# Systems of Linear Equations

In this chapter, we will deal with *systems of linear differential equations*

$$\begin{aligned} y_1' &= a_{11}(x)y_1 + a_{12}(x)y_2 + \cdots + a_{1n}(x)y_n + f_1(x) \\ y_2' &= a_{21}(x)y_1 + a_{22}(x)y_2 + \cdots + a_{2n}(x)y_n + f_2(x) \\ &\vdots \\ y_n' &= a_{n1}(x)y_1 + a_{n2}(x)y_2 + \cdots + a_{nn}(x)y_n + f_n(x) \end{aligned} \tag{1}$$

where $a_{ij}$, $f_i : I \to \mathbb{R}$, $i, j \in \{1, 2, \ldots, n\}$ are continuous functions on $I$. We will mostly use the matrix notation

$$Y' = A(x)Y + F(x), \tag{2}$$

where

$$A(x) = \begin{pmatrix} a_{11}(x) & a_{12}(x) & \ldots & a_{1n}(x) \\ a_{21}(x) & a_{22}(x) & \ldots & a_{2n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(x) & a_{n2}(x) & \ldots & a_{nn}(x) \end{pmatrix}, F(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{pmatrix}, \tag{3}$$

and $Y = (y_1, y_2, \ldots, y_n)^T$ is the vector of unknowns. If the functions $a_{ij}$ are independent of $x$, it is a *system of linear differential equations with constant coefficients*:

$$Y' = AY + F(x). \tag{4}$$

If $F(x) = 0$, $x \in I$, it is a *homogeneous system*:

$$Y' = A(x)Y. \tag{5}$$

If we supplement the system of equations with an *initial condition*

$$Y(a) = B, \tag{6}$$

where $a \in I$ and $B = (b_1, b_2, \ldots, b_n)^T \in \mathbb{R}^n$, we speak of an *initial value problem*.
Before showing how to solve systems of equations, we present several theorems that specify the structure of the solutions.

**Theorem 1** (Existence and Uniqueness)**.** *For each $a \in I$ and $B = (b_1, b_2, \ldots, b_n)^T \in \mathbb{R}^n$, there exists a unique maximal solution to the initial value problem* (2), (6)*. This solution is defined on the entire interval $I$.*

**Theorem 2** (Solution Space)**.** *1. The set of all maximal solutions to the homogeneous equation* (5) *forms an n-dimensional vector space.*

2. *If $Y_p$ is a maximal solution to the non-homogeneous equation* (2)*, then the set of all maximal solutions to equation* (2) *is*

$$\{Y_p + Y : Y \text{ is a maximal solution to } (5)\}.$$

A basis of the space of all maximal solutions from the previous theorem is called a *fundamental system* of equation (5). A matrix whose columns form the elements of a fundamental system is called a *fundamental matrix*. The general solution to the system (5) is then $Y(x) = \Phi(x)c$, $c \in \mathbb{R}^n$.
In this chapter, we will primarily deal with how to find a fundamental system of a homogeneous system, specifically for systems with constant coefficients. Finding a fundamental system for systems with non-constant coefficients is difficult to impossible. A brief overview of such problems can be found in the chapter on variation of parameters, where we also address finding the *particular solution* $Y_p$ from Theorem 2. In this chapter, we will only solve problems with right-hand sides of a special form, for which a particular solution can be found more easily.
In this part, we will focus on systems of equations with constant coefficients, i.e., primarily on the homogeneous system

$$Y' = AY, \tag{7}$$

where $A \in \mathbb{R}^{n\times n}$. We will demonstrate several methods for solving homogeneous (as well as non-homogeneous) systems with constant coefficients. We begin with the most elementary method and proceed to more sophisticated techniques.

## Solving Systems via Elimination

We manipulate the system of equations in such a way as to obtain a single equation (typically of higher order) in one unknown. In this manner (as we will see in the examples), it is also possible to solve non-homogeneous equations, provided that the function $F$ is sufficiently smooth. This method is suitable for systems of 2 equations, or when the system matrix is sufficiently sparse (contains many zeros). For larger systems (dense matrices), this approach becomes rather unwieldy.

**Example 1.** Find all maximal solutions of the system

$$\begin{aligned} y' &= 3y - 5z - 3e^x \\ z' &= y - z - e^x \end{aligned} \tag{8}$$

*Solution.* Differentiate the second equation (we therefore require that the "right-hand side", i.e., $-3e^x$, $-e^x$, is differentiable):

$$z'' = y' - z' - e^x \,.$$

From the first equation, subtract three times the second equation:

$$y' - 3z' = -2z \,.$$

Adding the two newly obtained equations and subtracting $y'$ from both sides yields

$$z'' - 3z' = -z' - 2z - e^x \,, \quad \text{or} \quad z'' - 2z' + 2z = -e^x \,. \tag{9}$$

The solutions to the homogeneous equation $z''-2z'+2z = 0$ are the functions

$$z(x) = ce^x \cos x + de^x \sin x \,, \quad x \in \mathbb{R} \,, c, d \in \mathbb{R}.$$

Since the right-hand side $-e^x$ is of a special form, there exists a particular solution of the form $z_p(x) = ce^x$. Substituting into the equation gives $c = -1$; the solutions to equation (9) are thus

$$z(x) = ce^x \cos x + de^x \sin x - e^x \,, \quad x \in \mathbb{R} \,, c, d \in \mathbb{R} \,.$$

From the second equation of the original system, we compute

$$y(x) = z'(x) + z(x) + e^x = (2c + d)e^x \cos x + (2d - c)e^x \sin x - e^x \,, \quad x \in \mathbb{R} \,.$$

