Problems on the Integrating Factor

An "integrating factor" is a suitably chosen function by which we multiply the given equation so that it becomes the derivative (more generally, the total differential) of "something". This yields an (implicitly defined) solution.

Solving first-order linear equations using an integrating factor reduces to computing two antiderivatives. Bernoulli equations can be formally and easily converted into first-order linear equations.

A generalization of first-order linear ODEs are equations in the form of a total differential (exact equations).

First-Order Linear ODEs

Definition 1. By a first-order linear ODE, we mean an equation of the form

$$y' + a(x)y = b(x)\,. \tag{1}$$

The equation is considered *linear* because the dependence (of the left-hand side) on the unknown function $y = y(x)$ is linear; the coefficients of the equation $a(x)$, $b(x)$ are generally *nonlinear* functions of x.

Theorem 1 (Solution of a first-order linear ODE)**.** *Consider equation* (1)*. Let* $a(x)$*,* $b(x)$ *be continuous on* I*, let* $A(x) = \int a(x)\,dx$*, and let* $B(x) = \int b(x)\exp A(x)\,dx$ *on* I*. Let* $c \in \mathbb{R}$ *be arbitrary. Then*

$$y(x) = \exp(-A(x))\big[B(x) + c\big]$$

is a solution of equation (1) *on* I*. Conversely, all solutions of equation* (1) *on* I *have this form.*

Remark. The expression $\exp A(x)$ is called the integrating factor. The proof of the theorem itself is very simple (and it is useful to know it when solving problems): We multiply the equation by $\exp A(x)$, and since $\big(\exp A(x)\big)' = a(x)\exp A(x)$, we obtain

$$\begin{aligned} y'(x)\exp A(x) + y(x)\big(\exp A(x)\big)' &= b(x)\exp A(x) \\ \big(y(x)\exp A(x)\big)' &= B'(x) \end{aligned}$$

We use the fact that $f'(x) = g'(x)$ on I if and only if $f(x) = g(x) + c$ for some $c \in \mathbb{R}$.

Remark. Since this is a linear problem, we can also use the *method of variation of constants*. The corresponding homogeneous equation is

$$y' = -a(x)y;$$

its general solution is thus of the form $y = c\exp(-A(x))$, where $A(x) = \int a(x)\,dx$ as above. Now we seek a particular solution in the form

$$y_p = c(x)\exp(-A(x)).$$

Since $y_p' = c'(x)\exp(-A(x)) - c(x)a(x)\exp(-A(x))$, substituting into the equation and simplifying gives

$$c'(x) = b(x)\exp A(x)$$

hence $c(x) = \int b(x)\exp A(x)\,dx$. The resulting solution has the same form as in Theorem 1.

Example 1. Equation $y' + y\cos x = (1+x)e^{-\sin x}$.

Solution. The integrating factor is $e^{\sin x}$, and after multiplying, the equation becomes

$$\begin{aligned} y'e^{\sin x} + y\cos x e^{\sin x} &= 1 + x \\ \left(ye^{\sin x}\right)' &= \left(x + \frac{x^2}{2}\right)'; \end{aligned}$$

whence the general solution is $y(x) = \left(x + \frac{x^2}{2} + C\right)e^{-\sin x}$, $x \in \mathbb{R}$.

Example 2. Equation $xy' - 3y = x^4$.

