Nonlinear Systems

In this chapter, we will demonstrate how to solve certain nonlinear autonomous systems of differential equations. Consider the system

$$X' = F(X), \tag{1}$$

where $X : (a, b) \subset \mathbb{R} \to \mathbb{R}^n$ is an unknown function and $F : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function. A special case will be systems of two equations

$$\begin{aligned} x' &= f(x, y), & (2)\\ y' &= g(x, y), & (3) \end{aligned}$$

where $X = \binom{x}{y}$. We will also examine certain higher-order nonlinear differential equations. These, as we know, can be converted into a system of first-order equations.

First Method of Solution

The first method of solution, suitable for systems of two equations (and exceptionally for more), is based on the following theorem (a proof sketch can be found among the exercises):

Theorem 1. *Let f, g be continuous functions, $x, y : (a, b) \to \mathbb{R}$, $t_0 \in (a, b)$, and $f(x(t_0), y(t_0)) \neq 0$.*

(i) Then there exists a neighborhood U of the point t_0 such that if $t \mapsto (x(t), y(t))$ is a solution of the system (2)–(3) on U, then the function $\tilde{y} := y \circ x^{-1}$ on $x(U)$ is a solution of the differential equation

$$\frac{d\tilde{y}}{dz} = \frac{g(z, \tilde{y})}{f(z, \tilde{y})}, \qquad \tilde{y}(x(t_0)) = y(t_0). \tag{4}$$

(ii) For every neighborhood U of t_0, if $\tilde{y}$ is a solution of equation (4) on $x(U)$ and x is a solution of the equation $x' = f(x, \tilde{y}(x))$ on U, then the function $\tilde{y} \circ x$ is a solution of equation (3) on U.

Notice that equation (4) is obtained by formally dividing equation (3) by equation (2):

$$\frac{dy}{dt} = g(x, y) \quad \& \quad \frac{dx}{dt} = f(x, y) \quad \Rightarrow \quad \frac{dy}{dx} = \frac{g(x, y)}{f(x, y)}.$$

Theorem 1 states that if we want to find a solution of the system, we can first solve equation (4), which yields the trajectories (orbits) of the solutions in the x-y plane, and then compute the explicit time dependence on t.

Example 1. Find the solutions of the system

$$\begin{aligned} x' &= x^2 y, \\ y' &= x y^2. \end{aligned}$$

Solution. First, we find the function $\tilde{y}$ from Theorem 1; thus, we solve the equation $d\tilde{y}/dx = \tilde{y}/x$. By separation of variables, we obtain $\tilde{y}(x) = kx$, $x \in \mathbb{R}$, $k \in \mathbb{R}$. The condition $f(x, y) \neq 0$ is violated if $x = 0$ or $y = 0$. We therefore exclude the origin and the case $k = 0$, which we will address later. If we sketch the trajectories in the x-y plane, the orbits lie on rays emanating from the origin. It remains to determine the direction in which solutions traverse these rays and whether they traverse them entirely or stop somewhere.

We now determine the dependence on t: From the first equation of the system, we have $x' = kx^3$, which yields $-1/2x^{-2} = kt + c$, i.e.,

$$x(t) = \frac{\pm 1}{\sqrt{-2(kt + c)}} \quad \text{and} \quad y(t) = \frac{\pm k}{\sqrt{-2(kt + c)}}.$$

The solution is defined on $(-\infty, -c/k)$ for $k > 0$ and on $(-c/k, +\infty)$ for $k < 0$. From this, we see that maximal solutions traverse the entire ray, some moving from the origin toward infinity, others moving from infinity toward the origin. Because they approach the origin only as $t \to \pm\infty$, the solutions cannot be extended through the origin.

Notice that from part (ii) of Theorem 1, the found pairs of functions indeed solve the system (2)–(3). From part (i), on the other hand, it follows that no other solutions exist on the set $\{(x, y) : \ x \neq 0 \text{ and } y \neq 0\}$ (if any existed, we would obtain corresponding additional solutions of equation (4), which is a contradiction, since separation of variables found all solutions of (4)).

It remains to find solutions on the coordinate axes, i.e., the cases where $y(t_0) = 0$ or $x(t_0) = 0$. In that case, we obtain stationary solutions

$$(x(t), y(t)) = (x(t_0), y(t_0)), \ t \in \mathbb{R}$$

(directly from the system (2)–(3), because x' and y' must vanish). By the uniqueness theorem, no other solutions exist on (or crossing) the coordinate axes.