**Exercises on Direct Elimination for Systems of Equations**

**1.**

$$\begin{aligned} x' &= 2x + y \,, \\ y' &= 2y \,. \end{aligned}$$

**2.**

$$\begin{aligned} x' &= 2x + y \,, \\ y' &= 3x + 4y \,. \end{aligned}$$

**3.**

$$\begin{aligned} x' &= -5x - 6y \,, \\ y' &= 8x + 9y \,. \end{aligned}$$

**4.**

$$\begin{aligned} x' &= -5x - 4y\,, \\ y' &= 10x + 7y\,. \end{aligned}$$

**5.**

$$\begin{aligned} x' &= 5x + 4y\,, \\ y' &= -9x - 7y\,. \end{aligned}$$

**6.**

$$\begin{aligned} x' &= -5x - y\,, \\ y' &= x - 3y - 9e^{2t}\,. \end{aligned}$$

**7.**

$$\begin{aligned} x' &= x + 3y + t\,, \\ y' &= x - y\,. \end{aligned}$$

**8.**

$$\begin{aligned} x' &= -4x - 4y + 2e^{2t}\,, \\ y' &= 6x + 6y + 2t\,. \end{aligned}$$

**9.**

$$\begin{aligned} x' &= 3x + y + e^{t}\,, \\ y' &= -4x - 2y + te^{t} \\ x(0) &= y(0) = 0\,. \end{aligned}$$

**10.**

$$\begin{aligned} x' &= -2x + y \\ y' &= -4x + 2y \end{aligned}$$

**11.**

$$\begin{aligned} x' &= -5x + 4y \\ y' &= -9x + 7y \end{aligned}$$

**12.**

$$\begin{aligned} x' &= x+y+z\,, \\ y' &= x+y-z\,, \\ z' &= 2x-y\,. \end{aligned}$$

**13.**

$$\begin{aligned} x' &= 2x-y+3z \\ y' &= -2x+y+5z \\ z' &= -x-y+6z \end{aligned}$$

**Solutions**

**1)** From the second equation, $y(t) = ce^{2t}$; from the first equation, $x' - 2x = ce^{2t}$. The solution of the equation with a special right-hand side is $x(t) = de^{2t} + x_p(t)$, where $x_p(t) = cte^{2t}$, $t \in \mathbb{R}$, $c, d \in \mathbb{R}$.

**2)** After elimination, $x''-6x'+5x = 0$, i.e., $\lambda_1 = 1$, $\lambda_2 = 5$, $x(t) = ce^t+de^{5t}$; from the first equation, $y(t) = -ce^t + 3de^{5t}$, $t \in \mathbb{R}$, $c, d \in \mathbb{R}$.

**3)** After elimination, we obtain $-x'' + 4x' - 3x = 0$, i.e., $\lambda_1 = 1$, $\lambda_2 = 3$, $x(t) = ce^t + de^{3t}$, and subsequently from the first equation, $y(t) = -ce^t - \frac{4}{3}de^{3t}$, $t \in \mathbb{R}$, $c, d \in \mathbb{R}$.

**4)** After elimination, we obtain $-x'' + 2x' - 5x = 0$, i.e., $\lambda_{1,2} = 1 \pm 2i$, $x(t) = ae^t \cos 2t + be^t \sin 2t$, and subsequently $y(t) = (-\frac{3}{2}a - \frac{1}{2}b)e^t \cos 2t + (\frac{1}{2}a - \frac{3}{2}b)e^t \sin 2t$, $t \in \mathbb{R}$, $a, b \in \mathbb{R}$.

**5)** After elimination, we have $x'' + 2x' + x = 0$, i.e., $\lambda_{1,2} = -1$, $x(t) = ae^{-t} + bte^{-t}$, and subsequently $y(t) = (-\frac{3}{2}a + \frac{1}{4}b - \frac{3}{2}bt)e^{-t}$, $t \in \mathbb{R}$, $a, b \in \mathbb{R}$.

**6)** After elimination, we get $x'' + 8x' + 16x = 9e^{2t}$, i.e., $\lambda_{1,2} = -4$; the particular solution (using the method of undetermined coefficients) is $\frac{1}{4}e^{2t}$, so $x(t) = ae^{-4t}+bte^{-4t}+\frac{1}{4}e^{2t}$, and subsequently $y(t) = (-a-b-bt)e^{-4t}-\frac{7}{4}e^{2t}$, $t \in \mathbb{R}$, $a, b \in \mathbb{R}$.

**7)** After elimination, $y'' - 4y = t$, i.e., $\lambda_{1,2} = \pm 2$; the particular solution is $-t/4$, so $y(t) = ae^{2t} + be^{-2t} - t/4$. From the second equation, $x(t) = 3ae^{2t} - be^{-2t} - 1/4(t + 1)$, $t \in \mathbb{R}$, $a, b \in \mathbb{R}$.

**8)** After elimination, we get $-x'' + 2x' = 8e^{2t} + 8t$, i.e., $\lambda_1 = 0, \lambda_2 = 2$; the particular solution (using undetermined coefficients) is $-4te^{2t} + 2t^2 + 2t$, so $x(t) = a + be^{2t} - 4te^{2t} + 2t^2 + 2t$, and subsequently $y(t) = 6te^{2t} + e^{2t}(\frac{3}{2} - \frac{3}{2}b) - 2t^2 - 3t - \frac{1}{2} - a$, $t \in \mathbb{R}$, $a, b \in \mathbb{R}$.

**9)** After elimination, we get $x'' - x' - 2x = 3e^t + te^t$, i.e., $\lambda_1 = 2, \lambda_2 = -1$; the particular solution (using undetermined coefficients) is $-\frac{7}{4}e^t - \frac{1}{2}te^t$, so the general solution is $x(t) = ae^{2t} + be^{-t} - \frac{7}{4}e^t - \frac{1}{2}te^t$, and subsequently $y(t) = -ae^{2t} - 4be^{-t} + 2e^t + te^t$, $t \in \mathbb{R}$, $a, b \in \mathbb{R}$. The solution satisfying the initial conditions is $x(t) = \frac{5}{3}e^{2t} + \frac{1}{12}e^{-t} - \frac{7}{4}e^t - \frac{1}{2}te^t$, $y(t) = -\frac{5}{3}e^{2t} - \frac{1}{3}e^{-t} + 2e^t + te^t$, $t \in \mathbb{R}$.