Solution. We write it in the form (1), which is possible on the intervals $(-\infty, 0)$ and $(0, +\infty)$:

$$y' - \frac{3}{x}y = x^3. \tag{2}$$

Since

$$\int \frac{-3}{x}\,dx = -3\ln|x| = \ln|x|^{-3},$$

we obtain the integrating factor $|x|^{-3}$; however, we prefer to work with the function x^{-3} (which is merely a sign change on the interval $(-\infty, 0)$). Multiplying equation (2) yields

$$\begin{aligned} \frac{y'}{x^3} - \frac{3}{x^4}y &= 1 \\ \left(\frac{y}{x^3}\right)' &= x' \\ \frac{y}{x^3} &= x + C \\ y &= x^3(x + C) \end{aligned}$$

The calculation was carried out under the assumption that $x \neq 0$; the validity of the equation is thus guaranteed on the intervals $(-\infty, 0)$ and $(0, +\infty)$. Due to the continuity of y and y', the equation is also satisfied at $x = 0$. Since $y(0) = y'(0) = 0$ regardless of C, solutions with different constants can be glued together at the origin. The general solution is therefore of the form

$$y(x) = \begin{cases} x^3(x + C_1), & x < 0 \\ x^3(x + C_2), & x \geq 0 \end{cases}$$

where $C_1, C_2 \in \mathbb{R}$ are arbitrary.

Solve the first-order linear ODEs.

1. $y' + y = \sin x \exp(-x)$

2. $x^3y' - xy = 1$

3. $xy' + (1 + x)y = \exp x$

4. $xy' - y = x^3 \exp x$

5. $y' + \alpha y = \exp \beta x$

6. $y' + \frac{x}{1+x^2} y = \frac{1}{x(x^2+1)}$

Answers.

1) $y = e^{-x}(c - \cos x)$, $x \in \mathbb{R}$; i.f. e^x.

2) $y = c \exp(-1/x) + 1 - 1/x$, $x \in (-\infty, 0)$ and $(0, +\infty)$; i.f. $\exp(1/x)$. The solution cannot be extended to $x = 0$ (infinite limits).

3) $y = x^{-1}(ce^{-x} + e^x/2)$, $x \in (-\infty, 0)$ and $(0, +\infty)$; i.f. xe^x. For $c \neq -1/2$, it cannot be extended to $x = 0$. If $c = -1/2$, setting $y(0) = 1$ produces a function (smooth, C^∞) that is a solution on the entire $\mathbb{R}$.

4) $y = x(x - 1)e^x + cx$, $x \in \mathbb{R}$; i.f. $1/x$. – Validity of the equation at $x = 0$ must be checked separately. We have $y(0) = 0$ for every c; however, since $y'(0) = c - 1$, solutions for different values of c cannot be glued together.

5) For $\alpha \neq -\beta$: $y = ce^{-\alpha x} + (\alpha + \beta)^{-1}e^{\beta x}$, $x \in \mathbb{R}$; for $\alpha = -\beta$: $y = e^{-\alpha x}(c + x)$, $x \in \mathbb{R}$; i.f. $e^{\alpha x}$.

6) $y = (\sqrt{x^2 + 1})^{-1}[c + \ln \sqrt{\frac{\sqrt{x^2+1}-1}{\sqrt{x^2+1}+1}}]$, $x \in (-\infty, 0)$ and $(0, +\infty)$; i.f. $\sqrt{x^2 + 1}$.

Bernoulli Equations

Definition 2. An equation of the form

$$y' + a(x)y = b(x)y^\alpha \,, \tag{3}$$

$\alpha \in \mathbb{R}$, is called a Bernoulli equation.

The procedure for solving a Bernoulli equation is as follows: multiplying by $(1-\alpha)y^{-\alpha}$ transforms it into the linear equation

$$z' + (1-\alpha)a(x)z = (1-\alpha)b(x) \tag{4}$$

for the new unknown function $z(x) = [y(x)]^{1-\alpha}$.

We may assume that $\alpha \neq 1$ (otherwise it would already be a linear equation); furthermore, we search only for *non-zero* solutions in this way. The zero function is always a solution (if the expression $0^\alpha = 0$); any possible gluing at points where $y = 0$ must then be discussed at the end.