Second Method of Solution — First Integral

This method can also be used for systems with more than two variables.

Definition 1. A function $V : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is called a *first integral* (or *constant of motion*) of the differential equation (1) if V is constant along every solution of equation (1) and is non-constant on Ω.

Remark. If we solve a system using the first method and then solve the resulting equation (4) by separation of variables, it may happen that after integrating both sides we cannot explicitly express the dependence of $\tilde{y}$ on x. Nonetheless, the obtained relation $G(y) = H(x) + c$ provides a first integral of the system: $V(x, y) = G(y) - H(x)$.

If V is a differentiable function, it must satisfy

$$0 = \frac{d}{dt}V(Y(t)) = \sum_{i=1}^{n} \frac{\partial V}{\partial Y_i} Y_i' = \sum_{i=1}^{n} \frac{\partial V}{\partial Y_i} F_i(Y).$$

Example 2. Find all solutions of the system of differential equations

$$\begin{aligned} x' \sin y &= 1, \\ y' \sin y &= \frac{y}{x}. \end{aligned}$$

Solution. Substituting the first equation into the second gives $y' = x'y/x$, or equivalently $y'/y = x'/x$. Integrating yields

$$\frac{d}{dt}(\ln|y|) = \frac{d}{dt}(\ln|x|),$$

i.e., $C = \ln|y| - \ln|x| = \ln|y/x|$, which gives $y/x = K$. The function $V(x, y) := y/x$ is therefore a first integral of the system.

Since $y = Kx$, substituting into the first equation yields $\sin(Kx)x' = 1$, i.e., $-\frac{1}{K}\cos(Kx) = t + c$, whence for $-Kt - Kc \in [-1, 1]$:

$$x(t) = K^{-1}(\arccos(-Kt - Kc) + 2m\pi),\ y(t) = \arccos(-Kt - Kc) + 2m\pi,$$

resp.

$$x(t) = -K^{-1}(\arccos(-Kt-Kc)+2m\pi),\ y(t) = -(\arccos(-Kt-Kc)+2m\pi).$$

Substituting into the system easily verifies that the found functions are solutions on the intervals $(-1/K - c, 1/K - c)$. These intervals cannot be extended because at the endpoints $\sin y = 0$, so the differential equations cannot be satisfied.

Example 3. Find all solutions of the system of differential equations

$$\begin{aligned} x' &= xy, \\ y' &= xz, \\ z' &= yz. \end{aligned}$$

Solution. Cross-multiplying the 1st and 3rd equations gives $x'z = z'x$, i.e., $x'/x = z'/z$, which for $x \neq 0 \neq z$ yields

$$z/x = K_1 \tag{5}$$

similarly to the previous example. Multiplying the 1st and 2nd equations gives $x'z = y'y$. At the same time, we know that $x'z = z'x$, so $z'x+x'z = 2yy'$, which after integration yields

$$y^2 - zx = K_2. \tag{6}$$

We thus have two independent first integrals: $V_1(x, y, z) = z/x$ and $V_2(x, y, z) = y^2 - zx$.

Substituting from (6) into the second equation gives $y' = y^2 - K_2$. For $K_2 > 0$, this yields

$$\ln\left(\frac{y+\sqrt{K_2}}{y-\sqrt{K_2}}\right) = t + c,$$

$$y = \sqrt{K_2}\,\frac{e^t e^c + 1}{e^t e^c - 1},$$

$t \in (-\infty, -c)$, $(-c, +\infty)$. For $K_2 < 0$, we have $1/\sqrt{-K_2}\,\mathrm{arctg}(y/\sqrt{-K_2}) = t+c$, i.e., $y = \sqrt{-K_2}\,\mathrm{tg}(\sqrt{-K_2}(t+c))$, $t \in (-\pi/2/\sqrt{-K_2}-c, \pi/2/\sqrt{-K_2}-c)$. For $K_2 = 0$, we have $y = -1/(t+c)$, $t \in (-\infty, -c)$, $(-c, +\infty)$, resp. $y(t) = 0$, $t \in \mathbb{R}$.

