Matrix Exponential and Other Matrix Functions

Motivation: You already know that the solutions to the equation

$$y' = ay$$

are the functions $y(t) = c \cdot e^{at}$, i.e., exponentials. For this function, $y(0) = c$, meaning the constant c is the initial condition at $t = 0$. Analogously, it might hold that the solution to a system of equations that can be written in matrix form as

$$Y' = AY \tag{1}$$

(see the chapter on systems of linear equations) is given by $Y(t) = e^{tA}C$, where $A \in \mathbb{R}^{n\times n}$ is a matrix, Y is an unknown function with values in $\mathbb{R}^n$, and $C \in \mathbb{R}^n$ is a vector of initial conditions. The matrix exponential can be defined as

$$e^B := \sum_{m=0}^{\infty} \frac{B^m}{m!}, \tag{2}$$

again drawing an analogy with the real exponential.

Before showing that this definition indeed yields a differentiable function that is a solution to the differential equation (1), we will discuss how to compute the matrix exponential (i.e., the sum of the infinite series (2)).

Properties and Computation of the Matrix Exponential

First, it should be noted that the series (2) converges for every matrix B. To justify this statement, let us introduce the *operator norm* of a matrix:

$$\|A\| := \sup\{\|Ax\| : \ x \in \mathbb{R}^n, \ \|x\| \le 1\}, \quad \text{where} \quad \|x\| := \sqrt{\sum_{i=1}^{n} x_i^2}.$$

For this norm, $\|AB\| \le \|A\| \cdot \|B\|$ holds, and thus $\|A^m\| \le \|A\|^m$. The series (2) therefore converges for every matrix B, because

$$\left\| \sum_{m=k}^{l} \frac{B^m}{m!} \right\| \le \sum_{m=k}^{l} \frac{\|B^m\|}{m!} \le \sum_{m=k}^{l} \frac{\|B\|^m}{m!} \le \sum_{m=k}^{\infty} \frac{\|B\|^m}{m!} \to 0 \tag{3}$$

as $k \to \infty$, i.e., the sequence of partial sums satisfies the Bolzano-Cauchy criterion. Now let us turn to the computation of the exponential.

Let $B \in \mathbb{R}^{n\times n}$; then there exists a Jordan canonical form $J \in \mathbb{R}^{n\times n}$ of the matrix B, i.e.,

$$B = VJV^{-1},$$

where J consists of Jordan blocks $J_1, \ldots, J_k$, and the columns of the matrix V are formed by eigenvectors, or generalized eigenvector chains. Then, of course, $B^m = VJ^mV^{-1}$ holds. Thus, we do not need to compute powers of the matrix B directly; it suffices to compute powers of its Jordan form. And since

$$J^2 = \begin{pmatrix} J_1 & 0 & \ldots & 0 \\ 0 & J_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \ldots & J_k \end{pmatrix} \begin{pmatrix} J_1 & 0 & \ldots & 0 \\ 0 & J_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \ldots & J_k \end{pmatrix} = \begin{pmatrix} J_1^2 & 0 & \ldots & 0 \\ 0 & J_2^2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \ldots & J_k^2 \end{pmatrix},$$

it is enough to know how to compute powers of a Jordan block.

For a Jordan block J_i corresponding to an eigenvalue λ_i, we have

$$J_i = \begin{pmatrix} \lambda_i & 1 & 0 & \ldots & 0 \\ 0 & \lambda_i & 1 & \ldots & 0 \\ 0 & 0 & \lambda_i & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & 0 & 0 & \ldots & \lambda_i \end{pmatrix} = \lambda_i I + D, \quad D = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ 0 & 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix},$$

so we can write

$$J_i^m = (\lambda_i I + D)^m = \sum_{k=0}^{m} \binom{m}{k} \lambda_i^{m-k} D^k,$$