Example 3. The equation

$$y' - \frac{4y}{x} = x\sqrt{y}\,, \tag{5}$$

i.e., $\alpha = 1/2$; the substitution $z = y^{1-1/2} = \sqrt{y}$ leads to

$$z' - \frac{2z}{x} = \frac{x}{2}\,. \tag{6}$$

This is a first-order linear ODE, which we easily solve using the integrating factor $1/x^2$; the general solution is $z = x^2(c + \ln\sqrt{|x|})$, valid on the intervals $(-\infty, 0)$ and $(0, \infty)$.

Now, ATTENTION: back-substitution

$$y = z^2 = x^4(c + \ln\sqrt{|x|})^2 \tag{7}$$

gives a function that satisfies the original equation (5) only under the additional assumption that $c + \ln\sqrt{|x|} > 0$ (verify in detail!); i.e., on $(-\infty, -\exp(-2c))$ and $(\exp(-2c), +\infty)$.

Where is the catch: at this moment, we are returning from equation (6) back to (5), but $y = z^2$ is the inverse operation to $z = \sqrt{y}$ only under the condition $z \geq 0$; for $z < 0$, we would have $z = -\sqrt{y}$, and hence (6) would look different.

It is perhaps simpler to remember that $z = \sqrt{y}$ is necessarily non-negative, and therefore we only look for non-negative solutions of equation (6). (In fact, we must consider only positive z, because in the transition from (5) to (6) we divide by the function $z = \sqrt{y}$.)

Furthermore, we observe that $y = 0$ is a solution to the original equation (5) on the maximal intervals $(-\infty, 0)$ and $(0, +\infty)$. This solution can be glued at the points $-\exp(-2c)$ and $\exp(-2c)$.

Remark. A similar situation is not uncommon when solving ODEs: a function calculated by a purely formal, thoughtless procedure is often a solution on a smaller set than its seemingly obvious "domain of definition".

Solve the Bernoulli equations:

7. $y' - \frac{xy}{2(x^2-1)} - \frac{x}{2y} = 0$

8. $xy' + y = xy^2 \ln x$

9. $y' - 2xy = 2x^3y^2$

10. $y' - \frac{y}{x} = \frac{1}{2y}$

11. $xy' + y = y^2 \ln x$, $y(1) = 1$.

12. $y' - xy = -y^3 e^{-x^2}$

13. $y' - 9x^2y = (x^5 + x^2)y^{2/3}$, $y(0) = 0$.

14. $y' + 4xy + 2e^{-x^2}\sqrt{y} = 0$, $y(0) = 0$.

15.

$$3y' + \frac{y}{(1+x^2)(\pi/2 - \operatorname{arccotg} x)} + \frac{1}{(1+x^2)y^2} = 0, \qquad y(1) = -\sqrt[3]{\pi}/2.$$

16. $2y' - 6y \operatorname{tg} x + 3\sqrt[3]{y} \sin x = 0$, $y(0) = (1/3)^{2/3}$.

17. $y' - xy + xy^2 = 0$

18. $y' + 2y/x^2 + 2\sqrt{y}/x^2 = 0$

19. $y' + y \operatorname{cotg} x + \cos x/2y = 0$, $y(\pi/2) = 2/\sqrt{3}$.

20. $y' - y - y^2 = 0$

21.

$$3y' + \frac{2y}{x \ln x} + \frac{2}{x\sqrt{y}} = 0$$

22.

$$4y' + \frac{3y}{\sqrt{x}} + \frac{3}{\sqrt{x}\sqrt[3]{y}} = 0$$

23. $4y' + 3y + 3(\cos^2 x - 1/2)e^{-x}/\sqrt[3]{y} = 0$

Answers.

7) $y = \pm\sqrt{(x^2-1) + c\sqrt{|x^2-1|}}$, on the intervals where the expression under the square root is positive; (subs. $z = y^2$).

8) $y = 0$, $y = \frac{2}{x(2c - \ln|x|)}$, on the intervals where the denominator is non-zero; (subs. $z = 1/y$).