Substituting for x from (5) into (6) gives $x = \pm\sqrt{(y^2 - K_2)/K_1}$ and $z = \pm\sqrt{K_1(y^2 - K_2)}$. The solutions are therefore:

$$\pm\sqrt{\frac{K_2}{K_1}}\frac{2e^{\frac{t+c}{2}}}{e^{t+c}-1}, \quad \sqrt{K_2}\,\frac{e^t e^c + 1}{e^t e^c - 1}, \quad \pm\sqrt{K_2 K_1}\frac{2e^{\frac{t+c}{2}}}{e^{t+c}-1}, \quad K_1, K_2 > 0$$

on the intervals $(-\infty, -c)$, $(-c, +\infty)$. Furthermore,

$$\pm\sqrt{\frac{-K_2}{K_1}(\mathrm{tg}(\sqrt{-K_2}(t+c)) - 1)}, \quad \sqrt{-K_2}\,\mathrm{tg}(\sqrt{-K_2}(t+c)),$$
$$\pm\sqrt{-K_2 K_1(\mathrm{tg}(\sqrt{-K_2}(t+c)) - 1)}, \quad K_1 > 0, K_2 < 0$$

on the intervals $(-\pi/2/\sqrt{-K_2}-c, \pi/4/\sqrt{-K_2}-c)$ and $(\pi/4/\sqrt{-K_2}-c, \pi/2/\sqrt{-K_2}-c)$. Furthermore,

$$\pm\sqrt{\frac{1}{K_1}}\frac{-1}{t+c}, \quad \frac{-1}{t+c}, \quad \pm\sqrt{K_1}\frac{-1}{t+c}, \quad K_1 > 0, K_2 = 0,$$

on the intervals $(-\infty, -c)$, $(-c, +\infty)$. And finally the stationary solutions $[x, 0, z]$, $t \in \mathbb{R}$, where $x = 0$ or $z = 0$.

It remains to resolve the cases where x or z is zero. If $x(t_0) = 0$, we have $x' = 0$, $y' = 0$, and $z' = yz$, which gives the (uniquely determined) solution $[0, c, de^{ct}]$, $t \in \mathbb{R}$. If $z(t_0) = 0$, we have $z' = 0$, $y' = 0$, and $x' = xy$, which gives the (uniquely determined) solution $[de^{ct}, c, 0]$, $t \in \mathbb{R}$.

Exercises on Systems of Differential Equations

Find the general solutions of the following systems:

1. $x' = xy$, $y' = xy$ $(x, y > 0)$

2. $x' = x/(x+y)^2$, $y' = y/(x+y)^2$ $(x, y > 0)$

3. $x' = x^2/y$, $y' = x$ $(x, y > 0)$

4. $x' = x/(x-y)$, $y' = y/(x-y)$ $(x > y > 0)$

5. $x' = y^2$, $y' = yz$, $z' = -z^2$ $(z, y > 0)$

6. $x' = -x^2$, $y' = xy - 2z^2$, $z' = xz$ $(x > 0)$

7. $x' = 1 + z$, $y' = y^2 e^{3x}$, $z' = (1+z)^2$ $(y > 0,\ z > -1)$

8. $x' = (1-x)^4$, $y' = (x-1)^3$, $z' = z^3 e^{-y}$ $(x > 1,\ z > 0)$

9. $x' = z^2 - y^2$, $y' = z$, $z' = -y$ $(y > z > 0)$

10. $x' = x - y^2$, $y' = y$, $z' = x + y^2 + z$ $(y > 0)$

11. Prove Theorem 1 (in the first part, first show that x has an inverse function on some neighborhood of t_0).

12. For the predator-prey system (which variable is which?) $x' = kx - axy$, $y' = -ly + bxy$, $x(0) > 0$, $y(0) > 0$, with $a, b, k, l > 0$:
(a) Prove that neither species goes extinct.
(b) Find a first integral of the system.
(c) Sketch the phase portrait of various solutions.

13. Find a first integral of the system $x' = x + y$, $y' = x^2 - y^2$.

14. Find a first integral of the system $x' = (x-y)x^2$, $y' = x^3 - y^3$ on the domain $x > 0$.

15. Let $F : \Omega \to \mathbb{R}^2$ and $\lambda : \Omega \to \mathbb{R}$ be continuous functions on an open set $\Omega \subset \mathbb{R}^2$, with $\lambda(X) \neq 0$. Show that the systems $X' = F(X)$ and $X' = \lambda(X)F(X)$ have the same orbits.