**10)**

$$\begin{bmatrix} 1-2\,t & t \\ -4\,t & 1+2\,t \end{bmatrix} \tag{10}$$

**11)**

$$\begin{bmatrix} e^t - 6\,te^t & 4\,te^t \\ -9\,te^t & e^t + 6\,te^t \end{bmatrix} \tag{11}$$

**12)** Stepwise elimination yields $x''' - 2x'' - 3x' + 6x = 0$. The roots of the characteristic polynomial are $2, \pm\sqrt{3}$, so $x(t) = ae^{2t} + be^{\sqrt{3}t} + ce^{-\sqrt{3}t}$. The functions $y$ and $z$ can be determined, for instance, from the equations $z = -x'' + x' + 3x$, $y = x' - x - z$, obtained during the computation. This gives $z(t) = ae^{2t} + \sqrt{3}be^{\sqrt{3}t} - \sqrt{3}ce^{-\sqrt{3}t}$ and $y(t) = -be^{\sqrt{3}t} - ce^{-\sqrt{3}t}$, $t \in \mathbb{R}$, $a, b, c \in \mathbb{R}$.

**13)**

$$\begin{bmatrix} 1/2\,e^{2\,t} - 1/2\,e^{4\,t} + e^{3\,t} & 1/2\,e^{2\,t} - 1/2\,e^{4\,t} & 2\,e^{4\,t} - e^{3\,t} - e^{2\,t} \\ -e^{3\,t} + 3/2\,e^{2\,t} - 1/2\,e^{4\,t} & 3/2\,e^{2\,t} - 1/2\,e^{4\,t} & 2\,e^{4\,t} + e^{3\,t} - 3\,e^{2\,t} \\ 1/2\,e^{2\,t} - 1/2\,e^{4\,t} & 1/2\,e^{2\,t} - 1/2\,e^{4\,t} & -e^{2\,t} + 2\,e^{4\,t} \end{bmatrix}$$

## Solving Systems via $\lambda$-Matrix Row Operations

This method differs from the previous one only in the notation used. Thanks to a clearer formalism, it allows solving larger systems as well. Let $\lambda$ denote the differentiation operator, i.e., $\lambda z := z'$, $\lambda^2 z = z''$, .... Using this notation, instead of performing operations with differential equations, we can perform row operations on a matrix containing polynomials in the variable $\lambda$. Specifically, these operations are:

- swapping rows of the matrix,
- multiplying a row of the matrix by a non-zero scalar,

- adding a $P(\lambda)$-multiple of one row to another row, where $P$ is a polynomial.

The matrix can easily be reduced to upper triangular form using Gaussian elimination. Furthermore, we can perform these operations on the augmented matrix including the column of right-hand sides, thereby solving non-homogeneous equations directly.
**Caution**: It is not permitted to multiply a row by a non-constant polynomial in $\lambda$, as this would increase the order of the system. Likewise, division of polynomials is not allowed.

**Example 2.** Find all maximal solutions of the system

$$\begin{aligned} x' &= 2x + y - z\,, \\ y' &= 7x + 4y - z\,, \\ z' &= 13x + 7y - 3z\,. \end{aligned}$$

*Solution.* Using operator notation in $\lambda$, we obtain

$$\begin{aligned} \lambda x - 2x - y + z &= 0\,, \\ -7x + \lambda y - 4y + z &= 0\,, \\ -13x - 7y + \lambda z + 3z &= 0\,. \end{aligned}$$

Let us reduce the matrix

$$\begin{pmatrix} \lambda - 2 & -1 & 1 \\ -7 & \lambda - 4 & 1 \\ -13 & -7 & \lambda + 3 \end{pmatrix}$$

Subtract the first row from the second row, and subtract $(\lambda + 3)$ times the first row from the third row:

$$\sim \begin{pmatrix} \lambda - 2 & -1 & 1 \\ -\lambda - 5 & \lambda - 3 & 0 \\ -(\lambda - 2)(\lambda + 3) - 13 & (\lambda + 3) - 7 & 0 \end{pmatrix} = \begin{pmatrix} \lambda - 2 & -1 & 1 \\ -\lambda - 5 & \lambda - 3 & 0 \\ -\lambda^2 - \lambda - 7 & \lambda - 4 & 0 \end{pmatrix}.$$

Subtract the third row from the second row, and then subtract $(\lambda - 4)$ times the second row from the third row:

$$\sim \begin{pmatrix} \lambda - 2 & -1 & 1 \\ \lambda^2 + 2 & 1 & 0 \\ -\lambda^2 - \lambda - 7 & \lambda - 4 & 0 \end{pmatrix} \sim \begin{pmatrix} \lambda - 2 & -1 & 1 \\ \lambda^2 + 2 & 1 & 0 \\ -\lambda^3 + 3\lambda^2 - 3\lambda + 1 & 0 & 0 \end{pmatrix}$$

In the last row, we now have the equation $-x''' + 3x'' - 3x' + x = 0$; the matrix directly gives the characteristic polynomial of this equation. We thus obtain the solution

$$x(t) = c_1 e^t + c_2 t e^t + c_3 t^2 e^t\,, \quad t \in \mathbb{R}\,, c_1, c_2, c_3 \in \mathbb{R}\,.$$

From the second row of the matrix, we determine $y$:

$$y(t) = -x'' - 2x = -e^t(3c_1 + 2c_2 + 2c_3 + t(3c_2 + 4c_3) + t^2 \cdot 3c_3)\,, \quad t \in \mathbb{R}\,.$$

From the first equation, we determine $z$:

$$z(t) = e^t(-2c_1 - 3c_2 - 2c_3 + t(-2c_2 - 6c_3) + t^2 \cdot (-2c_3))\,, \quad t \in \mathbb{R}\,.$$

**Example 3.** Solve the system

$$\begin{aligned} 2y'' + 3z'' - 7y - 6z &= t + 1 \\ 4y'' + 3z'' - 4y - 3z &= 2t. \end{aligned}$$