9) $y = 0$ and $y = (ce^{-x^2} + 1 - x^2)^{-1}$, on the intervals where the denominator is non-zero; (subs. $z = 1/y$).

10) $y = \pm\sqrt{x(c + \ln|x|)}$, $x \in (-e^{-c}, 0)$ and $x \in (0, e^{-c})$, (subs. $z = y^2$ ... only positive z).

11) $y = \frac{1}{1+\ln x}$, $x \in (1/e, \infty)$, (subs. $z = 1/y$).

12) $y = \pm(e^{-x^2}(2x + c))^{-1/2}$, $x > -c/2$, $y = 0$, $x \in \mathbb{R}$; (subs. $z = y^{-2}$).

13) $y = (ce^{x^3} - (x^3 + 2)/9)^3$, $x \in \mathbb{R}$. For a given c, there can be up to two zeros of this solution; at these points, the solution can be glued to $y = 0$, $x \in \mathbb{R}$. (subs. $z = \sqrt[3]{y}$).

14) $y = 0$, $x \in \mathbb{R}$, or $y = (x - c)^2 e^{-2x^2}$ for $x < c$ and zero elsewhere. $z = y^{1/2}$, necessarily $y \geq 0$, branching at $y = 0$.

15) $y = -\sqrt[3]{\operatorname{arctg} x/2}$, $x \in (0, \infty)$. $z = y^3$, necessarily $y \neq 0$, $x \neq 0$; note that $\pi/2 - \operatorname{arccotg} x = \operatorname{arctg} x$. General solution $y = \sqrt[3]{C/\operatorname{arctg} x - \operatorname{arctg} x/2}$.

16) $y = (\cos x/3)^{3/2}$, $x \in (-\pi/2, \pi/2)$. y is a solution $\implies$ $-y$ is a solution; $y = 0$ is a solution. $z = y^{2/3}$ under the assumption $y > 0$ yields $y = (C/\cos^2 x + \cos x/3)^{2/3}$, provided the expression inside the parentheses is positive.

17) $y = 0$, $x \in \mathbb{R}$. For $y \neq 0$, set $z = 1/y$, whence $y = 1/(Ce^{-x^2/2} + 1)$, $x \in I_C$, where $I_C = \mathbb{R}$ for $C > -1$; $I_C = (-\infty, 0)$ or $(0, \infty)$ for $C = -1$ and finally $I_C = (-\infty, -\sqrt{2\ln(-C)})$ or $(-\sqrt{2\ln(-C)}, \sqrt{2\ln(-C)})$ or $(\sqrt{2\ln(-C)}, \infty)$ for $C < -1$.

18) $y = 0$ for $x \in (-\infty, 0)$ or $(0, \infty)$. Necessarily $y \geq 0$. Set $z = \sqrt{y}$ for $y > 0$, whence $y = (Ce^{1/x} - 1)^2$, $x \in I_C$, where the expression in parentheses is positive, i.e., necessarily $C > 0$ and furthermore $I_C = (0, \infty)$ for $C \geq 1$ or $(-\infty, -1/\ln C)$ for $C > 1$ or finally $(0, -1/\ln C)$ for $C \in (0, 1)$. At the points $-1/\ln C$, the zero solution can be glued.

19) $y = \sqrt{(5 - \sin^3 x)/(3\sin^2 x)}$, $x \in (0, \pi)$. Necessarily $y \neq 0$, set $z = y^2$, whence $y = \pm\sqrt{(3C - \sin^3 x)/(3\sin^2 x)}$, $x \in I_C$ – from the condition $3C - \sin^3 x > 0$, i.e., $I_C = (0, \pi)$ for $C > 1/3$ and $I_C = (0, A)$ or $(\pi - A, \pi)$, where $A = \arcsin \sqrt[3]{3C}$ for $C \in (0, 1/3]$.