Solutions

1) $x(t) = c(1 - e^{c(t+d)})^{-1}$, $y(t) = ce^{c(t+d)}(1 - e^{c(t+d)})^{-1}$, $t \in (-\infty, -d)$ for $c > 0$, $t \in (-d, +\infty)$ for $c < 0$.

2) $x(t) = \pm\sqrt{2}/(k+1)\,\sqrt{t + c(k+1)^2}$, $y(t) = \pm\sqrt{2}k/(k+1)\,\sqrt{t + c(k+1)^2}$, $t \in (-c(k+1)^2, +\infty)$ for $k > 0$.

3) $x(t) = e^{t/k+c}$, $y(t) = ke^{t/k+c}$, $t \in \mathbb{R}$.

4) $x(t) = t/(1-k) + c$, $y(t) = kt(1-k) + ck$, $t \in (-c/(1-k), +\infty)$, $k \in (0, 1)$.

5) $z(t) = 1/(c_1 - t)$, $y(t) = c_2(c_1 - t)$, $x(t) = c_2^2 t(c^2 - ct + t^2/3) + c_3$, $t \in (-\infty, c_1)$, $c_2 > 0$.

6) $x(t) = 1/(c_1 - t)$, $z(t) = c_2(c_1 - t)$, $y(t) = 1/(c_1 - t) \cdot (2c_3 - c_2^2 t^4 + 4c_2^2 c_1 t^3 - 6c_2^2 c_1^2 t^2 + 4c_2^2 c_1^3 t - c_2^2 c_1^4)/2$, $t \in (-\infty, c_1)$.

7) $x(t) = -\ln(-t - c) + b$, $y(t) = \frac{2(t+c)^2}{e^{3b} + 2k(t+c)^2}$, $z(t) = -1 - 1/(t + c)$, $t \in (-\infty, -c)$ if $k \geq 0$. For $k < 0$, $t \in (-\sqrt{-e^{3b}/2k} - c, -c)$, where $c, b \in \mathbb{R}$.

8) $x(t) = 1 - 1/\sqrt[3]{3(t + c)}$, $y(t) = -\frac{1}{3}\ln|t+c| + b$, $z(t) = (-2d - 8/3e^{-b}|t + c|^{4/3})^{-1/2}$, $t \in (3de^b/4 - c, -c)$, where $d < 0$, $b, c \in \mathbb{R}$.

9) $x(t) = \frac{1}{2}(c_1 - c_2 - \sin(2t + 2c_3))$, $y(t) = \sqrt{c_1}\sin(t + c_3)$, $z(t) = \sqrt{c_1}\cos(t + c_3)$, $t \in \mathbb{R}$.

10) $x(t) = c_1 c_2 e^t - c_2^2 e^{2t}$, $y(t) = c_2 e^t$, $z(t) = (c_2 t + c_3)e^t$, $t \in \mathbb{R}$.

11) (i) Since $x' = f(x, y) \neq 0$, x is injective on a neighborhood of t_0. Denoting the inverse function by T, differentiate $y(T(x))$ with respect to x. Part (ii) follows from the chain rule.

12) (a) For the initial condition $x_0 = 0$, we have $x' = 0, y' = -ly$; for $y_0 = 0$, we have $y' = 0, x' = kx$. Thus, solutions lie entirely on the axes, and by uniqueness, solutions starting off the axes never reach the axes.
(b) We obtain

$$0 = \frac{x'}{x}(-l + bx) - \frac{y'}{y}(k - ay),$$

which after integration yields $-l\ln x + bx - k\ln y + ay = c$, providing a first integral.
(c) Fixing y and examining the function $-l\ln x + bx$, we find that it attains any given value at either two points, one point, or no points. This implies that solutions are periodic closed orbits. Analyzing the derivatives shows that they circulate counterclockwise around the stationary point $(l/b, k/a)$.

13) By Theorem 1, we obtain $y(x) = ce^{-x} + x - 1$. From this, $V(x, y) = e^x(y - x + 1)$.

