General Linear Problems

Variation of Constants

In the chapters on systems of linear equations and on n-th order linear differential equations, we learned how to solve equations (and systems) with zero right-hand sides, as well as with special right-hand sides. In this chapter, we will learn how to solve equations with arbitrary right-hand sides.

Consider an n-th order linear differential equation, i.e.,

$$a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_n(t)y = f(t)\,, \tag{1}$$

where $y : (a, b) \to \mathbb{R}$ is an unknown function, $a_i : (a, b) \to \mathbb{R}$ are the coefficients of the equation with $a_0(t) \neq 0$ for all $t \in (a, b)$, and $f : (a, b) \to \mathbb{R}$ is the right-hand side.

Using the notation

$$\mathscr{L}[y] = \sum_{k=0}^{n} a_k(t)y^{(n-k)},$$

we can rewrite equation (1) as

$$\mathscr{L}[y] = f. \tag{2}$$

Here, $\mathscr{L}$ is a linear operator from $V := C^n((a, b), \mathbb{R})$ to $W := C((a, b), \mathbb{R})$.

Now consider a system of equations, which can be written in vector form (see the chapter on systems of linear equations) as

$$Y' - A(t)Y = F(t), \tag{3}$$

where $Y : (a, b) \to \mathbb{R}^n$ is an unknown vector-valued function, $A : (a, b) \to \mathbb{R}^{n\times n}$ is the matrix of coefficients of the system, and $F : (a, b) \to \mathbb{R}^n$ is the right-hand side. Using the notation

$$\mathscr{L}[Y] = Y' - A(t)Y, \tag{4}$$

we can rewrite equation (3) as $\mathscr{L}[Y] = F$. In this case, $\mathscr{L}$ is a linear operator from $V := C^1((a, b), \mathbb{R}^n)$ to $W := C((a, b), \mathbb{R}^n)$.

In the following, we can discuss both problems simultaneously; f and y may be replaced by F and Y. The special case $f = 0$ is the so-called *homogeneous problem*

$$\mathscr{L}[y] = 0\,. \tag{5}$$

Recall the following theorem:

Theorem 1 (Solution Space)**.** *1. The set of all solutions of the homogeneous equation* (5) *defined on the interval* (a, b) *forms a vector space of dimension* n*.*

2. If y_p *is a solution of the non-homogeneous equation* (2) *on* (a, b)*, then the set of all solutions of equation* (2) *on* (a, b) *is* $\{y_p+y : y \text{ is a solution of } (5) \text{ on } (a, b)\}$*.*

The first part of the theorem states that the set of solutions of the homogeneous problem (5) forms a linear subspace of V (it is the kernel of the linear operator $\mathscr{L}$). By a *fundamental system* of equation (2), we mean any basis of this subspace. In other words, a fundamental system is an n-tuple of functions $\{y_1, \dots, y_n\}$ such that the general solution of the homogeneous problem (5) is of the form

$$y_h(t) = c_1 y_1(t) + \cdots + c_n y_n(t)\,, \tag{6}$$

where c_i are real constants. The second part of the theorem states that to find the general solution of (2), it suffices to determine a fundamental system and one (fixed) solution of the non-homogeneous problem, a so-called *particular solution*.

Remark. There is no general recipe for finding a fundamental system or a particular solution. However, once a fundamental system is found, a particular solution can also be found in the form

$$y(t) = c_1(t) y_1(t) + \cdots + c_n(t) y_n(t)\,, \tag{7}$$

where determining $c_i(t)$ reduces to solving a system of n linear algebraic equations followed by integration. This method is called the *variation of constants* (or *variation of parameters*).

Theorem 2 (Variation of Constants for n-th Order Equations)**.** *Let* $\{y_1, \dots, y_n\}$ *be a fundamental system of equation* (1)*. Let* $c_1(t), \dots, c_n(t)$ *satisfy the following system for all* $t \in (a, b)$ *(omitting the variable* t *for readability):*

$$\begin{aligned}
c_1' y_1 + c_2' y_2 + \cdots + c_n' y_n &= 0 \\
c_1' y_1' + c_2' y_2' + \cdots + c_n' y_n' &= 0 \\
&\vdots \\
c_1' y_1^{(n-2)} + c_2' y_2^{(n-2)} + \cdots + c_n' y_n^{(n-2)} &= 0 \\
c_1' y_1^{(n-1)} + c_2' y_2^{(n-1)} + \cdots + c_n' y_n^{(n-1)} &= \frac{f}{a_0}\,.
\end{aligned} \tag{8}$$

Then the function (7) *is a solution of problem* (1)*.*

Theorem 3 (Variation of Constants for Systems of Equations)**.** *Let* $\{Y_1, \dots, Y_n\}$ *be a fundamental system of the system* (3)*. Let* $c_1(t), \dots, c_n(t)$ *satisfy the following system for all* $t \in (a, b)$ *(omitting the variable* t *for readability):*

$$c_1' Y_1 + c_2' Y_2 + \cdots + c_n' Y_n = F. \tag{9}$$

Then the function $Y(t) = c_1(t)Y_1(t) + \cdots + c_n(t)Y_n(t)$ *is a solution of problem* (3)*.*

Remark. Let us emphasize that variation of constants works for general equations of types (1) and (3), although it is most commonly used for equations and systems with constant coefficients, since in that case we can effectively construct a fundamental system.

Remark. To conclude the introductory text, note that the n-th order equation (1) is "equivalent" to the first-order system (3), where

$$A(t) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -\frac{a_n(t)}{a_0(t)} & -\frac{a_{n-1}(t)}{a_0(t)} & -\frac{a_{n-2}(t)}{a_0(t)} & \dots & -\frac{a_1(t)}{a_0(t)} \end{pmatrix} \quad \text{and} \quad F(t) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \frac{f(t)}{a_0(t)} \end{pmatrix}.$$

Equivalent in the following sense: If y is a solution of (1) and we define $Y(t) := (y(t), y'(t), \dots, y^{(n-1)}(t))^T$, then Y is a solution of (3). Conversely, if Y is a solution of (3), then its first component Y_1 is a solution of equation (1). (Think this through in detail.)

Example 1. Find a particular solution of the equation $y'' + y = 1/\cos t$.

Solution. The fundamental system is $\{\cos t, \sin t\}$, so we seek a particular solution in the form $c_1(t)\cos t + c_2(t)\sin t$. The system (8) reduces to

$$\begin{aligned} c_1' \cos t + c_2' \sin t &= 0 \\ -c_1' \sin t + c_2' \cos t &= \frac{1}{\cos t}\,. \end{aligned}$$

Solving this (a neat trick is to multiply the first equation by $\cos t$ and the second by $-\sin t$): $c_1' = -\sin t/\cos t$, $c_2' = 1$. Integration yields

$$c_1(t) = \ln|\cos t|\,, \qquad c_2(t) = t\,.$$

The result holds on the intervals where $\cos t \neq 0$, i.e., $\big((2k-1)\frac{\pi}{2}, (2k+1)\frac{\pi}{2}\big)$.

Example 2. Solve the initial value problem $y''' - y' = \frac{e^t}{1+e^t}$.