where the binomial theorem can be applied because the matrices I and D commute ($ID = DI$). Raising the matrix D to powers shifts the diagonal of ones one row higher each time, i.e., $(D^k)_{ij} = 1$ for $j = i + k$ and $(D^k)_{ij} = 0$ otherwise (proven easily by induction). Specifically, if the size of the block is r, then D^{r-1} has a one in the upper right corner and zeros elsewhere, and $D^k = 0$ for $k \geq r$. Thus, for $m \geq r$ (where r is the size of the block), we have

$$J_i^m = \begin{pmatrix} \lambda_i^m & \binom{m}{1}\lambda_i^{m-1} & \binom{m}{2}\lambda_i^{m-2} & \ldots & \binom{m}{r-1}\lambda_i^{m-r+1} \\ 0 & \lambda_i^m & \binom{m}{1}\lambda_i^{m-1} & \ldots & \binom{m}{r-2}\lambda_i^{m-r+2} \\ 0 & 0 & \lambda_i^m & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \binom{m}{1}\lambda_i^{m-1} \\ 0 & 0 & 0 & \ldots & \lambda_i^m \end{pmatrix}.$$

For $m < r$, the negative powers of λ_i in the preceding expression are replaced by zeros.

Combining all this information, we have

$$e^B = e^{VJV^{-1}} = Ve^JV^{-1} = V\sum_{m=0}^{\infty}\frac{1}{m!}\begin{pmatrix} J_1^m & 0 & \dots & 0 \\ 0 & J_2^m & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_k^m \end{pmatrix}V^{-1} =$$

$$V\begin{pmatrix} \sum_{m=0}^{\infty}\frac{1}{m!}J_1^m & 0 & \dots & 0 \\ 0 & \sum_{m=0}^{\infty}\frac{1}{m!}J_2^m & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sum_{m=0}^{\infty}\frac{1}{m!}J_k^m \end{pmatrix}V^{-1} =$$

$$V\begin{pmatrix} e^{J_1} & 0 & \dots & 0 \\ 0 & e^{J_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{J_k} \end{pmatrix}V^{-1},$$

where e^{J_i} equals

$$\begin{pmatrix} \sum_{m=0}^{\infty}\frac{\lambda_i^m}{m!} & \sum_{m=1}^{\infty}\frac{\lambda_i^{m-1}}{(m-1)!1!} & \sum_{m=2}^{\infty}\frac{\lambda_i^{m-2}}{(m-2)!2!} & \dots & \sum_{m=r}^{\infty}\frac{\lambda_i^{m-r+1}}{(m-r+1)!(r-1)!} \\ 0 & \sum_{m=0}^{\infty}\frac{\lambda_i^m}{m!} & \sum_{m=1}^{\infty}\frac{\lambda_i^{m-1}}{(m-1)!1!} & \dots & \sum_{m=r-1}^{\infty}\frac{\lambda_i^{m-r+2}}{(m-r+2)!(r-2)!} \\ 0 & 0 & \sum_{m=0}^{\infty}\frac{\lambda_i^m}{m!} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \sum_{m=1}^{\infty}\frac{\lambda_i^{m-1}}{(m-1)!1!} \\ 0 & 0 & 0 & \dots & \sum_{m=0}^{\infty}\frac{\lambda_i^m}{m!} \end{pmatrix}$$

$$= \begin{pmatrix} e^{\lambda_i} & \frac{e^{\lambda_i}}{1!} & \frac{e^{\lambda_i}}{2!} & \dots & \frac{e^{\lambda_i}}{(r-1)!} \\ 0 & e^{\lambda_i} & \frac{e^{\lambda_i}}{1!} & \dots & \frac{e^{\lambda_i}}{(r-2)!} \\ 0 & 0 & e^{\lambda_i} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \frac{e^{\lambda_i}}{1!} \\ 0 & 0 & 0 & \dots & e^{\lambda_i} \end{pmatrix}.$$

This completes the computation of the matrix exponential. We will now apply the preceding derivations to the following example.