14) By Theorem 1, we obtain $y'(x) = 1 + y/x + y^2/x^2$. Hence $\operatorname{arctg}(y/x) = \ln x + c$, i.e., $V(x, y) = \operatorname{arctg}(y/x) - \ln x$.

15) Denoting the components of F as (f, g), equation (4) from Theorem 1 takes the form

$$\tilde{y}' = \frac{g(x,\tilde{y})}{f(x,\tilde{y})}, \text{ resp. } \quad \tilde{y}' = \frac{\lambda(x,\tilde{y})g(x,\tilde{y})}{\lambda(x,\tilde{y})f(x,\tilde{y})}.$$

The assertion follows directly from Theorem 1.

Polar Coordinates

In certain cases, a system of differential equations can be solved by transforming it into polar coordinates. Consider the system

$$\begin{aligned} x' &= f(x,y), \\ y' &= g(x,y) \end{aligned} \tag{7}$$

on an open set $\Omega \subset \mathbb{R}^2$. Introduce the variables r, φ via the transformations

$$\begin{aligned} x &= r\cos\varphi, \\ y &= r\sin\varphi. \end{aligned} \tag{8}$$

By the implicit function theorem, on a neighborhood of any point except the origin, these equations define C^∞ functions $r(x,y)$ and $\varphi(x,y)$. If $(x(\cdot), y(\cdot))$ is a C^1 solution of (7) in a disk not containing the origin, then the functions $r(\cdot) := r(x(\cdot), y(\cdot))$ and $\varphi(\cdot) := \varphi(x(\cdot), y(\cdot))$ are also of class C^1 and satisfy

$$\begin{aligned} x' &= r'\cos\varphi - r\sin\varphi\cdot\varphi', \\ y' &= r'\sin\varphi + r\cos\varphi\cdot\varphi'. \end{aligned}$$

The functions r and φ must therefore satisfy the system

$$\begin{aligned} r'\cos\varphi - r\sin\varphi\cdot\varphi' &= f(r\cos\varphi, r\sin\varphi), \\ r'\sin\varphi + r\cos\varphi\cdot\varphi' &= g(r\cos\varphi, r\sin\varphi). \end{aligned}$$

After simplification:

$$\begin{aligned} r' &= \cos\varphi\cdot f(r\cos\varphi, r\sin\varphi) + \sin\varphi\cdot g(r\cos\varphi, r\sin\varphi), \\ r\varphi' &= \cos\varphi\cdot g(r\cos\varphi, r\sin\varphi) - \sin\varphi\cdot f(r\cos\varphi, r\sin\varphi). \end{aligned} \tag{9}$$

Conversely, if r and φ are C^1 functions with values in a disk not containing the origin that satisfy (9), then the functions x, y defined in (8) are also of class C^1 and solve (7).

Transformation to polar coordinates is especially well-suited for systems containing expressions of the form $x^2 + y^2$ and/or rotational symmetries of the type $(-y, x)$.

Example 4. Find all solutions of the system

$$x' = x(x^2+y^2), \tag{10}$$
$$y' = y(x^2+y^2). \tag{11}$$

Solution. The origin is clearly a stationary solution. Away from the origin, we transform the system into polar coordinates:

$$r'\cos\varphi - r\sin\varphi\varphi' = r^3\cos\varphi, \tag{12}$$
$$r'\sin\varphi + r\cos\varphi\varphi' = r^3\sin\varphi. \tag{13}$$

Adding the first equation multiplied by $\cos\varphi$ to the second equation multiplied by $\sin\varphi$ yields

$$r' = r^3,$$

whence (by separation of variables)

$$r(t) = \frac{1}{\sqrt{-2t+2c}}, \quad t \in (-\infty, c).$$

Subtracting the first equation multiplied by $\sin\varphi$ from the second equation multiplied by $\cos\varphi$ gives

$$r\varphi' = 0,$$

meaning that φ is constant. Transforming back to Cartesian coordinates gives all solutions in the form

$$x(t) = a\frac{1}{\sqrt{-2t+2c}}\,, \quad y(t) = \pm\sqrt{1-a^2}\frac{1}{\sqrt{-2t+2c}}\,, \quad t \in (-\infty, c),$$

where $c \in \mathbb{R}$ and $a \in [-1, 1]$.