Solution. Fundamental system: $\{1, e^t, e^{-t}\}$. We seek a particular solution in the form $c_1(t) + c_2(t)e^t + c_3(t)e^{-t}$. The system for c_i' reduces to

$$\begin{aligned} c_1' + c_2'e^t + c_3'e^{-t} &= 0 \\ c_2'e^t - c_3'e^{-t} &= 0 \\ c_2'e^t + c_3'e^{-t} &= \frac{e^t}{1+e^t}\,. \end{aligned}$$

The solution is

$$c_1'(t) = -\frac{e^t}{1+e^t}\,, \qquad c_2'(t) = \frac{1}{2}\,\left(1+e^t\right)^{-1}\,, \qquad c_3'(t) = \frac{1}{2}\,\frac{e^{2\,t}}{1+e^t}\,.$$

Integration yields

$$\begin{gathered} c_1(t) = -\ln(1+e^t)\,, \qquad c_2(t) = -\frac{1}{2}\,\ln(1+e^t) + \frac{1}{2}\,\ln(e^t)\,, \\ c_3(t) = \frac{1}{2}\,e^t - \frac{1}{2}\,\ln(1+e^t)\,. \end{gathered}$$

Example 3. Find the general solution of the following system of equations:

$$\begin{aligned} y' &= 3y - 6z + \frac{e^{2t}}{2+e^t}\,, \\ z' &= y - 2z\,. \end{aligned}$$

Solution. First, let us solve the homogeneous system:

$$\begin{pmatrix} \lambda - 3 & 6 \\ -1 & \lambda + 2 \end{pmatrix} \sim \begin{pmatrix} 0 & \lambda^2 - \lambda \\ -1 & \lambda + 2 \end{pmatrix},$$

whence $z(t) = ce^t + d$ and $y(t) = z' + 2z = 3ce^t + 2d$. Theorem 3 gives

$$\begin{aligned} 3c'(t)e^t + 2d'(t) &= \frac{e^{2t}}{2+e^t} \\ c'(t)e^t + d'(t)e^{-t} &= 0, \end{aligned}$$

which gives $c(t) = \int \frac{e^t}{2+e^t}dt = \ln(2+e^t)$, $d(t) = -\int \frac{e^{2t}}{2+e^t}dt = -e^t + 2\ln(2+e^t)$. The resulting solution is:

$$\begin{aligned} y(t) &= 3ce^t + 2d + 3e^t\ln(2+e^t) - 2e^t + 4\ln(2+e^t) \\ z(t) &= ce^t + d + e^t\ln(2+e^t) - e^t + 2\ln(2+e^t). \end{aligned}$$

Exercises on n-th Order Equations

Find a particular solution using the variation of constants:

1. $y'' + y' = \frac{e^t}{1+e^{2\,t}}$

2. $y'' + y = \frac{1}{\sin^2 t}$

3. $y'' - 2y' = 5\,(3 - 4\,t)\,\sqrt{t}$

4. $y'' + 3y' = \frac{3t-1}{t^2}$

5. $y'' - y' = -\frac{t+1}{t^2}$

6. $y'' - 2y' + y = \frac{e^t}{t}$

7. $y'' + 4y = 2\tan t$

8. $y'' + y' = \frac{1}{1+e^t}$

9. $y'' - 3y' + 2y = \frac{1}{1+e^t}$

10. $y'' - y = \frac{e^t - e^{-t}}{e^t + e^{-t}}$

11. $y'' - 2y' + 10y = \frac{9e^t}{\cos 3t}$

12. $y'' + y = -\frac{\cos^2 t}{\sin^2 t}$

13. $y'' + 4y' + 4y = \frac{e^{-2t}}{t+1}$

14. $y'' - 4y' + 4y = \frac{2e^{2t}}{1+t^2}$

15. $y'' + 2y' + 2y = \frac{e^{-t}}{\sin t}$

16. $y'' + 2y' + 5y = \frac{2e^{-t}}{\cos 2t}$

17. $y'' - 2y' = \frac{1}{t} - 2 - 2\ln t$

18. $y^{(4)} + y'' = \frac{\sin t}{\cos^2 t}$

19. Find a particular solution of the equation

$$y'' + \frac{1}{t(1+\ln t)}y' + \frac{1}{t^2(1+\ln t)}y = t(1+\ln t),$$

given that the functions $\ln t$ and $-1/t$ form a fundamental system.

Exercises on Systems of Equations

Find all solutions of the following systems:

20.

$$\begin{aligned} x' &= 2x + y - \ln t\,, \\ y' &= -4x - 2y + \ln t\,. \end{aligned}$$

21.

$$\begin{aligned} x' &= -2x + 4y + \frac{1}{1+e^t}\,, \\ y' &= -2x + 4y - \frac{1}{1+e^t}\,. \end{aligned}$$

22.

$$\begin{aligned} z' &= -y + 3z + e^{2t}\ln t\,, \\ y' &= y + z + te^{2t}\,. \end{aligned}$$

23.

$$\begin{aligned} y' &= 5y - 6z + \frac{3e^{2t}}{\cos^3 3t}\,, \\ z' &= 3y - z\,. \end{aligned}$$

24.

$$\begin{aligned} y' &= 4y - 2z\,, \\ z' &= 8y - 4z + \sqrt{t}\,. \end{aligned}$$

25.

$$\begin{aligned} y' &= -7y + 2z\,, \\ z' &= -15y + 4z + \frac{e^{-2t}}{1+e^{2t}}\,. \end{aligned}$$

26.

$$\begin{aligned} x' &= x + y - z - t\sqrt{t}\,, \\ y' &= -3x - 3y + 3z\,, \\ z' &= -2x - 2y + 2z + \sqrt{t}\,. \end{aligned}$$

27.

$$\begin{aligned} 2x' &= x - 3y + 4z\,, \\ 2y' &= x - 3y + 4z + \operatorname{tg} t\,, \\ z' &= -y + z\,. \end{aligned}$$

Solutions

1) $y_p(t) = c_1(t) + c_2(t)e^{-t}$, $c_1(t) = \int \frac{e^t}{1+e^{2\,t}}dt = \arctan(e^t)$, $c_2(t) = \int -\frac{e^{2\,t}}{1+e^{2\,t}}dt = -1/2\,\ln(1+e^{2\,t})$.

2) $y_p(t) = c_1(t)\cos(t) + c_2(t)\sin(t)$, $c_1(t) = -\int \frac{dt}{\sin t} = -\ln|\tan(t/2)|$, $c_2(t) = \int \frac{\cos t}{\sin^2 t}\,dt = -\frac{1}{\sin t}$.

3) $y_p(t) = c_1(t) + c_2(t)e^{2\,t}$, $c_1(t) = \int 5/2\,\sqrt{t}\,(-3+4\,t)\,dt = -5\,t^{3/2} + 4\,t^{5/2}$, $c_2(t) = \int -5/2\,\sqrt{t}\,(-3+4\,t)\,e^{-2\,t}dt = 5\,\frac{t^{3/2}}{(e^t)^2}$, $y_p(t) = 4\,t^{5/2}$.

4) $y_p(t) = c_1(t) + c_2(t)e^{-3\,t}$, $c_1(t) = \int 1/3\,\frac{3\,t-1}{t^2}dt = 1/3\,t^{-1} + \ln(t)$, $c_2(t) = \int -1/3\,\frac{(3\,t-1)e^{3\,t}}{t^2}dt = -1/3\,\frac{e^{3\,t}}{t}$, $y_p(t) = \ln|t|$.

5) $y_p(t) = c_1(t)+c_2(t)e^t$, $c_1(t) = \int \frac{t+1}{t^2}dt = -t^{-1}+\ln(t)$, $c_2(t) = \int -\frac{(t+1)e^{-t}}{t^2}dt = \frac{e^{-t}}{t}$, $y_p(t) = \ln|t|$.

6) $y_p(t) = c_1(t)e^t + c_2(t)e^t t$, $c_1(t) = \int -1dt = -t$, $c_2(t) = \int t^{-1}dt = \ln|t|$, $y_p(t) = t\ln|t|e^t$.

7) $y_p(t) = c_1(t)\cos(2\,t) + c_2(t)\sin(2\,t)$, $c_1(t) = \int -2 + 2\,\cos^2 t dt = -t + \sin(t)\cos(t)$, $c_2(t) = \int \sin(t)(-1+2\,\cos^2(t))/\cos(t)dt = -\cos^2(t)+\ln|\cos(t)|$.