Example 1. Compute e^A for

$$A = \begin{pmatrix} -2 & 4 & 3 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Solution. We compute the eigenvalues:

$$\det(\lambda I - A) = (\lambda + 2)(\lambda - 1)^2,$$

i.e., $\lambda_{1,2} = 1$, $\lambda_3 = -2$. The eigenvalue -2 has the eigenvector $(1, 0, 0)^T$, and the eigenvalue 1 has the eigenvector $(1, 0, 1)^T$, for which we find a generalized eigenvector satisfying $(A-I)v = (1, 0, 1)^T$, e.g., $v = (1, 1, 0)^T$. Thus, we have

$$V = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad V^{-1} = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

and $A = VJV^{-1}$. Then

$$e^A = Ve^JV^{-1} = V \begin{pmatrix} e^{-2} & 0 & 0 \\ 0 & e & e \\ 0 & 0 & e \end{pmatrix} V^{-1} = \begin{pmatrix} e^{-2} & e^{-2} + 2e & -e^{-2} + e \\ 0 & e & 0 \\ 0 & e & e \end{pmatrix}$$

Exercises

1. (a) Let v be an eigenvector of the matrix A corresponding to the eigenvalue λ. Then v is also an eigenvector of the matrix e^A corresponding to the eigenvalue e^λ.
(b) Prove the spectral mapping theorem: $\sigma(e^A) = \{e^\lambda;\ \lambda \in \sigma(A)\}$.

2. Prove that $\det e^A = e^{\lambda_1 + \cdots + \lambda_n} = e^{\mathrm{Tr}(A)}$, where λ_i are the eigenvalues of A, and $\mathrm{Tr}(A)$ is the trace (sum of diagonal elements).

3. Matrix logarithm. Let $A \in \mathbb{R}^{n \times n}$.
(a) If $A^3 = 0$ and $L := A - A^2/2$, then $e^L = I + A$.
(b) If the series $L := A - A^2/2 + A^3/3 + \ldots$ converges, then $e^L = I + A$.
* (c) For a given matrix A, find a matrix B such that $A = e^B$.

4. For the following matrices A, find matrices B such that $A = e^B$ (using the result of the previous exercise):

$$\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}, \quad \begin{pmatrix} 3 & 2 \\ 0 & 3 \end{pmatrix}, \quad \begin{pmatrix} e & 1 & 2 \\ 0 & e & 3 \\ 0 & 0 & e \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & -1 \\ 2 & 5 & 0 \end{pmatrix}.$$

5. (a) Given a matrix A, does there always exist a matrix B such that $B^2 = A$?
(b) Find a square root of the matrix

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

6. For an arbitrary square matrix A, define ($A^0 = I$)

$$\sin A := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} A^{2n+1} \qquad \cos A := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} A^{2n}$$

(a) Show that these series converge.
(b) Show that $(\sin A)^2 + (\cos A)^2 = I$.
(c) Does $e^{A+iB} = e^A(\cos B + i \sin B)$ hold for all pairs of matrices A, B, or only for some?
(d) Does $\cos(A+B) = \cos A \cos B - \sin A \sin B$ hold for all pairs of matrices A, B, or only for some?

7. Compute $\sin A$ (without using the Jordan canonical form) for the following matrices A:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}.$$

8. Let A be skew-symmetric (i.e., $A^T = -A$). Then e^A is orthogonal.

9. Let $A \in \mathbb{R}^{n \times n}$ have distinct (simple) eigenvalues $\lambda_1, \ldots, \lambda_n$. Let P be a polynomial such that $P(\lambda_i) = e^{\lambda_i}$, $i = 1, \ldots, n$. Then $e^A = P(A)$. Prove this.

10. !! Cauchy's integral formula states that $f(a) = \frac{1}{2\pi i} \int_\gamma f(z)(z-a)^{-1} dz$, where γ is a positively oriented circle centered at a.
(a) Show that

$$e^A = \frac{1}{2\pi i} \int_\gamma e^z (zI - A)^{-1} dz,$$

where γ is a circle centered at 0 with radius greater than $\|A\|$.
(b) Use this method to compute e^A for the following matrices:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$

Solutions

1) (a) $Av = \lambda v$, hence $A^k v = \lambda^k v$, so $(e^A)v = \ldots$. (b) Using the Jordan canonical form. Can it be done without it?