*Solution.* We write down the augmented $\lambda$-matrix with the right-hand side:

$$\left(\begin{array}{cc|c} 2\lambda^2 - 7 & 3\lambda^2 - 6 & t+1 \\ 4\lambda^2 - 4 & 3\lambda^2 - 3 & 2t \end{array}\right) \sim \left(\begin{array}{cc|c} 2\lambda^2 - 7 & 3\lambda^2 - 6 & t+1 \\ 2\lambda^2 + 3 & 3 & t-1 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|c} 2\lambda^2 - 7 - (\lambda^2 - 2)(2\lambda^2 + 3) & 0 & t + 1 - (t-1)'' + 2(t-1) \\ 2\lambda^2 + 3 & 3 & t - 1 \end{array}\right)$$

$$= \left(\begin{array}{cc|c} -2\lambda^4 + 3\lambda^2 - 1 & 0 & 3t - 1 \\ 2\lambda^2 + 3 & 3 & t - 1 \end{array}\right).$$

The roots of the characteristic polynomial in the first row are $\lambda_{1,2} = \pm 1$, $\lambda_{3,4} = \pm\sqrt{2}/2$. Thus, the solutions to the homogeneous equation

$$-2y^{(4)} + 3y'' - y = 0$$

are

$$ae^t + be^{-t} + ce^{\sqrt{2}/2t} + de^{-\sqrt{2}/2t}.$$

The particular solution to the non-homogeneous equation will be of the form $y_p(t) = rt + s$, which, upon substitution into the equation, gives $y_p(t) = -3t + 1$. Thus,

$$y(t) = ae^t + be^{-t} + ce^{\sqrt{2}/2t} + de^{-\sqrt{2}/2t} - 3t + 1\,.$$

From the second equation, we have

$$\begin{aligned} 3z(t) &= t - 1 - 2y'' - 3y = t - 1 - 2(ae^t + be^{-t} + \frac{1}{2}ce^{\sqrt{2}/2t} + \frac{1}{2}de^{-\sqrt{2}/2t}) \\ &\quad -3(ae^t + be^{-t} + ce^{\sqrt{2}/2t} + de^{-\sqrt{2}/2t} - 3t + 1) \\ &= 10t - 4 - 5ae^t - 5be^{-t} - 4ce^{\sqrt{2}/2t} - 4de^{-\sqrt{2}/2t}\,. \end{aligned}$$

**Exercises on $\lambda$-Matrix Row Operations**

**14.**

$$\begin{aligned} 5z' - 2y' + 4z - y &= e^{-t}\,, \\ z' + 8z - 3y &= 5e^{-t}\,. \end{aligned}$$

**15.**

$$\begin{aligned} z'' + y' + z &= e^{t}\,, \\ z' + y'' &= 1\,. \end{aligned}$$

**16.**

$$\begin{aligned} x' &= 5x + y - z\,, \\ y' &= x + 3y + z\,, \\ z' &= 7x + 3y + z\,. \end{aligned}$$

**17.**

$$\begin{aligned} x' &= x - 6y + 3z\,, \\ y' &= -8y + 6z\,, \\ z' &= 3x - 12y + 7z\,. \end{aligned}$$

**18.**

$$\begin{aligned} x' &= 7x - 10y - 4z\,, \\ y' &= 4x - 7y - 4z\,, \\ z' &= -6x + 7y + z\,. \end{aligned}$$

**19.**

$$\begin{aligned} x' &= 6x - 7y + 4z\,, \\ y' &= x + z\,, \\ z' &= -2x + 3y\,. \end{aligned}$$

**20.**

$$\begin{aligned} x' &= 2x - y + z + \cos t\,, \\ y' &= 5x - 4y + 3z + \sin t\,, \\ z' &= 4x - 4y + 3z + 2\sin t - 2\cos t\,. \end{aligned}$$

**21.**

$$\begin{aligned} x' &= x - 2y - z - 2e^t\,, \\ y' &= -x + y + z + 2e^t\,, \\ z' &= x - z - e^t\,. \end{aligned}$$

**22.**

$$\begin{aligned} x' &= 2x - 3y + t\,, \\ y' &= x - 2z - 3t^2\,, \\ z' &= -y + 2z + 3t - 2\,. \end{aligned}$$

**23.**

$$\begin{aligned} x' &= 2x + y + e^{2t}\,, \\ y' &= 2y + 4z - 4e^{-t}\,, \\ z' &= x - z\,, \\ x(0) &= 0, y(0) = -1, z(0) = 1\,. \end{aligned}$$

**Solutions**

**14)** Stepwise reduction yields $z'' + z' - 2z = -4e^{-t}$. The roots of the characteristic polynomial are 1 and $-2$; the particular solution (via undetermined coefficients) is $2e^{-t}$, so $z(t) = 2e^{-t} + ae^{-2t} + be^t$. We determine the function $y$ from $z' + 8z - 3y = 5e^{-t}$. This gives $y(t) = 3e^{-t} + 2ae^{-2t} + 3be^t$, $t \in \mathbb{R}$, $a, b \in \mathbb{R}$.

**15)** Stepwise reduction yields $-z''' = 1 - e^t$. Zero is a triple root of the characteristic polynomial; the particular solution (via undetermined coefficients) is $e^t - \frac{1}{6}t^3$, so $z(t) = e^t - \frac{1}{6}t^3 + a + bt + ct^2$ (the same result is obtained by direct integration of the reduced equation). We then compute the function $y$ from the equation $z'' + y' + z = e^t$. This gives $y'(t) = (-2-b)e^t + \frac{1}{6}t^3 - t^2 - (a+c)$, and direct integration yields $y(t) = d - t(a+c) - \frac{1}{3}t^3 + \frac{1}{24}t^4 - (2+b)e^t$, $t \in \mathbb{R}$, $a, b, c, d \in \mathbb{R}$.

**16)** Stepwise reduction yields $-x'' + 5x' - 6x = 0$. The roots of the characteristic polynomial are 2 and 3, so $x(t) = ae^{3t} + be^{2t}$. Functions $y$ and $z$ are determined from the equations $x' - 6x + y' - 4y = 0$ and $x' - 5x - y + z = 0$, obtained during the computation. From the first, we get $y' - 4y = 3ae^{3t} + 4be^{2t}$, i.e., $\lambda_1 = 4$; the particular solution is $-3ae^{3t} - 2be^{2t}$, so $y(t) = -3ae^{3t} - 2be^{2t} + ce^{4t}$. From the second, by substitution, $z(t) = -ae^{3t} + be^{2t} + ce^{4t}$, $t \in \mathbb{R}$, $a, b, c \in \mathbb{R}$.