8) $y_p(t) = c_1(t) + c_2(t)e^{-t}$, $c_1(t) = \int (1+e^t)^{-1}\,dt = -\ln(1+e^t) + t$, $c_2(t) = \int -\frac{e^t}{1+e^t}dt = -\ln(1+e^t)$.

9) $y_p(t) = c_1(t)e^t + c_2(t)e^{2\,t}$, $c_1(t) = \int -\frac{e^{-2\,t}}{e^{-t}+1}dt = e^{-t} + t - \ln(1+e^t)$, $c_2(t) = \int \frac{e^{-2\,t}}{1+e^t}dt = -\frac{1}{2}e^{-2t} + e^{-t} + t - \ln(1+e^t)$.

10) $y_p(t) = c_1(t)\sinh(t) + c_2(t)\cosh(t)$, $c_1(t) = \int -1/2\,\left(-1+e^{-2\,t}\right)e^t dt = e^t/2 + e^{-t}/2$, $c_2(t) = \int -1/2\,\frac{\left(-1+e^{-2\,t}\right)^2 e^t}{1+e^{-2\,t}}dt = 2\arctan e^t - e^t/2 + e^{-t}/2$.

11) $y_p(t) = c_1(t)e^t\cos(3\,t)+c_2(t)e^t\sin(3\,t)$, $c_1(t) = \int -3\,\frac{\sin(3\,t)}{\cos(3\,t)}dt = \ln|\cos(3\,t)|$, $c_2(t) = \int 3dt = 3\,t$.

12) $y_p(t) = c_1(t)\cos(t) + c_2(t)\sin(t)$, $c_1(t) = \int \frac{\cos^2 t}{\sin t}\,dt = \ln|\tan(t/2)| + \cos t$, $c_2(t) = \int \frac{\cos^3 t}{\cos^2 t - 1}\,dt = \sin t + 1/\sin t$.

13) $y_p(t) = c_1(t)e^{-2\,t} + c_2(t)e^{-2\,t}t$, $c_1(t) = \int -\frac{t}{t+1}dt = -t + \ln|t+1|$, $c_2(t) = \int (t+1)^{-1}\,dt = \ln|t+1|$.

14) $y_p(t) = c_1(t)e^{2\,t} + c_2(t)e^{2\,t}t$, $c_1(t) = \int \frac{-2t}{t^2+1}dt = -\ln(t^2+1)$, $c_2(t) = \int 2\,(t^2+1)^{-1}\,dt = 2\,\arctan(t)$.

15) $y_p(t) = c_1(t)e^{-t}\cos(t) + c_2(t)e^{-t}\sin(t)$, $c_1(t) = \int -1dt = -t$, $c_2(t) = \int \frac{\cos(t)}{\sin(t)}dt = \ln|\sin(t)|$.

16) $y_p(t) = c_1(t)e^{-t}\cos(2\,t)+c_2(t)e^{-t}\sin(2\,t)$, $c_1(t) = \int -\frac{\sin(2\,t)}{\cos(2\,t)}dt = 1/2\,\ln|\cos(2\,t)|$, $c_2(t) = \int 1dt = t$.

17) $y_p(t) = c_1(t)+c_2(t)e^{2\,t}$, $c_1(t) = \int(-1+2\,t+2\,\ln(t)t)/2tdt = -1/2\,\ln(t)+\ln(t)t$, $c_2(t) = -\int(-1+2\,t+2\,\ln(t)t)e^{2t}/2tdt = 1/2\,e^{-2\,t} + 1/2\,e^{-2\,t}\ln(t)$;

$y_p(t) = \ln(t)t + 1/2$.

18) $y_p(t) = c_1(t)\sin t + c_2(t)\cos t + c_3(t) + c_4(t)\cdot t$, where $c_1(t) = \int -\frac{\sin t}{\cos t}dt = \ln|\cos t|$, $c_2(t) = \int \frac{\sin^2 t}{\cos^2 t}dt = \operatorname{tg} t - t$, $c_3(t) = -\int t\frac{\sin t}{\cos^2 t}dt = -\frac{t}{\cos t} + \frac{1}{2}\ln|\frac{1+\sin t}{1-\sin t}|$, $c_4(t) = \int \frac{\sin t}{\cos^2 t}dt = \frac{1}{\cos t}$; the solution is defined on the intervals $(-\pi/2 + k\pi, \pi/2 + k\pi)$.

19) $y_p = c_1(t)\ln t - c_2(t)\frac{1}{t}$, where $c_1(t) = \frac{1}{2}t^2$, $c_2(t) = \frac{1}{3}t^3\ln t - \frac{1}{9}t^3$.

20) After transformations, $x'' = -\frac{1}{t} - \ln t$, i.e., $\lambda_{1,2} = 0$; the particular solution (via variation of constants) is $t + \frac{3}{4}t^2 - t\ln t - \frac{t^2}{2}\ln t$, so $x(t) = a + t(b+1) + \frac{3}{4}t^2 - t\ln t - \frac{t^2}{2}\ln t$ and subsequently $y(t) = b - 2a - t(1 + 2b) - \frac{3}{2}t^2 + (t + t^2)\ln t$, $t \in (0, \infty)$, $a, b \in \mathbb{R}$. One can also proceed directly by integrating $x'' = -\frac{1}{t} - \ln t$ twice.

21) The eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 2$, with eigenvectors $v_1 = (2,1)^T$, $v_2 = (1,1)^T$, so the fundamental system is

$$\begin{pmatrix}2\\1\end{pmatrix}, e^{2t}\begin{pmatrix}1\\1\end{pmatrix}.$$

The particular solution (via variation of constants) is

$$\frac{1}{2}\begin{pmatrix}3+8t\\3+4t\end{pmatrix} + \begin{pmatrix}-3\\-3\end{pmatrix}(e^t + te^{2t}) + \ln(1+e^t)\left[\begin{pmatrix}-4\\-2\end{pmatrix} + \begin{pmatrix}3\\3\end{pmatrix}e^{2t}\right].$$

22) We seek a solution in the form

$$\begin{pmatrix}y(t)\\z(t)\end{pmatrix} = c(t)\begin{pmatrix}e^{2t}\\e^{2t}\end{pmatrix} + d(t)\begin{pmatrix}te^{2t}\\e^{2t}(t+1)\end{pmatrix}.$$

This gives $c'(t) = t + t^2 - t\ln t$ and $d'(t) = \ln t - t$, whence $c(t) = \frac{3}{4}t^2 + \frac{1}{2}t^3 - \frac{1}{2}t^2\ln t$ and $d(t) = t\ln t - t - \frac{1}{2}t^2$. The solution is defined on $(0, +\infty)$.

23) $y(t) = (c+d)e^{2t}\cos 3t + (c-d)e^{2t}\sin 3t + e^{2t}\operatorname{tg} 3t(\sin 3t + \cos 3t) + \frac{e^{2t}}{2\cos^2 3t}(\sin 3t - \cos 3t)$, $z(t) = ce^{2t}\sin 3t + de^{2t}\cos 3t - \frac{e^{2t}\cos 6t}{\cos 3t}$.

24) $y(t) = c + dt - \frac{8}{15}t^{5/2}$, $z(t) = 2c + d(t - 1/2) - \frac{16}{15}t^{5/2} + \frac{2}{3}t^{3/2}$.

25) $y(t) = 2ce^{-2t} + de^{-t} - 2e^{-2t} - e^{-2t}\ln(1+e^{-2t}) - 2e^{-t}\operatorname{arctg} e^t$, $z(t) = 5ce^{-2t} + 3de^{-t} - \frac{5}{2}e^{-2t}\ln(1+e^{-2t}) - 6e^{-t}\operatorname{arctg} e^t$.