2) Using the Jordan canonical form or Liouville's formula.

3) (a) By direct substitution. (b) Note that $\exp(x - x^2/2 + x^3/3 + \ldots) = 1 + x$ for $x \in (0, 1)$, and argue using the possibility of rearranging absolutely

convergent series. (c)

$$B = V \begin{pmatrix} C_1 & 0 & \dots & 0 \\ 0 & C_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & C_k \end{pmatrix} V^{-1},$$

where V is a matrix whose columns form a basis of generalized eigenvectors of the matrix A, and the block C_i equals

$$- \begin{pmatrix} \sum_{m=1}^{\infty} \binom{m}{0} \frac{(-1)^m \lambda_i^m}{m} & \sum_{m=1}^{\infty} \binom{m}{1} \frac{(-1)^m \lambda_i^{m-1}}{m} & \dots & \sum_{m=r}^{\infty} \binom{m}{r-1} \frac{(-1)^m \lambda_i^{m-r+1}}{m} \\ 0 & \sum_{m=1}^{\infty} \binom{m}{0} \frac{(-1)^m \lambda_i^m}{m} & \dots & \sum_{m=r-1}^{\infty} \binom{m}{r-2} \frac{(-1)^m \lambda_i^{m-r+2}}{m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sum_{m=1}^{\infty} \binom{m}{0} \frac{(-1)^m \lambda_i^m}{m} \end{pmatrix},$$

where λ_i are the eigenvalues of the matrix $A - I$.

4) (a) The matrix $A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$ has eigenvalues -2 and 4, and therefore there does not exist a real matrix B such that $e^B = A$.
(b) Due to the convergence of the Taylor series of $\ln(1+x)$ on the interval $(0, 2)$, it is convenient to write the matrix A as: $A = \begin{pmatrix} 3 & 2 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{3}{2} & 1 \\ 0 & \frac{3}{2} \end{pmatrix}$ and compute the exponential separately for both matrices (the first being trivial and the second using the result of exercise 3(c)). The result is then

$$B = \begin{pmatrix} \ln 3 & \frac{2}{3} \\ 0 & \ln 3 \end{pmatrix}.$$

(c) $A = \begin{pmatrix} e & 1 & 2 \\ 0 & e & 3 \\ 0 & 0 & e \end{pmatrix} = \begin{pmatrix} e & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & e \end{pmatrix} \begin{pmatrix} 1 & e^{-1} & 2e^{-1} \\ 0 & 1 & 3e^{-1} \\ 0 & 0 & 1 \end{pmatrix}$ and the result is:

$$\begin{pmatrix} 1 & e^{-1} & 2e^{-1} - \frac{3}{2}e^{-2} \\ 0 & 1 & 3e^{-1} \\ 0 & 0 & 1 \end{pmatrix}.$$

(d) The eigenvalues of the matrix $\begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & -1 \\ 2 & 5 & 0 \end{pmatrix}$ are 0 and 2, and thus there is no matrix B such that $e^B = A$ (there is no $c \in \mathbb{R}$ such that $e^c = 0$).

5) It suffices to consider a Jordan block. In the case of zero eigenvalues, a matrix square root generally does not exist.

6) (a) Proceed analogously to the case of the exponential.
(b) Use the formula for the Cauchy product of series, i.e.,

$$\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} a_{n-m}b_m\right).$$

(c) The statement holds for pairs of matrices satisfying $AB = BA$. In the proof, first show from the definition of the exponential that $e^{iB} = \cos B + i\sin B$, and it then remains to show that $e^{A+iB} = e^A e^{iB}$. For this, use the formula

$$(A+B)^n = \sum_{m=0}^{n} \binom{n}{m} A^{n-m}B^m,$$

which holds if $AB = BA$. If $AB \neq BA$, the formula might not hold. Verify that this is the case for the matrices

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

(d) Holds for pairs of matrices satisfying $AB = BA$. The proof uses the formula for the product of series (see item (b)) and the binomial theorem (see item (c)), relying on the commutativity of matrices A and B.