**17)** Stepwise reduction yields $y''' + 6y' + 20y = 0$. The roots of the characteristic polynomial are $-2$ and $1 \pm 3i$, so $y(t) = ae^{-2t} + be^t \cos 3t + ce^t \sin 3t$. Functions $x$ and $z$ are determined from $y' + 8y - 6z = 0$ and $-3x + 12y + z' - 7z = 0$. This gives $z(t) = ae^{-2t} + (\frac{c}{2} + \frac{3b}{2})e^t \cos 3t + (\frac{3c}{2} - \frac{b}{2})e^t \sin 3t$, $x(t) = ae^{-2t} + \frac{c+b}{2}e^t \cos 3t + \frac{c-b}{2}e^t \sin 3t$, $t \in \mathbb{R}$, $a, b, c \in \mathbb{R}$.

**18)** Stepwise reduction yields $x''' - x'' - 5x' - 3x = 0$. The roots of the characteristic polynomial are 3 and $-1$ (double root), so $x(t) = ae^{3t} + be^{-t} + cte^{-t}$. Functions $y$ and $z$ are determined from $-x'' - 2x' + 47x - 48y = 0$ and $x' - 7x + 10y + 4z = 0$. This gives $y(t) = \frac{2}{3}ae^{3t} + be^{-t} + cte^{-t}$, $z(t) = -\frac{2}{3}ae^{3t} + (-\frac{1}{2}b - \frac{1}{4}c - \frac{1}{2}ct)e^{-t}$, $t \in \mathbb{R}$, $a, b, c \in \mathbb{R}$.

**19)** Stepwise reduction yields $y'' - 4y' + 4y = 0$. The double root of the characteristic polynomial is 2, so $y(t) = ae^{2t} + bte^{2t}$. Functions $x$ and $z$ are determined from $x' - 2x + 7y - 4y' = 0$ and $-x + y' - z = 0$. From the first, $x' - 2x = ae^{2t} + 4be^{2t} + bte^{2t}$, i.e., $\lambda_1 = 2$; the particular solution is $(a + 4b)te^{2t} + \frac{b}{2}t^2e^{2t}$, so $x(t) = ce^{2t} + (a + 4b)te^{2t} + \frac{b}{2}t^2e^{2t}$. From the second, $z(t) = (2a + b - c)e^{2t} - (a + 2b)te^{2t} - \frac{b}{2}t^2e^{2t}$, $t \in \mathbb{R}$, $a, b, c \in \mathbb{R}$.

**20)** Stepwise reduction yields $x'' - 2x' + x = -2\cos t$. The double root of the characteristic polynomial is 1; the particular solution is $\sin t$, so $x(t) = ae^t + bte^t + \sin t$. Functions $y$ and $z$ are determined from $x - 3x' + y' + y = \sin t - 3\cos t$ and $x' - 2x + y - z = \cos t$. From the first, $y' + y = (2a + 3b)e^t + 2bte^t$, i.e., $\lambda_1 = -1$; the particular solution is $(2a + b + 2bt)e^t$, so $y(t) = ce^{-t} + (2a + b + 2bt)e^t$. From the second, $z(t) = ce^{-t} + (a + 2b + bt)e^t - 2\sin t$, $t \in \mathbb{R}$, $a, b, c \in \mathbb{R}$.

**21)** Stepwise reduction yields $-x'' + 2x' = 3e^t$. The roots of the characteristic polynomial are 0 and 2; the particular solution is $3e^t$, so $x(t) = a + be^{2t} + 3e^t$. Functions $y$ and $z$ are determined from $x' + y' + y = 0$ and $x' - x + 2y + z = -2e^t$. From the first, $y' + y = -2be^{2t} - 3e^t$, i.e., $\lambda_1 = -1$; the particular solution is $-\frac{3}{2}e^t - \frac{2}{3}be^{2t}$, so $y(t) = ce^{-t} - \frac{3}{2}e^t - \frac{2}{3}be^{2t}$. From the second, $z(t) = a - 2ce^{-t} + e^t + \frac{1}{3}be^{2t}$, $t \in \mathbb{R}$, $a, b, c \in \mathbb{R}$.

**22)** Stepwise reduction yields $y'' - 2y' + y = 6t^2 - 11t + 4$. The double root of the characteristic polynomial is 1; the particular solution is $6t^2 + t - 6$, so $y(t) = ae^t + bte^t + 6t^2 + t - 6$. Functions $x$ and $z$ are determined from $y + z' - 2z = 3t - 2$ and $-x + y' + 2z = -3t^2$. From the first, $z' - 2z = -ae^t - bte^t - 6t^2 + 2t + 4$, i.e., $\lambda_1 = 2$; the particular solution is $bte^t + (a + b)e^t + 3t^2 + 4t$, so $z(t) = ce^{2t} + bte^t + (a + b)e^t + 3t^2 + 4t$. From the second, $x(t) = 3bte^t + (3a + 3b)e^t + 2ce^{2t} + 3t^2 + 20t + 1$, $t \in \mathbb{R}$, $a, b, c \in \mathbb{R}$.

**23)** Stepwise reduction yields $z''' - 3z'' = -4e^{-t}$. The roots of the characteristic polynomial are 3 and 0 (double root); the particular solution is $e^{-t}$, so $z(t) = a + bt + ce^{3t} + e^{-t}$. Functions $x$ and $y$ are determined from

$-x + z' + z = 0$ and $x' - 2x - y = e^{2t}$. This gives $x(t) = a + b + bt + 4ce^{3t}$, $y(t) = -2a - b - 2bt + 4ce^{3t} - e^{2t}$, $t \in \mathbb{R}$, $a, b, c \in \mathbb{R}$. The solution satisfying the initial conditions is $x(t) = 0$, $y(t) = -e^{2t}$, $z(t) = e^{-t}$, $t \in \mathbb{R}$.

## Solving Systems via Eigenvalues and Eigenvectors

The third method for solving homogeneous linear systems is based on the matrix exponential, which is discussed in detail in another chapter. Here, we state the key theorems that outline how to construct the fundamental system.