26) $x(t) = -\frac{4}{35}t^{7/2} - \frac{2}{3}t^{5/2} + ct + d$, $y(t) = \frac{12}{35}t^{7/2} + \frac{4}{5}t^{5/2} - 3ct + e$, $z(t) = \frac{8}{35}t^{7/2} + \frac{8}{15}t^{5/2} + \frac{2}{3}t^{3/2} - 2ct + d - c + e$.

27) $x(t) = a(1/2 + 1/2\sin t + 1/2\cos t) + b(-1/2 + 1/2\cos t - 3/2\sin t) + c(2\sin t) + 1/2\ln(\cos t) - 1/2 + 1/2\sin t\ln((1+\sin t)/\cos t) + 3/2\cos t\ln((1+$

$\sin t)/\cos t)$, $y(t) = a(-1/2 + 1/2\sin t + 1/2\cos t) + b(-1/2 + 1/2\cos t - 3/2\sin t) + c(2\sin t) - 1/2\ln(\cos t) - 1/2 + 1/2\sin t\ln((1+\sin t)/\cos t) + 3/2\cos t\ln((1+\sin t)/\cos t)$, $z(t) = a(-1/2 + 1/2\cos t) + b(1/2 - \sin t - 1/2\cos t) + c(\sin t + \cos t) - 1/2\ln(\cos t) + 1/2 - 1/2\sin t\ln((1+\sin t)/\cos t) + \cos t\ln((1+\sin t)/\cos t)$.

Variation of Constants - Integral Form

Sometimes, variation of constants refers to expressing a particular solution as a convolution of the right-hand side with a suitable solution of the homogeneous problem. The resulting explicit formula is also useful for incorporating initial conditions directly.

For simplicity, let us state the result for equations with constant coefficients:

$$\mathscr{K}[y] = f(t)\,, \tag{10}$$

where $\mathscr{K}[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y$.

Theorem 4. *Let $\omega(t)$ be the solution of the homogeneous problem $\mathscr{K}[y] = 0$ with the initial conditions*

$$\begin{aligned} \omega(0) = \omega'(0) = \cdots = \omega^{(n-2)}(0) = 0\,, \\ \omega^{(n-1)}(0) = \frac{1}{a_0}\,. \end{aligned} \tag{11}$$

Then

$$y_p(t) = \int_0^t \omega(t-s) f(s)\, ds \tag{12}$$

is the solution of problem (10) *with zero initial conditions:*

$$y(0) = y'(0) = \cdots = y^{(n-1)}(0) = 0\,. \tag{13}$$

Before the proof, let us look at two illustrative examples.

Example 4. Find the general solution of the equation $y'' + a^2 y = f(t)$.

Solution. Choose the fundamental system $y_1(t) = \cos at$, $y_2(t) = \frac{1}{a}\sin at$. Notice that $y_2(0) = 0$, $y_2'(0) = 1$, so y_2 is precisely the required function ω – the solution of the homogeneous problem with initial conditions (11). Thus $y_p(t) = \frac{1}{a}\int_0^t \sin[a(t-s)] f(s)\, ds$, and the general solution is of the form

$$y(t) = c_1 \cos at + \frac{c_2}{a}\sin at + \int_0^t \frac{\sin[a(t-s)]}{a} f(s)\, ds\,.$$

By matching initial conditions, we find that $c_1 = y(0)$ and $c_2 = y'(0)$.

Example 5. Let $x(t)$ be a C^2 function such that $x''(t) + x'(t) + x(t) \to 0$ as $t \to \infty$. Then $x(t) \to 0$ as $t \to \infty$.

Solution. We can easily solve this problem using Theorem 1. Notice that $x(t)$ is a solution of

$$y'' + y' + y = f(t) \tag{14}$$

where $f(t) := x''(t) + x'(t) + x(t)$. The characteristic polynomial has roots $(-1 \pm i\sqrt{3})/2$, so the fundamental system is formed by the functions $\{y_1(t), y_2(t)\}$, where $y_1(t) = \exp(-t/2)\cos(\sqrt{3}t/2)$, $y_2(t) = \exp(-t/2)\sin(\sqrt{3}t/2)$. Furthermore, $\omega(t) := 2y_2(t)/\sqrt{3}$ clearly satisfies conditions (11). The general solution of equation (14) (and thus also $x(t)$) has the form

$$c_1 y_1(t) + c_2 y_2(t) + \frac{2}{\sqrt{3}} \int_0^t y_2(t-s) f(s)\, ds\,.$$

We claim that this tends to 0 as $t \to \infty$. The first two terms clearly vanish; it remains to estimate the integral, which is done by splitting it into two parts. Since $f(t) \to 0$ as $t \to \infty$, choose $K > 0$ such that $|f(t)| \le \varepsilon$ for $t > K$. Using the bound $|y_2(t-s)| \le \exp(-(t-s)/2)$, we have for $t > K$:

$$\begin{aligned} \Big| \int_0^t y_2(t-s) f(s)\, ds \Big| &\le \int_0^K e^{-(t-s)/2} |f(s)|\, ds + \varepsilon \int_K^t e^{-(t-s)/2}\, ds \\ &= e^{-t/2} \int_0^K e^{s/2} |f(s)|\, ds + 2\varepsilon \underbrace{\left(1 - e^{-(t-K)/2}\right)}_{\le 1}\,. \end{aligned}$$

For sufficiently large t, this is bounded by 3ε, which completes the proof.

To prove Theorem 1, we need the following auxiliary result, which is of independent interest.

Lemma 5. *Let*

$$F(t) = \int_0^t \phi(t,s)\, ds\,,$$

where ϕ is a C^1 function. Then

$$F'(t) = \phi(t,t) + \int_0^t \frac{\partial \phi}{\partial t}(t,s)\, ds\,.$$

Now we can proceed with the proof of Theorem 4.

Proof. Substitute (12) directly into (10). By Lemma 5,

$$y_p'(t) = \underbrace{\omega(0)}_{=0} f(t) + \int_0^t \omega'(t-s) f(s)\, ds = \int_0^t \omega'(t-s) f(s)\, ds$$

using (11). Repeating this argument yields

$$y_p^{(k)}(t) = \int_0^t \omega^{(k)}(t-s)f(s)\,ds\,, \qquad k = 0, \dots, n-1 \tag{15}$$

and finally

$$y_p^{(n)}(t) = \underbrace{\omega^{(n-1)}(0)}_{=1/a_0} f(t) + \int_0^t \omega^{(n)}(t-s)f(s)\,ds.$$

Therefore,

$$\mathscr{K}[y_p](t) = a_0\frac{1}{a_0}f(t) + \int_0^t \underbrace{\mathscr{K}[\omega]}_{=0}(t-s)f(s)\,ds = f(t)$$

since ω solves the homogeneous problem. The satisfaction of the initial conditions (13) follows from (15). □

Now let us state the variation of constants formula in integral form for a general system of equations:

$$Y' = A(t)Y + F(t), \tag{16}$$

where $A : (a,b) \subset \mathbb{R} \to \mathbb{R}^{n\times n}$ and $F : (a,b) \to \mathbb{R}^n$ are continuous functions. Recall that a fundamental matrix of the system is a matrix whose columns form a fundamental system of solutions.

Theorem 6 (Variation of Constants - Integral Form)**.** *Let $t_0 \in (a,b)$ and let Φ be a fundamental matrix of the system* (16)*. Then the solution of the non-homogeneous problem* (16) *with initial condition $Y(t_0) = Y_0$ is*

$$Y(t) = \Phi(t)\Phi(t_0)^{-1}Y_0 + \Phi(t)\int_{t_0}^t \Phi(s)^{-1}F(s)\,ds.$$

Example 6. Consider the constant-coefficient system $Y' = AY + b(t)$, where b is bounded on $[0,+\infty)$. Let the eigenvalues of the matrix A have positive real parts. Then there exists at most one solution that is bounded on $[0,+\infty)$.