7) (a) Verify by computation that $A^3 = A$, and then easily compute that $\sin A$ is equal to the matrix

$$\begin{pmatrix} 0 & \sin 1 \\ \sin 1 & 0 \end{pmatrix}.$$

(b) Verify by computation that $A^3 = 0$, and therefore simply: $\sin A = A$.

8) M is orthogonal if and only if $MM^T = I$.

9) Follows from the computation using the Jordan canonical form.

10) (a) Use the formula $(I - A/z)^{-1} = \sum_{k=0}^{\infty}(A/z)^k$, interchange the summation and integral, and apply Cauchy's integral formula for the k-th derivative of a function. (b) Use the formula for the inverse of a 2×2 matrix:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Using this, compute $(zI - A)^{-1}$ for both given matrices. For the first and second matrices, this yields:

$$\begin{pmatrix} \frac{z}{z^2-1} & \frac{1}{z^2-1} \\ \frac{1}{z^2-1} & \frac{z}{z^2-1} \end{pmatrix}, \text{ resp. } \begin{pmatrix} \frac{z+1}{z^2-1} & \frac{1}{z^2-1} \\ 0 & \frac{z+1}{z^2-1} \end{pmatrix}.$$

Subsequently, substituting into the formula for computing the exponential from the previous item and applying Cauchy's integral formula separately to each entry of the resulting resolvent matrix (using partial fraction decomposition), you find that the exponential of the first and second matrix, respectively, is:

$$\begin{pmatrix} \frac{1}{2}(e+e^{-1}) & \frac{1}{2}(e-e^{-1}) \\ \frac{1}{2}(e-e^{-1}) & \frac{1}{2}(e+e^{-1}) \end{pmatrix}, \text{ resp. } \begin{pmatrix} e & \frac{1}{2}(e-e^{-1}) \\ 0 & e \end{pmatrix}.$$

Matrix Exponential and Solutions to Systems of Linear Equations

In this subsection, we will investigate the properties of the function

$$\Phi : \mathbb{R} \to \mathbb{R}^{n\times n}, \quad \Phi : t \mapsto e^{tA}.$$

Among other things, we will show that ΦC is a solution to the system $Y' = AY$ for every vector $C \in \mathbb{R}^n$.

First, recall that from the estimate (3) (in the previous subsection), for $t \in [-K, K]$ it follows that

$$\left\| \sum_{m=k}^{l} \frac{A^m t^m}{m!} \right\| \leq \sum_{m=k}^{\infty} \frac{(K\|A\|)^m}{m!}. \tag{4}$$

Thus, the series converges locally uniformly, and the function Y is therefore continuous. Differentiating the m-th term of the series with respect to t, we obtain

$$f'_m(t) = \frac{1}{m!} A^m m t^{m-1} = A \frac{A^{m-1} t^{m-1}}{(m-1)!}.$$

The estimate (4) thus also gives the uniform convergence of the derivatives. It follows that the limit function is differentiable and can be differentiated term-by-term. From this we immediately obtain

$$\Phi'(t) = \sum_{m=1}^{\infty} A \frac{A^{m-1} t^{m-1}}{(m-1)!} = A\Phi(t),$$

meaning that the function Φ satisfies the equation (1) and $Y(t) = \Phi(t)C$ is a solution to the system of equations (1).

Example 2. Find a fundamental matrix of the following system using the matrix exponential:

$$\begin{aligned} x' &= 10x - 6y \\ y' &= 18x - 11y. \end{aligned}$$

Solution. We compute the eigenvalues of the system matrix:

$$\det(\lambda I - A) = (\lambda - 10)(\lambda + 11) + 6 \cdot 18 = \lambda^2 + \lambda - 2 = (\lambda + 2)(\lambda - 1).$$