**Theorem 3.** *Let the matrix $A \in \mathbb{R}^{n\times n}$ have $n$ linearly independent eigenvectors $q_1, q_2, \ldots, q_n$ corresponding to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, respectively. Then the functions*

$$e^{\lambda_1 t}q_1,\ e^{\lambda_2 t}q_2,\ \ldots,\ e^{\lambda_n t}q_n \tag{12}$$

*form a fundamental system of* (7).

*Remark.* The preceding theorem does not require that $\lambda_1, \ldots, \lambda_n$ are distinct. If $A$ has $n$ distinct eigenvalues, it definitely possesses $n$ linearly independent eigenvectors. If an eigenvalue is repeated (multiplicity greater than one), there may or may not exist $n$ linearly independent eigenvectors. If there are fewer than $n$ linearly independent eigenvectors, generalized eigenvectors must be used, as discussed below.

**Example 4.** Find all maximal solutions of the system

$$\begin{aligned} x' &= -8x + 21y\,, \\ y' &= -3x + 8y\,. \end{aligned}$$

*Solution.* Compute the eigenvalues of the matrix:

$$\det(\lambda I - A) = (\lambda + 8)(\lambda - 8) + 3 \cdot 21 = \lambda^2 - 1.$$

The eigenvalues are $\lambda_{1,2} = \pm 1$. An eigenvector $v$ corresponding to $\lambda_1 = 1$ satisfies

$$(I - A)v = 0\,, \quad \text{i.e.,} \quad 9v_1 - 21v_2 = 0.$$

Thus, we can choose $v = (7, 3)^T$. Similarly, an eigenvector $w$ corresponding to $\lambda_2 = -1$ satisfies

$$(-I - A)w = 0\,, \quad \text{i.e.,} \quad 7w_1 - 21w_2 = 0.$$

Thus, we can choose $w = (3, 1)^T$. The fundamental system of the system is therefore

$$e^t \begin{pmatrix} 7 \\ 3 \end{pmatrix},\ e^{-t} \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

Now we address the case where there do not exist $n$ linearly independent eigenvectors. We say that $v_1, \ldots, v_k$ is a *chain of generalized eigenvectors* corresponding to the eigenvalue $\lambda$ if

$$(A - \lambda I)v_1 = 0,\ (A - \lambda I)v_2 = v_1,\ \ldots,\ (A - \lambda I)v_k = v_{k-1}.$$

**Theorem 4.** *Let $v_1, \ldots, v_k$ be a chain of generalized eigenvectors corresponding to the eigenvalue $\lambda$. Then the functions*

$$e^{\lambda t}v_1,\ e^{\lambda t}(v_1 t + v_2),\ e^{\lambda t}\left(v_1\frac{t^2}{2} + v_2 t + v_3\right),$$

$$\ldots,\ e^{\lambda t}\left(v_1\frac{t^{k-1}}{(k-1)!} + \cdots + v_{k-1}t + v_k\right)$$

*are linearly independent solutions to the system* (7).

Each Jordan block of size $k$ corresponds to an eigenvector and a chain of $k-1$ generalized eigenvectors, yielding $k$ linearly independent solutions. Combining these solutions across all Jordan blocks gives a fundamental system. If the matrix has non-real complex eigenvalues, this fundamental system will be complex-valued. To obtain a real fundamental system, for a pair of complex conjugate eigenvalues $\lambda = a + bi$, $\bar{\lambda} = a - bi$ and corresponding eigenvectors $v = v_1 + iv_2$, $\bar{v} = v_1 - iv_2$, we replace the complex pair

$$e^{\lambda t}v,\ e^{\bar{\lambda} t}\bar{v}$$

with the real solutions

$$e^{at}(\cos(bt)v_1 - \sin(bt)v_2),\ e^{at}(\cos(bt)v_2 + \sin(bt)v_1).$$

**Example 5.** Solve the system

$$\begin{aligned} x' &= x + 4z \\ y' &= x + y + z \\ z' &= -4x + z. \end{aligned}$$

*Solution.* Compute the eigenvalues of the matrix:

$$\det\begin{pmatrix} 1-\lambda & 0 & 4 \\ 1 & 1-\lambda & 1 \\ -4 & 0 & 1-\lambda \end{pmatrix} = (1-\lambda)^3 + 16(1-\lambda) = (1-\lambda)(17 - 2\lambda + \lambda^2).$$

Thus $\lambda_1 = 1$, $\lambda_{2,3} = \frac{1}{2}(2 \pm \sqrt{4 - 4 \cdot 17}) = 1 \pm 4i$. An eigenvector corresponding to $\lambda_1 = 1$:

$$\begin{pmatrix} 0 & 0 & 4 \\ 1 & 0 & 1 \\ -4 & 0 & 0 \end{pmatrix},$$

which gives $u = (0, 1, 0)^T$. For $\lambda_2 = 1 + 4i$:

$$\begin{pmatrix} -4i & 0 & 4 \\ 1 & -4i & 1 \\ -4 & 0 & -4i \end{pmatrix} \sim \begin{pmatrix} -4i & 0 & 4 \\ 1 & -4i & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

which gives $v = (4, 1-i, 4i)^T$. For $\lambda_3 = 1-4i$, the eigenvector is the complex conjugate $w = \bar{v} = (4, 1+i, -4i)^T$. The complex fundamental system is

$$e^t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, e^t(\cos(4t) + i\sin(4t)) \begin{pmatrix} 4 \\ 1-i \\ 4i \end{pmatrix}, e^t(\cos(4t) - i\sin(4t)) \begin{pmatrix} 4 \\ 1+i \\ -4i \end{pmatrix}.$$

The real fundamental system is then

$$e^t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, e^t \cos(4t) \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} - e^t \sin(4t) \begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix}, e^t \cos(4t) \begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix} + e^t \sin(4t) \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}.$$

**Example 6.** Find all maximal solutions of the system

$$\begin{aligned} x' &= 10x - 13y + 6z - 19w, \\ y' &= -3x + 7y - 2z + 7w, \\ z' &= 5x - 8y + 6z - 12w, \\ w' &= 7x - 11y + 6z - 15w. \end{aligned}$$