Exercises on Polar Coordinates

16. Find all solutions of the differential equation $y' = \frac{x-y}{x+y}$.

17. Find all solutions of the system

$$x' = x^2y, \tag{14}$$
$$y' = xy^2. \tag{15}$$

18. Find all solutions of the system

$$x' = y + ax(x^2+y^2), \tag{16}$$
$$y' = -x + ay(x^2+y^2). \tag{17}$$

19. Find all solutions of the system

$$x' = -y + x(x^2 + y^2 - 1), \tag{18}$$
$$y' = x + y(x^2 + y^2 - 1). \tag{19}$$

Solutions

16) By Theorem 1, it is useful to solve the system $\frac{dy}{dt} = x - y$, $\frac{dx}{dt} = x + y$. Converting to polar coordinates yields $r' = r$, $\varphi' = 1$. Thus $\varphi(t) = t + c$, $r(t) = de^t$, and therefore $x(t) = de^t \cos(t + c)$, $y(t) = de^t \sin(t + c)$. The solution curves are portions of logarithmic spirals.

17) In polar coordinates: $\varphi' = 0$, $r' = r^3 \cos\varphi \sin\varphi$, hence $\varphi(t) = \varphi_0$, $r(t) = (-2t \sin\varphi_0 \cos\varphi_0 + d)^{-1/2}$.

18) In polar coordinates: $\varphi' = -1$, $r' = ar^3$, hence $r(t) = 1/\sqrt{2a(c - t)}$, $\varphi(t) = \varphi_0 - t$, with $t \in (-\infty, c)$ for $a > 0$ and $t \in (c, +\infty)$ for $a < 0$.

19) In polar coordinates: $\varphi' = 1$, $r' = r(r^2 - 1)$, hence $\varphi(t) = t + \varphi_0$, $r(t) = (1 - ke^{2t})^{-1/2}$ for $k \leq 0$ ($t \in \mathbb{R}$), and for $k > 0$ ($t \in (-\infty, -\frac{1}{2} \ln k)$).

Kepler's Laws

In this section, we will show how Kepler's laws of planetary motion follow directly from Newton's law of universal gravitation.

Newton's Equations of Motion

Let $X = (x, y, z)$ denote the position of a planet relative to the Sun, and let $r = \sqrt{x^2 + y^2 + z^2}$. Newton's law yields

$$mX'' = -\frac{\kappa m M}{r^2} \frac{X}{r},$$

where m, M are the masses of the planet and the Sun, respectively, and κ is the gravitational constant. Written in components:

$$x'' = -\frac{kx}{r^3}, \quad y'' = -\frac{ky}{r^3}, \quad z'' = -\frac{kz}{r^3},$$

where we define

$$k = \kappa M. \tag{20}$$

We consider this system with the initial conditions

$$X(0) = (x_0, 0, 0), \qquad X'(0) = (v_1, v_2, 0),$$

where $x_0 > 0$ and $v_2 \neq 0$. From the initial conditions and uniqueness theorems, it follows that $z \equiv 0$; thus, it suffices to consider the problem in the planar (x, y) coordinates.

Transformation to the Complex Plane

We switch to polar coordinates (r, φ) using the complex representation

$$Z = x + iy = re^{i\varphi}.$$

Differentiating, we obtain expressions for velocity and acceleration:

$$Z' = \big(r' + ir\varphi'\big)e^{i\varphi}, \tag{21}$$

$$Z'' = \Big\{\big(r'' - r(\varphi')^2\big) + i\big(2r'\varphi' + r\varphi''\big)\Big\}e^{i\varphi}. \tag{22}$$

Newton's law in complex form,

$$Z'' = -\frac{k}{r^2}e^{i\varphi}, \tag{23}$$

then implies (dividing by $e^{i\varphi}$ and equating real and imaginary parts):

$$r'' - r(\varphi')^2 = -\frac{k}{r^2}, \tag{24}$$

$$2r'\varphi' + r\varphi'' = 0. \tag{25}$$

Kepler's Second Law

Equation (25) implies that the quantity $r^2\varphi'$ has zero derivative, meaning

$$r^2\varphi' = c_1 \tag{26}$$

for some constant c_1. Recall the formula for area under a curve in polar coordinates:

$$\lambda_2(M) = \frac{1}{2}\int_\alpha^\beta f^2(\varphi)\,d\varphi,$$

where

$$M = \big\{(r\cos\varphi, r\sin\varphi);\ \alpha \le \varphi \le \beta,\ 0 \le r \le f(\varphi)\big\}.$$