The eigenvector corresponding to $\lambda_1 = 1$ is v_1 satisfying $(I - A)v_1 = 0$, i.e., $v_1 = (2, 3)^T$. The eigenvector corresponding to $\lambda_2 = -2$ satisfies $(-2I - A)v_2 = 0$, i.e., $v_2 = (1, 2)^T$. Thus,

$$A = VJV^{-1} = V \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} V^{-1},$$

where

$$V = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \quad \text{and} \quad V^{-1} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}.$$

We therefore have

$$\Phi(t) = e^{tA} = Ve^{\begin{pmatrix} t & 0 \\ 0 & -2t \end{pmatrix}} V^{-1} = V \begin{pmatrix} e^t & 0 \\ 0 & e^{-2t} \end{pmatrix} V^{-1} =$$

$$\begin{pmatrix} 4e^t - 3e^{-2t} & -2e^t + 2e^{-2t} \\ 6e^t - 6e^{-2t} & -3e^t + 4e^{-2t} \end{pmatrix},$$

which is the fundamental matrix of the system.

Remark. Note that for every invertible matrix M, the matrix-valued function $\Phi(t)M$ is also a fundamental matrix of the system (multiplying from the right replaces the columns with linear combinations of the columns of the original matrix). Thus, e^{tJ} is also a fundamental matrix of the system. However, $\Phi(t) = Ve^{tJ}V^{-1}$ is the unique fundamental matrix satisfying $\Phi(0) = I$. Therefore, $\Phi(t)C$ is the solution to the system satisfying the initial condition $Y(0) = C$.

For the asymptotic behavior of the solution to the system as $t \to +\infty$, the following estimate is crucial, which will be used extensively in the chapter on stability. Its proof is left as an exercise.

Proposition 1. *For every $\epsilon > 0$, there exists $M \geq 1$ such that*

$$\|e^{tA}\| \leq Me^{t(\alpha+\epsilon)} \quad \textit{for all } t \geq 0,$$

where $\alpha := \max\{\operatorname{Re}\lambda : \ \lambda \in \sigma(A)\}$.

Exercises on Solving Systems of Linear Equations

Find the general solution of the following systems using the matrix exponential.

11.

$$x' = -6x + 8y$$
$$y' = -4x + 6y$$

12.

$$x' = -2x - 3y$$
$$y' = 6x + 7y$$

13.

$$x' = -12x - 8y$$
$$y' = 20x + 12y$$

14.

$$x' = -5x - 10y$$
$$y' = 5x + 5y$$

15.

$$x' = 5x - 6y$$
$$y' = 3x - y$$

16.

$$x' = -5x + 4y$$
$$y' = -x - y$$

17.

$$x' = -2x + 8y + 6z$$
$$y' = -4x + 10y + 6z$$
$$z' = 4x - 8y - 4z$$

18. Find the fundamental matrix of the system $Y' = AY$, where

$$A = \begin{pmatrix} 1 & -2 & -1 \\ -1 & 2 & -5 \\ -1 & 2 & -3 \end{pmatrix}.$$

Verify that $A^3 = 0$ and use this fact to compute the fundamental matrix.

Solutions

11) The fundamental matrix of the system is

$$\begin{pmatrix} 2\,e^{-2\,t} - e^{2\,t} & 2\,e^{2\,t} - 2\,e^{-2\,t} \\ -e^{2\,t} + e^{-2\,t} & -e^{-2\,t} + 2\,e^{2\,t} \end{pmatrix}$$

12) The fundamental matrix of the system is

$$\begin{pmatrix} 2\,e^{t} - e^{4\,t} & -e^{4\,t} + e^{t} \\ 2\,e^{4\,t} - 2\,e^{t} & -e^{t} + 2\,e^{4\,t} \end{pmatrix}$$

13) The fundamental matrix of the system is

$$\begin{pmatrix} \cos(4\,t) - 3\,\sin(4\,t) & -2\,\sin(4\,t) \\ 5\,\sin(4\,t) & \cos(4\,t) + 3\,\sin(4\,t) \end{pmatrix}$$

