decide, whether they exist for $\mu > 0$ or $\mu < 0$. **(5 points)**
Verify the assumptions of the theorems you apply. Hint: $16a = f_{xxx}+f_{xyy}+g_{xxy}+g_{yyy}+\frac{1}{\omega_0}[f_{xy}(f_{xx}+f_{yy})-g_{xy}(g_{xx}+g_{yy})-f_{xx}g_{xx}+f_{yy}g_{yy}]$.