Solution. A fundamental matrix of the system is the matrix exponential e^{tA}. Thus,

$$Y(t) = e^{tA}Y_0 + \int_0^t e^{(t-s)A}b(s)\,ds.$$

Multiplying by e^{-tA} gives

$$e^{-tA}Y(t) = Y_0 + \int_0^t e^{-sA}b(s)\,ds.$$

If the solution Y is bounded, we must have

$$0 = Y_0 + \lim_{t\to\infty}\int_0^t e^{-sA}b(s)\,ds = Y_0 + \int_0^{+\infty} e^{-sA}b(s)\,ds.$$

(The integral converges because the eigenvalues of A have positive real parts, as follows from the Jordan canonical form.) There exists at most one such Y_0.

Exercises

28. Find all solutions of the equation $y'' + 2y' + y = f(t)$.

29. Consider the equation $y'' - 2y' + y = t^2 + 1$. By how much does the solution with initial conditions $y(0) = 1$, $y'(0) = 0$ differ from the solution with initial conditions $y(0) = 1/2$, $y'(0) = 0$ at time $t = 5$?

30. Consider the initial value problem $y'' - 2y' + y = f(t)$, $y(0) = 1, y'(0) = 0$. By how much does the solution with right-hand side $f(t) = 1 + t$ differ from the solution with right-hand side $\tilde{f}(t) = 1 + \sin t$ (and the same initial conditions) at time $t = 1$?

31. Show that if $x(0) \geq 0$, $x'(0) > -x(0)$, and $x''(t) \geq x(t)$ for all $t > 0$, then $x(t) \geq 0$ for all $t > 0$.

32. Show that if $x(0) = 0$, $x'(0) = 1$, and $x''(t) \geq -x(t)$ for all $t > 0$, then $x(t) \geq \sin t$ for all $t \in [0, \pi]$. Show that this does not hold in general if $x(0) > 0$.

33. Let $e^{-t}(x'''(t) - x'(t)) \geq 1/100$ for $t \in (100, +\infty)$. Prove or find a counterexample to the statement that $x(t) \to +\infty$.

34. Let $x'''(t) - x'(t) \to +\infty$ as $t \to +\infty$. Prove or find a counterexample to the statement that $x(t) \to +\infty$.

35. Consider a process described by $y'' - 3y' + 2y = t$ with measured initial state $y(0) = y_0$, $y'(0) = 0$. With what precision must y_0 be known to guarantee that the perturbed solution does not differ from the exact solution on the interval $[0, 10]$ by more than 1?

36. Consider an oscillation described by $y'' + y = f(t)$, $y(0) = 1$, $y'(0) = 1$, where f is a controllable external force. Show that a small change in the external force results in a small change in the oscillation. Formulate the problem rigorously.

37. Show that for a system with constant coefficients, the solution satisfies

$$y(t) = \Phi(t - t_0)y_0 + \int_{t_0}^{t} \Phi(t - s)f(s)\, ds,$$

where $\Phi(t) = e^{tA}$.

38. Show that Theorem 4 is a special case of Theorem 6.

39. Decide whether in the setting of Example there exists a unique bounded solution.

40. Consider the nonlinear system of equations $Y' = AY + F(Y)$, where F is L-Lipschitz. For each initial condition $Y(t_0) = Y_0$:
(a) there exists a solution defined at least on $(t_0 - 1/L, t_0 + 1/L)$.
(b) there exists a unique solution defined on all of $\mathbb{R}$.

41. Consider the system $Y' = AY + b(t)$, where b is T-periodic. Show that:
(a) the corresponding homogeneous equation has a non-trivial S-periodic solution if and only if $1 \in \sigma(e^{SA})$ (1 is an eigenvalue of the matrix e^{SA}).
(b) if $1 \notin \sigma(e^{TA})$, then there exist $d \in \mathbb{R}^n$, T-periodic functions $F : \mathbb{R} \to \mathbb{R}^{n\times n}$, $f : \mathbb{R} \to \mathbb{R}^n$, and a matrix $C \in \mathbb{R}^{n\times n}$ such that the solution of the non-homogeneous equation with initial condition $Y(0) = y_0$ is $Y(t) = F(t)C^n(y_0 + d) - F(t)d + f(t)$ for $t = nT + \sigma$, $\sigma \in [0, T)$.
(c) if $1 \notin \sigma(e^{TA})$, then there exists a unique T-periodic solution of the non-homogeneous equation.

Solutions

28) The fundamental system is e^{-t}, te^{-t}. We seek a solution $y(t) = ae^{-t} + bte^{-t}$ satisfying $y(0) = 0$, $y'(0) = 1$. This is te^{-t}. The solutions of the non-homogeneous equation are therefore

$$y(t) = ce^{-t} + dte^{-t} + \int_0^t (t - s)e^{-(t-s)}f(s)\, ds.$$

29) $y(t)-\tilde{y}(t) = (e^t-te^t)-(e^t/2-te^t/2)$, i.e., at $t = 5$ we have $|\frac{1}{2}e^5-\frac{5}{2}e^5| = 2e^5$.

30) $y(t) - \tilde{y}(t) = \int_0^t (t - s)e^{t-s}(s - \sin s)\, ds$, i.e., at $t = 1$ we have $3 - e - \frac{1}{2}\cos 1$.

31) Let $f(t) := x''(t) - x(t) \geq 0$. Then $x(t) = ce^t + de^{-t} + \int_0^t \frac{1}{2}(e^{t-s} - e^{s-t})f(s)\, ds \geq 0$, and from the initial conditions, $c > 0$ and $c + d \geq 0$. The assertion follows directly.

32) Let $f(t) := x''(t) + x(t) \geq 0$. Then $x(t) = \sin t + x(0)\cos t + \int_0^t \sin(t - s)f(s)\, ds \geq \sin t$ if $x(0) = 0$. If $x(0) > 0$ and $f(t) \equiv 0$, then $x(t) < 0$ on $(\pi/2, \pi)$.

33) Let $f(t) := e^{-t}(x'''(t) - x'(t))$, then

$$\begin{aligned} x(t) &= c_1 + c_2 e^t + c_3 e^{-t} + \int_0^t \frac{1}{2}(e^{t-s} + e^{s-t} - 2)e^s f(s)\, ds \\ &= c_1 + c_2 e^t + c_3 e^{-t} + e^t \int_0^t \frac{1}{2}(1 + e^{2s-2t} - e^{s-t}) f(s)\, ds. \end{aligned}$$

The function $1 + e^{2s-2t} - e^{s-t}$ attains a minimum of $1 + 1/4 - 1/2 = 3/4$, so we have

$$x(t) \geq e^t \left(c_2 + \frac{3}{8} \int_0^t f(s)\, ds - \varepsilon \right)$$

and since f is bounded below by a positive constant, the expression in parentheses approaches $+\infty$.

34) Let $f(t) := x'''(t) - x'(t)$, then

$$x(t) = c_1 + c_2 e^t + c_3 e^{-t} + \int_0^t \frac{1}{2}(e^{t-s} + e^{s-t} - 2) f(s)\, ds.$$

If, for instance, $f(t) = t$, we have

$$x(t) = c_1 + c_2 e^t + c_3 e^{-t} + e^t + e^{-t} - 2 - t^2 \to -\infty,$$

whenever $c_2 \leq -1$.

35) The solution is $y(t) = (2e^t - e^{2t})y_0 + \int_0^t (e^{2t-2s} - e^{t-s}) s\, ds$, so $|y(t) - \tilde{y}(t)| = |2e^t - e^{2t}||y_0 - \tilde{y}_0| \leq 1$. The function $|2e^t - e^{2t}|$ attains its maximum on $[0, 10]$ at $t = 10$, so $|y_0 - \tilde{y}_0| \leq (e^{20} - 2e^{10})^{-1}$.