*Solution.* First, compute the eigenvalues of the matrix:

$$\det \begin{pmatrix} 10-\lambda & -13 & 6 & -19 \\ -3 & 7-\lambda & -2 & 7 \\ 5 & -8 & 6-\lambda & -12 \\ 7 & -11 & 6 & -15-\lambda \end{pmatrix} =$$

$$\det \begin{pmatrix} 1-\lambda & 8-3\lambda & 0 & 2 \\ -3 & 7-\lambda & -2 & 7 \\ -8+3\lambda & 26-13\lambda+\lambda^2 & 0 & 18-7\lambda \\ -2 & 10-3\lambda & 0 & 6-\lambda \end{pmatrix}$$

$$= 2 \cdot \det \begin{pmatrix} 1-\lambda & 8-3\lambda & 2 \\ -8+3\lambda & 26-13\lambda+\lambda^2 & 18-7\lambda \\ -2 & 10-3\lambda & 6-\lambda \end{pmatrix},$$

which evaluates to

$$\lambda^4 - 8\lambda^3 + 24\lambda^2 - 32\lambda + 16 = (\lambda - 2)^4,$$

so $\lambda = 2$ is an eigenvalue of multiplicity 4.
Now, find the eigenvectors:

$$\begin{pmatrix} 8 & -13 & 6 & -19 \\ -3 & 5 & -2 & 7 \\ 5 & -8 & 4 & -12 \\ 7 & -11 & 6 & -17 \end{pmatrix} \sim \begin{pmatrix} -1 & 2 & 0 & 2 \\ -3 & 5 & -2 & 7 \\ -1 & 2 & 0 & 2 \\ -2 & 4 & 0 & 4 \end{pmatrix} \sim \begin{pmatrix} -1 & 2 & 0 & 2 \\ -3 & 5 & -2 & 7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We find only two linearly independent eigenvectors, e.g., $u = (2, 0, 1/2, 1)^T$ and $v = (2, 1, -1/2, 0)^T$. This means the Jordan form has either two blocks of size 2, or one block of size 1 and one block of size 3. We look for generalized eigenvectors by solving $(A - 2I)v_2 = au + bv$:

$$\left(\begin{array}{cccc|c} 8 & -13 & 6 & -19 & 2a+2b \\ -3 & 5 & -2 & 7 & b \\ 5 & -8 & 4 & -12 & \frac{a}{2} - \frac{b}{2} \\ 7 & -11 & 6 & -17 & a \end{array}\right) \sim \left(\begin{array}{cccc|c} -1 & 2 & 0 & 2 & 2a+5b \\ -3 & 5 & -2 & 7 & b \\ -1 & 2 & 0 & 2 & \frac{a}{2} + \frac{3b}{2} \\ -2 & 4 & 0 & 4 & a+3b \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} -1 & 2 & 0 & 2 & 2a+5b \\ -3 & 5 & -2 & 7 & b \\ 0 & 0 & 0 & 0 & -\frac{3}{2}a - \frac{7}{2}b \\ 0 & 0 & 0 & 0 & -3a-7b \end{array}\right).$$

This requires $3a = -7b$. Setting $a = 7, b = -3$, the initial vector of the chain is $v_1 = (8, -3, 5, 7)^T$. Since the space of such vectors is 1-dimensional, there is only one chain of length $> 1$ (hence one block of size 3 and one of size 1). Solving for $v_2$:

$$\left(\begin{array}{cccc|c} -1 & 2 & 0 & 2 & -1 \\ -3 & 5 & -2 & 7 & -3 \end{array}\right),$$

which gives $v_2 = (3, 1, -1/2, 0)^T + a_2(2, 0, 1/2, 1)^T + b_2(2, 1, -1/2, 0)^T$. Now find $v_3$ by solving $(A - 2I)v_3 = v_2$:

$$\left(\begin{array}{cccc|c} 8 & -13 & 6 & -19 & 2a_2+2b_2+3 \\ -3 & 5 & -2 & 7 & b_2+1 \\ 5 & -8 & 4 & -12 & \frac{1}{2}a_2 - \frac{1}{2}b_2 - \frac{1}{2} \\ 7 & -11 & 6 & -17 & a_2 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} -1 & 2 & 0 & 2 & 2a_2+5b_2+6 \\ -3 & 5 & -2 & 7 & b_2+1 \\ -1 & 2 & 0 & 2 & \frac{1}{2}a_2+\frac{3}{2}b_2+\frac{3}{2} \\ -2 & 4 & 0 & 4 & a_2+3b_2+3 \end{array}\right) \sim \left(\begin{array}{cccc|c} -1 & 2 & 0 & 2 & 2a_2+5b_2+6 \\ -3 & 5 & -2 & 7 & b_2+1 \\ 0 & 0 & 0 & 0 & -\frac{3}{2}a_2-\frac{7}{2}b_2-\frac{9}{2} \\ 0 & 0 & 0 & 0 & -3a_2-7b_2-9 \end{array}\right).$$

A solution is given by $b_2 = 0, a_2 = -3$ (hence $v_2 = (-3, 1, -2, -3)^T$), and $v_3$ satisfies

$$\left(\begin{array}{cccc|c} -1 & 2 & 0 & 2 & 0 \\ -3 & 5 & -2 & 7 & 1 \end{array}\right),$$

e.g., $v_3 = (0, 0, -1/2, 0)^T$. The fundamental system of the system is therefore:

$$e^{2t}\begin{pmatrix} 2 \\ 0 \\ \frac{1}{2} \\ 1 \end{pmatrix},\ e^{2t}\begin{pmatrix} 8 \\ -3 \\ 5 \\ 7 \end{pmatrix},\ e^{2t}\left(\begin{pmatrix} 8 \\ -3 \\ 5 \\ 7 \end{pmatrix} t + \begin{pmatrix} -3 \\ 1 \\ -2 \\ -3 \end{pmatrix}\right),$$

$$e^{2t}\left(\begin{pmatrix} 8 \\ -3 \\ 5 \\ 7 \end{pmatrix} \frac{t^2}{2} + \begin{pmatrix} -3 \\ 1 \\ -2 \\ -3 \end{pmatrix} t + \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{2} \\ 0 \end{pmatrix}\right).$$