14) The fundamental matrix of the system is

$$\begin{pmatrix} \cos(5\,t) - \sin(5\,t) & -2\,\sin(5\,t) \\ \sin(5\,t) & \cos(5\,t) + \sin(5\,t) \end{pmatrix}$$

15) The fundamental matrix of the system is

$$\begin{pmatrix} e^{2\,t}\cos(3\,t) + e^{2\,t}\sin(3\,t) & -2\,e^{2\,t}\sin(3\,t) \\ e^{2\,t}\sin(3\,t) & e^{2\,t}\cos(3\,t) - e^{2\,t}\sin(3\,t) \end{pmatrix}$$

16) The fundamental matrix of the system is

$$\begin{pmatrix} e^{-3\,t} - 2\,te^{-3\,t} & 4\,te^{-3\,t} \\ -te^{-3\,t} & e^{-3\,t} + 2\,te^{-3\,t} \end{pmatrix}$$

17) The fundamental matrix of the system is

$$\begin{pmatrix} 2 - e^{2\,t} & 4\,e^{2\,t} - 4 & 3\,e^{2\,t} - 3 \\ -2\,e^{2\,t} + 2 & -4 + 5\,e^{2\,t} & 3\,e^{2\,t} - 3 \\ 2\,e^{2\,t} - 2 & -4\,e^{2\,t} + 4 & 3 - 2\,e^{2\,t} \end{pmatrix}$$

18) Since $A^3 = 0$ (easily verified by computation), the fundamental matrix of the system is $\phi(t) = e^{tA} = I + tA + \frac{t^2}{2}A^2$, hence:

$$\phi(t) = \begin{pmatrix} 1+t+2t^2 & -2t-4t^2 & -t+6t^2 \\ -t+t^2 & 1+2t-2t^2 & -5t+3t^2 \\ -t & 2t & 1-3t \end{pmatrix}.$$

More Theoretical Exercises

19. Prove Proposition 1.

20. Sketch the qualitative behavior of the system $x' = Ax$ with a constant matrix A, if A is similar to the matrix

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Distinguish the cases: (a) $\lambda_1 > 0 > \lambda_2$, (b) $0 > \lambda_1 > \lambda_2$.

21. Compute the matrix exponential of

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

(if possible, without a routine reduction to Jordan form). Sketch the behavior of the solutions to the system with the matrix above; how do the cases $a > 0$, $a < 0$, and $a = 0$ differ?

22. (a) Compute e^{tA} directly from the definition, given that $A^2 = -I$.
(b) Show that $A^2 = -I$ implies $\sigma(A) \subset \{i, -i\}$.
(c) Using (a), determine the fundamental matrix of the system $u' = Au$, where

$$A = \begin{pmatrix} 2 & -5 & 8 & -12 \\ 1 & -2 & 4 & -8 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 1 & -2 \end{pmatrix}$$

23. (a) Compute e^{tA} directly from the definition, given that $A^2 = c^2 I$, $c > 0$.
(b) Show that $A^2 = c^2 I$ implies $\sigma(A) \subset \{c, -c\}$.
(c) Using item (a), find the fundamental matrix for the system $u' = Au$, where

$$A = \begin{pmatrix} 3 & 3 & -3 \\ 6 & 0 & 6 \\ -3 & 3 & 3 \end{pmatrix}.$$

24. Evaluate

$$\int_0^\infty e^{tA}\,dt\,,$$

given that the matrix A has only negative eigenvalues (real parts). (Why does this integral converge?)

25. Show that $[\sin(tA)]' = A\cos(tA)$ and $[\cos(tA)]' = -A\sin(tA)$.

26. Find all solutions of the system $Y'' = AY$ for

$$A = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

using the matrix sine and cosine.

Solutions

19) Follows easily from the form of e^J for a Jordan block J.

20) The solution is a linear combination of $e^{\lambda_i t}v$, where v is the corresponding eigenvector.

21) $e^{a+ib} = e^a[\cos b + i\sin b]$; the matrix "represents" the complex number $a+ib$. The result is therefore

$$\begin{pmatrix} e^{at}\cos bt & -e^{at}\sin bt \\ e^{at}\sin bt & e^{at}\cos bt \end{pmatrix}$$

For $a < 0$, the matrix exponential will converge to 0; for $a = 0$, it will be bounded; and for $a > 0$, it will be unbounded.