36) Directly from the integral form of variation of constants:

$$|y_1(t) - y_2(t)| \leq \int_0^t |\sin(t-s)|\, |f_1(s) - f_2(s)|\, ds.$$

37) A homogeneous system with constant coefficients is autonomous; if $t \mapsto \Phi(t)C$ is a solution, then $t \mapsto \Phi(s+t)C$ is also a solution. At $t = 0$, this solution has value $\Phi(s)C$, so $\Phi(s+t)C = \Phi(t)\Phi(s)C$ for all C. Hence $\Phi(t)\Phi(s)^{-1} = \Phi(t-s)$.

38) Divide equation (10) by a_0. It suffices to show that the fundamental matrix $\Phi(t)$ with $\Phi(0) = I$ for the companion system has the function $\omega(t)$ in position $(\Phi)_{1n}$, which matches the initial conditions defining ω.

39) It remains to show that the function $f : t \mapsto e^{tA}\left(Y_0 - \int_0^t e^{-sA} b(s)\, ds\right)$ for Y_0 defined in Example is bounded. Indeed,

$$f(t) = \int_t^{+\infty} e^{(t-s)A} b(s)\, ds = \int_0^{+\infty} e^{-sA} b(t+s)\, ds,$$

so $|f(t)| \leq \sup|b| \cdot \int_0^{+\infty} \|e^{-sA}\|\, ds < C < +\infty$.

40) (a) Define x_{n+1} as the solution of $Y' = AY + F(x_n)$ with the given initial condition. By variation of constants,

$$\|x_{n+1}(t) - x_n(t)\| \leq \int_{t_0}^{t} \|e^{(t-s)A}\| L \|x_n(s) - x_{n-1}(s)\|\, ds.$$

On an interval smaller than $(t_0 - 1/L, t_0 + 1/L)$, the map is a contraction in the supremum norm. Thus, a solution exists on $(t_0 - 1/L, t_0 + 1/L)$.
(b) Show that the Picard operator is a contraction in an exponentially weighted supremum norm on $[t_0, +\infty)$ and $(-\infty, t_0]$.

41) (a) $1 \in \sigma(e^{SA})$ if and only if $2\pi ik/S \in \sigma(A)$ for some $k \in$.
(b) From variation of constants, $Y(t) = e^{nTA}e^{\sigma A}y_0 + \int_0^{nT+\sigma} e^{(nT+\sigma-s)A}b(s)\, ds$. Splitting the integral over intervals of length T and summing the geometric series $\sum_{k=1}^{n} C^{k-1} = (C^n - I)(C - I)^{-1}$ for $C = e^{TA}$ yields the desired expression.
(c) Setting $y_0 = -d$ eliminates the C^n dependence, yielding the unique T-periodic solution.

Method of Reduction of Order

Consider a second-order homogeneous linear equation with variable coefficients:

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0,$$

for which one non-zero solution $y_1(x)$ is already known or guessed. Then a second linearly independent solution of the fundamental system can be computed as follows:

Set $y(x) = y_1(x)z(x)$, so $y' = y_1'z + y_1z'$ and $y'' = y_1''z + 2y_1'z' + y_1z''$. Substituting these into the equation gives:

$$\begin{aligned} &a_2(x)(y_1''z + 2y_1'z' + y_1z'') + a_1(x)(y_1'z + y_1z') + a_0(x)y_1z \\ &\quad = z(a_2(x)y_1'' + a_1(x)y_1' + a_0(x)y_1) + a_2(x)y_1z'' + (a_1(x)y_1 + 2a_2(x)y_1')z' \\ &\qquad\qquad = a_2(x)y_1z'' + (a_1(x)y_1 + 2a_2(x)y_1')z', \end{aligned}$$

where the first group vanishes because y_1 solves the homogeneous equation. Setting $v = z'$, we obtain a first-order linear equation for v:

$$a_2(x)y_1v' + (a_1(x)y_1 + 2a_2(x)y_1')v = 0,$$

which we can readily solve.

Example 7. Find all solutions of the equation

$$y'' + \frac{x-3}{x}y' + \frac{x-3}{x^2}y = 0.$$

Solution. We guess the solution $y_1(x) = x$. Substitute $y(x) = xz(x)$. This yields the equation $xv' = (1-x)v$, i.e., $\ln|v| = \ln x - x + c$, so $v = \pm e^c x e^{-x}$, and integrating yields $z = ke^{-x}(-x-1)$. The second function of the fundamental system is $e^{-x}(-x^2-x)$, and the general solution is $y(x) = cx + de^{-x}(x^2+x)$.

Exercises

42. Find all solutions of the equation

$$y'' - \frac{4-2x^2}{4-x^2}y' + \frac{x^2}{4-x^2}y = (x-2)e^x,$$

given that e^x solves the homogeneous equation.

43. Find all solutions of the equation

$$y'' + \frac{2}{x}y' + \frac{2}{x(x+1)^2}y = x + 2 + \frac{1}{x},$$

given that $\frac{1}{x+1}$ solves the homogeneous equation.

44. Find all solutions of the equation

$$x^2y'' + x(x^2+3)y' - (x^2+3)y = 0.$$

45. Find all solutions of the equation $x^3y'' - x^2y' + (x^2-x^3)y = 0$ that are defined on all of $\mathbb{R}$.

46. Find all solutions of the equation

$$(1+x^2)y'' + xy' - y = 0.$$

47. Find the solution of the equation

$$(1+x^2)y'' - 2xy' + 2y = 0$$

satisfying the initial condition $y(0) = 1$, $y'(0) = 0$.

Solutions

42) Using reduction of order with $y(x) = z(x)e^x$, we get $z''(x) + 4z'(x)/(4-x^2) = 0$, which yields $z(x) = a(x - 4\ln(x+2)) + b$. The second solution is $(x - 4\ln(x+2))e^x$. By variation of constants, $y_p = c(x)e^x + d(x)(x - 4\ln(x+2))e^x$, where $c(x) = \int -(x+2)(x-4\ln(x+2))dx = -1/3x^3 - 2x^2 - 4x - 8/3 + 2(x+2)^2\ln(x+2)$ and $d(x) = \int (x+2)dx = 1/2x^2 + 2x$.

43) Using reduction of order with $y(x) = z(x)/(x+1)$, substitution gives $x(x+1)z''(x) - 2z'(x) = 0$, which yields $z(x) = a(x - 1/x + 2\ln x) + b$. The second solution is $(x - 1/x + 2\ln x)/(x+1)$. By variation of constants, $y_p = c(x)/(x+1) + d(x)(x - 1/x + 2\ln x)/(x+1)$, where $c(x) = \int -x(x+1)(x - 1/x + 2\ln x)dx = -1/4x^4 + x^2 - 2/3x^3\ln x - 1/9x^3 + x - x^2\ln x$ and $d(x) = \int x(x+1)dx = 1/2x^2 + 1/3x^3$.

44) Guess $y_1 = x$. Compute the second solution: $x^3v' - (x^4 + x^2)v = 0$, $v = xe^{x^2/2}$, $z = e^{x^2/2}$, so $y_2(x) = xe^{x^2/2}$.

45) Guess $y_1 = e^x$. Compute the second solution: $y_2(x) = e^x \int_1^x e^{-2t}/t\,dt$ on $(0, +\infty)$. Because the integral diverges as $x \to 0$, this solution cannot be extended to $x = 0$. Thus, all solutions defined on $\mathbb{R}$ are $y(x) = c \cdot e^x$.

46) Guess $y_1 = x$. Compute the second solution: $(x^2+1)v' + (x + 2x(1 + x^2)^{-1})v = 0$, giving $v = x^{-2}(1+x^2)^{-1/2}$, $z = -\sqrt{1+x^2}/x$, so $y_2(x) = -\sqrt{1+x^2}$.