**Exercises on Solving Systems via Eigenvectors**

**24.**

$$\begin{aligned} x' &= -2x + y\,, \\ y' &= -4x + 2y\,. \end{aligned}$$

**25.**

$$\begin{aligned} x' &= 2x + 2y - 2z\,, \\ y' &= 2x + 5y - 4z\,, \\ z' &= -2x - 4y + 5z\,. \end{aligned}$$

**26.**

$$\begin{aligned} x' &= 2x + 4y - z\,, \\ y' &= -2x - 4y + 4z\,, \\ z' &= -4x + 2y - z\,. \end{aligned}$$

**27.**

$$\begin{aligned} x' &= 5x - y + 2z\,, \\ y' &= -x + 3y - z\,, \\ z' &= -4x + 2y - z\,. \end{aligned}$$

**28.**

$$\begin{aligned} x' &= x + 2y + 3z \\ y' &= 2x + 4y + 6z \\ z' &= 3x + 6y + 9z \end{aligned}$$

**29.**

$$\begin{aligned} x' &= y - z \\ y' &= -y + z \\ z' &= x - z \end{aligned}$$

**30.**

$$\begin{aligned} x' &= -3x + z \\ y' &= -3y + 2z \\ z' &= 3x - 2y - 3z \end{aligned}$$

**31.**

$$\begin{aligned} x' &= 6x - 7y + 4z \\ y' &= x + z \\ z' &= -2x + 3y \end{aligned}$$

**Solutions**

**24)** Zero is a double eigenvalue. An eigenvector corresponding to the eigenvalue 0 is $v_1 = (c, 2c)^T$, $c \in \mathbb{R}$, $c \neq 0$, and a generalized eigenvector $v_2 = (a, b)^T$ must satisfy $-2a + b = c$; e.g., choosing $c = 1, b = 0$ gives the chain $v_1 = (1, 2)^T$, $v_2 = (-\frac{1}{2}, 0)^T$. The fundamental system is

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \left[ \begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix} \right].$$

**25)** The eigenvalues are $\lambda_{1,2} = 1$, $\lambda_3 = 10$, with corresponding eigenvectors $v_1 = (-2, 1, 0)^T$, $v_2 = (2, 0, 1)^T$, $v_3 = (1, 2, -2)^T$. The fundamental system is

$$e^t \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, e^t \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, e^{10t} \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}.$$

**26)** The eigenvalues are $\lambda_1 = -1$, $\lambda_2 = 3i$, $\lambda_3 = -3i$, with corresponding eigenvectors $v_1 = (-1, 1, 1)^T$, $v_2 = (-\frac{2}{5} - \frac{1}{5}i, \frac{3}{5} - \frac{1}{5}i, 1)^T$, and $v_3 = (-\frac{2}{5} + \frac{1}{5}i, \frac{3}{5} + \frac{1}{5}i, 1)^T$. The real fundamental system is

$$e^{-t}\begin{pmatrix}-1\\1\\1\end{pmatrix},\ \cos 3t\begin{pmatrix}-\frac{2}{5}\\\frac{3}{5}\\1\end{pmatrix} - \sin 3t\begin{pmatrix}-\frac{1}{5}\\-\frac{1}{5}\\0\end{pmatrix},\ \cos 3t\begin{pmatrix}-\frac{1}{5}\\-\frac{1}{5}\\0\end{pmatrix} + \sin 3t\begin{pmatrix}-\frac{2}{5}\\\frac{3}{5}\\1\end{pmatrix}.$$

**27)** The eigenvalues are $\lambda_{1,2} = 2$, $\lambda_3 = 3$; to $\lambda_1 = 2$ corresponds the chain of generalized eigenvectors $v_1 = (-1, 1, 2)^T$, $v_2 = (0, 1, 0)^T$, and to $\lambda_3 = 3$ corresponds $v_3 = (1, 0, -1)^T$. The fundamental system is

$$e^{3t}\begin{pmatrix}1\\0\\-1\end{pmatrix},\ e^{2t}\begin{pmatrix}-1\\1\\2\end{pmatrix},\ e^{2t}\left[\begin{pmatrix}-1\\1\\2\end{pmatrix}t + \begin{pmatrix}0\\1\\0\end{pmatrix}\right].$$

**28)** The eigenvalue 0 has eigenvectors $(-2, 1, 0)^T$ and $(-3, 0, 1)^T$; the eigenvalue 14 has eigenvector $(1, 2, 3)^T$. The fundamental system is

$$\begin{pmatrix}-2\\1\\0\end{pmatrix},\ \begin{pmatrix}-3\\0\\1\end{pmatrix},\ e^{14t}\begin{pmatrix}1\\2\\3\end{pmatrix}.$$

**29)** The fundamental system is

$$\begin{pmatrix}1\\1\\1\end{pmatrix},\ e^{-t}\cos t\begin{pmatrix}-1\\1\\0\end{pmatrix} - e^{-t}\sin t\begin{pmatrix}0\\0\\-1\end{pmatrix},\ e^{-t}\cos t\begin{pmatrix}0\\0\\-1\end{pmatrix} + e^{-t}\sin t\begin{pmatrix}-1\\1\\0\end{pmatrix}.$$

**30)** The fundamental system is

$$e^{-3t}\begin{pmatrix}2\\3\\0\end{pmatrix},\ e^{-3t}\cos t\begin{pmatrix}1\\2\\0\end{pmatrix} - e^{-3t}\sin t\begin{pmatrix}0\\0\\1\end{pmatrix},\ e^{-3t}\cos t\begin{pmatrix}0\\0\\1\end{pmatrix} + e^{-3t}\sin t\begin{pmatrix}1\\2\\0\end{pmatrix}.$$

**31)** The fundamental system is

$$e^{2t}\begin{pmatrix}1\\0\\-1\end{pmatrix},\ e^{2t}\begin{pmatrix}t+4\\1\\-t-2\end{pmatrix},\ e^{2t}\begin{pmatrix}t^2/2+4t+1\\t\\-t^2/2-2t\end{pmatrix}.$$