20) $y = 0$, $x \in \mathbb{R}$. For $y \neq 0$, set $z = 1/y$, whence $y = 1/(Ce^{-x} - 1)$, $x \in I_C$, where $I_C = \mathbb{R}$ for $C \leq 0$ and $I_C = (-\infty, \ln C)$ or $(\ln C, \infty)$ for $C > 0$. – Gluing is not possible (local uniqueness of solutions!)

21) Necessarily $y > 0$, $x > 0$, and $x \neq 1$. Set $z = y^{3/2}$, whence $y = ((2C - \ln^2 x)/(2\ln x))^{2/3}$, $x \in I_C$, where $I_C = (0, 1)$ for $C \leq 0$, while for $C > 0$, $I_C = (0, \exp(-\sqrt{2C}))$ or $(1, \exp\sqrt{2C})$.

22) Necessarily $x > 0$, $y \neq 0$; y is a solution $\implies$ $-y$ is a solution. For $y > 0$, set $z = y^{4/3}$, whence $y = \pm(C\exp(-2\sqrt{x}) + 1)^{3/4}$, $x \in I_C$, where $I_C = (0, \infty)$ for $C \geq 0$ and $I_C = (\ln^2(-C)/4, \infty)$ for $C < 0$.

23) Necessarily $y \neq 0$; y is a solution $\implies$ $-y$ is a solution. For $y > 0$, set $z = y^{4/3}$, whence $y = \pm(C - \sin 2x/4)^{3/4}e^{-3/4x}$, $x \in I_C + k\pi$, $k \in \mathbb{Z}$. Here $I_C = \mathbb{R}$ for $C > 1/4$, while $I_C = (A/2, \pi - A/2)$, $A = \arcsin 4C$ for $C \in [-1/4, 1/4]$.

Equations in Total Differential Form (Exact Equations)

We study the equation

$$M(x, y)\,dx + N(x, y)\,dy = 0, \tag{8}$$

where $M(x, y), N(x, y) : \Omega \to \mathbb{R}$ are continuous and $\Omega \subset \mathbb{R}^2$ is an open set.

Remark. Geometrically speaking: the solutions of (8) are curves whose tangent vector (dx, dy) is perpendicular to the vector (M, N) at every point in the plane.

Equation (8) can be understood as the combined representation of two (under certain assumptions equivalent) first-order ODEs:

$$\frac{dy}{dx} = -\frac{M(x, y)}{N(x, y)} \tag{9}$$

for the unknown function $y = y(x)$, or

$$\frac{dx}{dy} = -\frac{N(x, y)}{M(x, y)} \tag{10}$$

for the unknown function $x = x(y)$.

If there exists a function $V = V(x, y)$ such that

$$\partial_x V = M\,, \qquad \partial_y V = N \tag{11}$$

(in this sense, (8) is the total differential of V), the solutions of equations (9), (10) are implicitly given by the condition $V(x, y) = \text{const}$.

Definition 3. Equation (8) is called exact if $M, N \in C^1$ and

$$\partial_y M = \partial_x N\,. \tag{12}$$

Remark. If there exists $V \in C^2$ such that $\nabla V = (M, N)$, then (8) is necessarily exact.

The converse also holds: if (8) is exact, there exists (at least locally) $V \in C^2$ such that (11) holds.

The procedure for solving equation (8) is as follows:

1. If the equation is exact, we compute V such that (11) holds; this V determines the solution in the sense of the previous discussion.

2. If it is not exact, we multiply it by a suitable non-zero function (a so-called *integrating factor*) so that it becomes exact, and then proceed as in Step 1.

Example 4. Consider the equation $(e^x + y)\,dx + (x + 2y\cos y^2)\,dy = 0$, i.e., $M = e^x + y$, $N = x + 2y\cos y^2$. The equation is exact; we want to find $V = V(x, y)$ satisfying (11).

$$\begin{aligned} \partial_x V(x,y) &= e^x + y \\ V(x,y) &= e^x + xy + C(y) \end{aligned}$$

What is happening: we temporarily fix y and use the theorem that two functions (here $V(x, y)$ and $e^x + xy$) have the same derivative (with respect to x) and therefore differ by a constant C – which, however, depends in general on y, hence $C(y)$.