47) Guess $y_1 = x$. Then $y_2(x) = x^2 - 1$. The solution satisfying the initial condition is $y(x) = 1 - x^2$.

Euler's Equations

An equation of the form

$$\mathscr{E}[y] = f(x)\,, \qquad \mathscr{E}[y] = \sum_{k=0}^{n} b_k x^{n-k} y^{(n-k)}\,, \tag{17}$$

with $b_k \in \mathbb{C}$ and $b_0 \neq 0$, is called an Euler (or Cauchy-Euler) differential equation.

This is a special class of n-th order linear equations with variable coefficients, where the coefficient of the $(n-k)$-th derivative is $b_k x^{n-k}$. The leading coefficient is non-zero on the intervals $(-\infty, 0)$ and $(0, \infty)$, where the equation is considered. Euler's equations are one of the few classes of variable-coefficient equations for which a fundamental system can always be found explicitly.

First Method of Solution

We seek solutions of the form $y(x) = |x|^\lambda$. We obtain $\mathscr{E}[y] = |x|^\lambda p(\lambda)$, where

$$p(\lambda) = \sum_{k=0}^{n} \lambda(\lambda-1)\ldots(\lambda-(n-k)+1)b_k \tag{18}$$

is the characteristic polynomial of equation (17). Specifically: y corresponds to 1, xy' corresponds to λ, x^2y'' corresponds to $\lambda(\lambda-1)$, x^3y''' corresponds to $\lambda(\lambda-1)(\lambda-2)$, etc.

Thus: if λ_0 is a root of $p(\lambda)$ of multiplicity k, then the functions

$$|x|^{\lambda_0},\ |x|^{\lambda_0}\ln|x|,\ \ldots,|x|^{\lambda_0}\ln^{k-1}|x| \tag{19}$$

solve the homogeneous problem $\mathscr{E}[y]=0$. Taking the functions of type (19) for all roots of the characteristic polynomial yields a fundamental system.

In the case of complex conjugate roots $\lambda = a \pm ib$, we replace the pair of functions $|x|^{a\pm ib}$ with their real and imaginary parts:

$$|x|^a \sin(b\ln|x|)\,, \quad |x|^a\cos(b\ln|x|)\,.$$

Example 8. Find all solutions of the equation

$$x^3y^{(3)} - 2xy' + 4y = 0.$$

Solution. The characteristic polynomial is

$$\lambda(\lambda-1)(\lambda-2) - 2\lambda + 4 = \lambda^3 - 3\lambda^2 + 4 = (\lambda+1)(\lambda-2)^2\,.$$

The fundamental system consists of the functions $|x|^{-1}$, x^2, and $x^2\ln|x|$.

Second Method of Solution — Substitution

Using the substitution $z(t) := y(e^t) = y(x)$, we transform the Euler equation into a linear equation with constant coefficients.

Example 9. Solve the equation

$$x^2y'' + y = 0.$$

Solution. The substitution $z(t) := y(e^t) = y(x)$ on the interval $x \in (0,+\infty)$ gives $z'(t) = y'(e^t)e^t = xy'(x)$, $z''(t) = y''(e^t)e^{2t} + y'(e^t)e^t = x^2y''(x) + z'(t)$. Substituting into the equation gives

$$z'' - z' + z = 0.$$

The characteristic polynomial is

$$\lambda(\lambda-1) + 1 = \lambda^2 - \lambda + 1\,.$$

The roots are $\lambda = 1/2 \pm i\sqrt{3}/2$, and the real fundamental system consists of $e^{t/2}\cos(\sqrt{3}t/2)$, $e^{t/2}\sin(\sqrt{3}t/2)$, which after back-substitution yields

$$\sqrt{x}\sin(\frac{\sqrt{3}}{2}\ln x)\,,\ \sqrt{x}\cos(\frac{\sqrt{3}}{2}\ln x)\,.$$

On $(-\infty,0)$, we use the substitution $z(t) = y(-e^t)$. Altogether, the fundamental system is

$$\sqrt{|x|}\sin(\frac{\sqrt{3}}{2}\ln|x|)\,,\ \sqrt{|x|}\cos(\frac{\sqrt{3}}{2}\ln|x|)$$

on $(-\infty,0)$ and $(0,+\infty)$.

Non-homogeneous equations can be solved using variation of constants or the method of undetermined coefficients for special right-hand sides.

Theorem 7 (Special Right-Hand Side for Euler Equations)**.** *Let*

$$f(x) = |x|^{\alpha}\big[q_1(\ln|x|)\cos(\beta\ln|x|) + q_2(\ln|x|)\sin(\beta\ln|x|)\big], \qquad (20)$$

where q_1, q_2 are polynomials of degree $\le m$. Let $k \ge 0$ denote the multiplicity of $\lambda = \alpha + \beta i$ as a root of the characteristic polynomial (18) *of the Euler equation* (17)*.*

Then there exist polynomials r_1, r_2 of degree $\le m$ such that

$$y(x) = (\ln|x|)^k|x|^{\alpha}\big[r_1(\ln|x|)\cos(\beta\ln|x|) + r_2(\ln|x|)\sin(\beta\ln|x|)\big]$$

is a solution of (17) *on the interval* $(0, +\infty)$*.*

Example 10. Find all solutions of the equation

$$x^3y''' - 2xy' + 4y = x^2.$$

Solution. From Example 1, the fundamental system is $|x|^{-1}$, x^2, and $x^2\ln|x|$. The characteristic polynomial has a simple root -1 and a double root 2. The right-hand side is of special form with $\alpha + i\beta = 2$, so $k = 2$. The polynomials q_1, q_2 have degree 0. Thus, a particular solution exists in the form $y_p(x) = c\cdot(\ln|x|)^2x^2$. Substituting into the equation yields $c = 1/6$, so the general solution is

$$y(x) = \frac{1}{6}x^2\ln^2|x| + \frac{a}{|x|} + bx^2 + cx^2\ln|x|, \quad a, b, c \in \mathbb{R}$$

on $(0, +\infty)$ or $(-\infty, 0)$.

Exercises

48. Complete Example 9 in detail on the interval $(-\infty, 0)$.

49. $x^2y'' - 2xy' + 2y = 0$

50. $x^2y'' - 5xy' + 8y = 0$

51. $2x^2y'' + 3xy' - y = 0$

52. $9x^2y'' + 9xy' - y = 0$

53. $x^2y'' + 7xy' + 9y = 0$

54. $x^2y'' + xy' = 0$

55. $x^2y'' - xy' + 2y = 0$

56. $x^3y''' + 9x^2y'' + 18xy' + 6y = 0$

57. $x^3y''' + 3x^2y'' = 0$

58. $x^3y''' - 3x^2y'' + 7xy' - 8y = 0$

59. $x^3y''' + 3x^2y'' + 2xy' = 0$

60. $x^3y''' + 4x^2y'' - xy' - 3y = 0$

61. $x^3y''' - 7x^2y'' - 9xy' = 0$

62. $x^3y''' + 5x^2y'' + 5xy' + y = 0$

63. $x^3y''' + 7x^2y'' + 13xy' = 0$

64. $x^3y''' - 4x^2y'' + 19xy' - 39y = 0$

65. $x^2y'' + (3i + 1)xy' - 2y = 0$

66. $x^2y'' + (1 - 20i)xy' - 100y = 0$

67. $x^2y'' + 2\sqrt{2}xy' + 3y = 0$

68. $x^2y'' + 2xy' - 12y = 17 \ln x \cos(\ln x)$

69. $2x^2y'' - 3xy' + 3y = x\sqrt{x}$

70. $x^2y'' + xy' + y = -2 \sin(\ln x)$

71. $x^2y'' - 6y = -16x^2 \ln x$

72. Find a particular solution of the equation $x^2y'' - xy' + y = \frac{x}{1+\ln x}$.

73. Find a particular solution of the equation $x^3y''' + xy' - y = \frac{x}{\ln^2 x+1}$.

74. Verify in detail that the first solution method indeed yields a fundamental system of solutions.

75. Prove Theorem 7.

76. Theorem 7 also holds on the interval $(-\infty, 0)$. Are the polynomials r_1, r_2 on $(-\infty, 0)$ the same as on $(0, +\infty)$ (if the right-hand side is given by (20) for all $x \in \mathbb{R} \setminus \{0\}$)? Prove your answer.