22) (a) $e^{tA} = (\cos t)I + (\sin t)A$. (b) Note that $\lambda \in \sigma(M)$ if and only if there exists a non-zero vector v such that $Av = \lambda v$. (c) Verify by computation that $A^2 = -I$. It then follows from item (a) that the fundamental matrix $\phi(t)$ is $e^{tA} = I\cos t + A\sin t$, hence:

$$\phi(t) = \begin{pmatrix} \cos t + 2\sin t & -5\sin t & 8\sin t & -12\sin t \\ \sin t & \cos t - 2\sin t & 4\sin t & -8\sin t \\ 0 & 0 & \cos t + 2\sin t & -5\sin t \\ 0 & 0 & \sin t & \cos t - 2\sin t \end{pmatrix}.$$

23) (a) $e^{tA} = \cosh(tc)I + c^{-1}\sinh(tc)A$
(b) $\sigma(c^2 I) = \{c^2\}$. Apply the spectral mapping theorem with $f(x) = x^2$. Then $f(\sigma(A)) = \sigma(f(A)) = \sigma(c^2 I) = \{c^2\}$, which implies that $\sigma(A) \subset \{-c, c\}$.

(c) Verify by computation that $A^2 = 36I$. By item (a), the fundamental matrix is equal to $e^{tA} = I\cosh(6t) + \frac{1}{6}A\sinh(6t)$, hence:

$$\phi(t) = \begin{pmatrix} \cosh 6t + \frac{1}{2}\sinh 6t & \frac{1}{2}\sinh 6t & -\frac{1}{2}\sinh 6t \\ \sinh 6t & \cosh 6t & \sinh 6t \\ -\frac{1}{2}\sinh 6t & \frac{1}{2}\sinh 6t & \cosh 6t + \frac{1}{2}\sinh 6t \end{pmatrix}.$$

24) Compute it first for a 1×1 matrix, i.e., a negative real number. This suggests the result. Definition: $B = A^{-1}$ if and only if $BA = I$.

25) Just as in the introduction to Section 1.2, show that the series representing the sine and cosine can be differentiated term-by-term. After differentiating, factor out appropriate terms and simplify.

26) Based on Exercise 25, show that $(\sin tB)'' = -B^2 \sin tB$ and $(\cos tB)'' = -B^2 \cos tB$ for any matrix B. If a matrix B is found such that $-B^2 = A$, then $\{\sin Bt, \cos Bt\}$ forms a fundamental system of solutions for the given system. The matrix B can be found, for example, using Cauchy's integral formula (as in Exercise 10) by setting $f(z) = \sqrt{-z}$ and evaluating $f(A)$. Then $B = f(A) = \sqrt{-A}$, giving

$$B = \begin{pmatrix} \sqrt{2} & -\frac{1}{2\sqrt{2}} & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then the fundamental system is given by the matrices:

$$\sin Bt = \begin{pmatrix} \sin\sqrt{2}t & -\frac{t}{2\sqrt{2}}\cos\sqrt{2}t & 0 \\ 0 & \sin\sqrt{2}t & 0 \\ 0 & 0 & \sin t \end{pmatrix}$$

$$\text{and } \cos Bt = \begin{pmatrix} \cos\sqrt{2}t & \frac{t}{2\sqrt{2}}\sin\sqrt{2}t & 0 \\ 0 & \cos\sqrt{2}t & 0 \\ 0 & 0 & \cos t \end{pmatrix},$$

which can be computed, for instance, via Cauchy's integral formulas:
$f(a) = \frac{1}{2\pi i}\int_\gamma f(z)(z-a)^{-1}dz$,
$f'(a) = \frac{1}{2\pi i}\int_\gamma f(z)(z-a)^{-2}dz$,
following the same procedure as in Exercise 10, replacing exp with sin and cos, respectively.