The second equation in (11) implies

$$\begin{aligned} \partial_y V &= x + C'(y) = x + 2y\cos y^2 \\ C'(y) &= 2y\cos y^2 \\ C(y) &= \sin y^2 + c \end{aligned}$$

The fact that x "miraculously" dropped out of the equation for $C(y)$ is precisely a consequence of exactness. Result: $V(x, y) = e^x + xy + \sin y^2$, so the solutions are curves implicitly defined by the equation

$$e^x + xy + \sin y^2 = c.$$

Example 5. Consider the equation $(x^2 + y)\,dx - x\,dy = 0$. The equation is not exact. We try an integrating factor of the form $\mu = \mu(x)$. The exactness condition for the multiplied equation is

$$\partial_y\big\{(x^2 + y)\mu(x)\big\} = \partial_x\big\{-x\mu(x)\big\}\,,$$

which reduces to $x\mu'(x) = -2\mu(x)$. Here we do not seek the general solution, but rather any *non-zero* solution, e.g., $\mu(x) = 1/x^2$. The multiplied equation $(1 + y/x^2)\,dx - dy/x = 0$ is exact, and we find $V = x - y/x$.

Example 6. Consider the equation $(2x^2/y + xy^2)\,dx + 3x^2y\,dy = 0$. The equation is not exact. We try an integrating factor of the form $\mu = \mu(x/y)$. The exactness condition for the multiplied equation is

$$\partial_y\big\{\mu(x/y)\big(\frac{2x^2}{y} + xy^2\big)\big\} = \partial_x\big\{\mu(x/y)3x^2y\big\}\,. \tag{13}$$

Recall that $\partial_x\mu(x/y) = \mu'(x/y)/y$ and $\partial_y\mu(x/y) = -\mu'(x/y)x/y^2$ by the chain rule. After simplification, we obtain

$$-\mu'(x/y)\cdot\frac{x}{y} = \mu(x/y)\,.$$

Thus, denoting the variable of the function μ by 't', we have $-\mu'(t)t = \mu(t)$. Again, any non-zero solution suffices, e.g., $\mu(t) = 1/t$. The integrating factor is therefore y/x.

The multiplied equation is $(2x + y^3)\,dx + 3xy^2\,dy = 0$; we easily integrate to find $V = xy^3 + x^2$.

Remark. The key is finding the integrating factor in the correct form. In standard textbook problems, μ typically works as a function of x, y, $x + y$, xy, or x/y.

Solve the equations in total differential form (exact equations).

Unless stated otherwise, search for an integrating factor as a function of x or y.

24. $\frac{dx}{y} - \frac{x\,dy}{y^2} = 0$.

25. $\frac{y\,dx}{2\sqrt{x}} + \sqrt{x}\,dy = 0$.

26. $-\frac{y\,dx}{x^2+y^2} + \frac{x\,dy}{x^2+y^2} = 0$.

27. $\cos(x+1)\,dx = 0$.

28. $\frac{\sqrt{y}\,dx}{\sqrt{-(x-1)(x+1)}} + \frac{\left(\arcsin(x)+2\sqrt{y}\right)dy}{2\sqrt{y}} = 0$.

29. $-ye^{\frac{y}{x}}x^{-2}\,dx + e^{\frac{y}{x}}x^{-1}\,dy = 0$.

30. $\left(\frac{y}{x^2} - x^{-2}\right)dx + \frac{dy}{x} = 0$.

31. $(y^2 - y)\,dx + xy\,dy = 0$.

32. $-\sin(x)\cos(y)\,dx - (\cos(y))^2\,dy = 0$.