Solutions

48) ...

49) $\lambda^2 - 3\,\lambda + 2$, $\{x, x^2\}$.

50) $\lambda^2 - 6\,\lambda + 8$, $\{x^2, x^4\}$.

51) $2\,\lambda^2 + \lambda - 1$, $\{1/x, \sqrt{|x|}\}$.

52) $9\,\lambda^2 - 1$, $\{\sqrt[3]{x}, 1/\sqrt[3]{x}\}$.

53) $\lambda^2 + 6\,\lambda + 9$, $\{x^{-3}, x^{-3} \ln |x|\}$.

54) λ^2, $\{1, \ln |x|\}$.

55) $\lambda^2 - 2\,\lambda + 2$, $\{x \sin(\ln |x|), x \cos(\ln |x|)\}$.

56) $\lambda^3 + 6\,\lambda^2 + 11\,\lambda + 6$, $\{x^{-1}, x^{-2}, x^{-3}\}$.

57) $\lambda^3 - \lambda$, $\{x, 1/x, 1\}$.

58) $\lambda^3 - 6\,\lambda^2 + 12\,\lambda - 8$, $\{x^2, x^2 \ln x, x^2 \ln^2 x\}$.

59) $\lambda^3 + \lambda$, $\{1, \sin(\ln|x|), \cos(\ln|x|)\}$.

60) $\lambda^3 - 3\,\lambda + \lambda^2 - 3$, $\{1/x, |x|^{\sqrt{3}}, |x|^{-\sqrt{3}}\}$.

61) $\lambda^3 - 10\,\lambda^2$, $\{1, \ln|x|, x^{10}\}$.

62) $\lambda^3 + 2\,\lambda^2 + 2\,\lambda + 1$, $\{1/x, |x|^{-1/2}\sin(\sqrt{3}/2 \cdot \ln|x|), |x|^{-1/2}\cos(\sqrt{3}/2 \cdot \ln|x|)\}$.

63) $\lambda^3 + 4\,\lambda^2 + 8\,\lambda$, $\{1, \frac{\sin(\ln x^2)}{x^2}, \frac{\cos(\ln x^2)}{x^2}\}$.

64) $\lambda^3 - 7\,\lambda^2 + 25\,\lambda - 39$, $\{x^3, x^2\sin(\ln x^3), x^2\cos(\ln x^3)\}$.

65) $\lambda^2 + 3i\lambda - 2$, $\{|x|^{-i}, |x|^{-2i}\}$.

66) $\lambda^2 - 20i\lambda - 100$, $\{|x|^{10i}, |x|^{10i}\ln|x|\}$.

67) $\lambda^2 - 2\sqrt{2}\lambda + 3 = 0$, $\{|x|^{-\sqrt{2}}\sin(\ln x), |x|^{-\sqrt{2}}\cos(\ln x)\}$.

68) The characteristic polynomial $\lambda^2 + \lambda - 12$ has roots 3 and -4, so the fundamental system is x^3 and x^{-4}. We seek a particular solution of the form $y_p(x) = (a\ln x + b)\cos(\ln x) + (c\ln x + d)\sin(\ln x)$. This gives $c = 1/10$, $a = -13/10$, $b = -29/425$, and $d = 181/850$. All solutions on $(0, +\infty)$ are

$$\left(\frac{1}{10}\ln x + \frac{181}{850}\right)\sin(\ln x) - \left(\frac{13}{10}\ln x + \frac{29}{425}\right)\cos(\ln x) + cx^3 + \frac{d}{x^4}.$$

69) The characteristic polynomial $2\lambda^2 - 5\lambda + 3$ has roots 1 and $3/2$, so the fundamental system is x and $x\sqrt{x}$. We seek a particular solution of the form $y_p(x) = ax^{3/2}\ln x$ ($\alpha = 3/2$, $\beta = 0$, $k = 1$, $m = 0$). This gives $a = 1$, so all solutions on $(0, +\infty)$ are

$$x\sqrt{x}\ln x + cx + dx\sqrt{x}.$$

70) The characteristic polynomial $\lambda^2 + 1$ has roots $\pm i$, so the fundamental system is $\cos(\ln x)$, $\sin(\ln x)$. We seek a particular solution of the form $y_p(x) = \ln x(a\sin(\ln x) + b\cos(\ln x))$ ($\alpha = 0$, $\beta = 1$, $k = 1$, $m = 0$). This gives $a = 1$, $b = 0$, so all solutions on $(0, +\infty)$ are

$$\ln x\sin(\ln x) + c\sin(\ln x) + d\cos(\ln x).$$

71) The characteristic polynomial $\lambda^2 - \lambda - 6$ has roots 3 and -2, so the fundamental system is x^3, $1/x^2$. We seek a particular solution of the form $y_p(x) = (a\ln x + b)x^2$ ($\alpha = 2$, $\beta = 0$, $k = 0$, $m = 1$). This gives $a = 4$, $b = 3$, so all solutions on $(0, +\infty)$ are

$$(4\ln x + 3)x^2 + cx^{-2} + dx^3.$$

72) Euler equation; substitute $x = e^t$, obtaining

$$z'' - 2z' + z = \frac{e^t}{t+1}.$$

We seek a solution in the form $c(t)e^t + d(t)te^t$. By variation of constants, $c(t) = -\int \frac{t}{t+1}dt = -t + \ln|t+1|$, $d(t) = \int \frac{1}{t+1}dt = \ln|t+1|$. Thus,

$$y(x) = x(\ln|1 + \ln x| - \ln x) + x \ln x \ln|1 + \ln x| + cx + dx \ln x.$$

73) Euler equation; substitute $x = e^t$, obtaining

$$z''' - 3z'' + 3z' - z = \frac{e^t}{t^2+1}.$$

We seek a solution in the form $c(t)e^t + d(t)te^t + f(t)t^2e^t$. By variation of constants, $f(t) = \int \frac{1}{2}\ \frac{1}{t^2+1}dt = \frac{1}{2}\operatorname{arctg} t$, $d(t) = \int -\frac{t}{t^2+1}dt = -\frac{1}{2}\ln|t^2+1|$, $c(t) = \int \frac{1}{2}\ \frac{t^2}{t^2+1}dt = \frac{1}{2}(t - \operatorname{arctg} t)$. Thus,

$$y_p(x) = \frac{1}{2}x(\ln x - \operatorname{arctg} \ln x) - \frac{1}{2}x \ln x \ln(\ln^2 x + 1) + \frac{1}{2}x \ln^2 x \operatorname{arctg} \ln x.$$

74) One must show that all functions $|x|^\lambda \ln^r |x|$ solve the equation and are linearly independent. The proof is analogous to that for constant-coefficient equations.

75) Apply the substitution $z(t) = y(e^t)$ and use the theorem on special right-hand sides for constant-coefficient equations.

76) If y_1 is a solution on $(0, +\infty)$ and we define $y_2(x) := y_1(-x)$ for $x < 0$, we have

$$\mathscr{E}[y_2](x) = \mathscr{E}[y_1(-x)] = f(-x) = f(x),$$

so the polynomials r_1, r_2 are identical on both the positive and negative half-lines.