33. $-\sin(xy)\,dx - \frac{\sin(xy)x\,dy}{y} = 0$.

34. $\left(x^{-1} + \frac{y}{x}\right)dx + (1 + x^{-1})\,dy = 0$.

35. $\left(\frac{2x}{y^2} - \frac{1}{yx^2}\right) dx + \left(3 + \frac{1}{y^2 x}\right) dy = 0.$

36. $\left(-\frac{y^2}{x^2} + 1\right) dx + \left(2\,y^3 - \frac{2x}{y}\right) dy = 0.$

37. $\frac{dx}{xy} + 2\,\frac{dy}{x} = 0,\ \mu = \mu(xy).$

38. $-y^3 x^2\,dx + \left(-x^2 - x^3 y^2 - \frac{2x^2}{y}\right) dy = 0,\ \mu = \mu(xy).$

39. $\left(-\frac{1}{x^4 y^3} + \frac{1}{x^4 y^2}\right) dx - \frac{dy}{y^4 x^3} = 0,\ \mu = \mu(xy).$

40. $\frac{dx}{xy^3} + \frac{dy}{yx^3} = 0,\ \mu = \mu(xy).$

41. $\frac{x^3 dx}{y^2} - \frac{x^2 dy}{y^5} = 0,\ \mu = \mu(x/y).$

42. $\left(-\frac{y}{x^3} + \frac{y^3}{x}\right) dx + 2\,y^2 dy = 0,\ \mu = \mu(y/x).$

43. $\frac{x\,dx}{y} + \left(-\frac{x^3}{y^5} - \frac{2x^2}{y^2}\right) dy = 0,\ \mu = \mu(x/y).$

44. $\left(x^{-1} + \frac{y^2}{x^3}\right) dx + \left(-y^{-1} - \frac{y}{x^2}\right) dy = 0,\ \mu = \mu(y/x).$

45. $\left(\frac{y}{x} - \frac{y\cos(x)}{x}\right) dx + \frac{y\,dy}{x} = 0.$

46. $\left(y^{-1} + x^{-1}\right) dx + \left(y^{-1} + x^{-1}\right) dy = 0,\ \mu = \mu(xy).$

Answers.

24) $V = \frac{x}{y}.$

25) $V = y\sqrt{x}.$

26) $V = \arctan(\frac{y}{x}).$

27) $V = \sin(x + 1).$

28) $V = \arcsin(x)\sqrt{y} + y.$

29) $V = e^{\frac{y}{x}}.$

30) $\mu = x^2,\ V = xy - x.$

31) $\mu = y^{-1},\ V = xy - x.$

32) $\mu = (\cos(y))^{-1},\ V = \cos(x) - \sin(y).$

33) $\mu = y,\ V = \cos(xy).$

34) $\mu = x,\ V = x + y + xy.$

35) $\mu = y^2,\ V = x^2 + y^3 + \frac{y}{x}.$

36) $\mu = y^{-2},\ V = x^{-1} + y^2 + \frac{x}{y^2}.$

37) $\mu = xy,\ V = x + y^2.$

38) $\mu = \frac{1}{x^2y^2}$, $V = y^{-1} - xy + y^{-2}$.

39) $\mu = x^2y^2$, $V = \frac{1}{xy} - x^{-1}$.

40) $\mu = x^3y^3$, $V = x^3 + y^3$.

41) $\mu = \frac{y^2}{x^2}$, $V = x^2 + y^{-2}$.

42) $\mu = \frac{x}{y}$, $V = x^{-1} + xy^2$.

43) $\mu = \frac{y^3}{x^3}$, $V = y^{-1} - \frac{y^2}{x}$.

44) $\mu = \frac{x}{y}$, $V = \frac{x}{y} - \frac{y}{x}$.

45) $\mu = \frac{x}{y}$, $V = x + y - \sin(x)$.

46) $\mu = xy$, $V = (x + y)^2$.