Substituting $\varphi = \varphi(t)$, with

$$\alpha = \varphi(t_1), \quad \beta = \varphi(t_2), \quad d\varphi = \varphi'\,dt,$$

it follows that the area swept out by the position vector between times t_1 and t_2 is

$$\mathcal{P}(t_1, t_2) = \frac{1}{2}\int_{t_1}^{t_2} r^2\varphi'\,dt = \frac{c_1}{2}(t_2 - t_1). \tag{27}$$

The area therefore increases *linearly* with time, which is Kepler's second law.

Kepler's First Law

Assume henceforth that $c_1 > 0$. Then (26) and (23) yield

$$Z'' = -\frac{k}{c_1} e^{i\varphi} \varphi';$$

whence

$$Z' = \frac{ik}{c_1} e^{i\varphi} + ic_2 e^{i\varphi_0}.$$

The second term is an integration constant in polar form; we may assume $c_2 \geq 0$. Comparing this with (21) gives

$$\begin{aligned} \frac{ik}{c_1} e^{i\varphi} + ic_2 e^{i\varphi_0} &= \big(r' + ir\varphi'\big) e^{i\varphi}, \\ \frac{ik}{c_1} + ic_2 e^{-i(\varphi - \varphi_0)} &= r' + ir\varphi'. \end{aligned}$$

Taking only the imaginary part and applying (26) once again:

$$\begin{aligned} \frac{k}{c_1} + c_2 \cos(\varphi - \varphi_0) &= r\varphi' = \frac{c_1}{r}, \\ r &= \frac{c_1^2}{k + c_1 c_2 \cos(\varphi - \varphi_0)}. \end{aligned}$$

Without loss of generality, assume $\varphi_0 = 0$. We write the equation as

$$r = \frac{A}{1 + \varepsilon \cos \varphi}, \tag{28}$$

where we define

$$A := \frac{c_1^2}{k}, \qquad \varepsilon := \frac{c_1 c_2}{k}.$$

This is the polar equation of a conic section with a focus at the origin; the cases $\varepsilon = 0$ (resp. $\varepsilon \in (0,1)$, $\varepsilon = 1$, and $\varepsilon > 1$) correspond to a circle (resp. ellipse, parabola, and hyperbola).

Kepler's first law concerns bounded planetary motion along an ellipse, i.e., $\varepsilon \in (0,1)$. Let a, b denote the semi-major and semi-minor axes, $E = \sqrt{a^2 - b^2}$ the linear eccentricity, and $\varepsilon = E/a$ the numerical eccentricity. In Cartesian coordinates, (28) becomes

$$\left(\frac{x + E}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1,$$

where

$$\begin{aligned} E &= \lambda c_1^3 c_2, \\ a &= \lambda k c_1^2, \\ b &= \sqrt{\lambda} c_1^2, \\ \lambda &= \frac{1}{k^2 - c_1^2 c_2^2}. \end{aligned}$$

Kepler's Third Law

If T is the orbital period, the formula for the area of an ellipse together with (27) yields

$$\pi a b = \frac{c_1}{2} T;$$

applying the relations for b and a above then gives

$$T = 2\pi a \sqrt{\lambda} c_1 = \frac{2\pi}{\sqrt{k}} a^{3/2}.$$

Recalling that the constant k depends neither on the mass of the planet nor on its initial position, we obtain Kepler's third law ($T^2 \propto a^3$).

Backward Derivation of Newton's Law

Assume Kepler's 1st and 2nd laws hold, i.e.,

$$r = \frac{A}{1 + \varepsilon \cos \varphi}, \qquad r^2 \varphi' = c_1.$$

Using again $Z = re^{i\varphi}$, we have

$$Z' = \frac{c_1}{A} i e^{i\varphi} + \frac{c_1 i \varepsilon}{A}$$

and

$$Z'' = -\frac{c_1^2}{A r^2} e^{i\varphi};$$

thus the acceleration points directly toward the origin and is inversely proportional to r^2, in full agreement with Newton's law of gravitation.

Second-Order Differential Equations

The techniques discussed above can also be applied to second-order (and higher-order) equations, which can be converted into systems of first-order equations.

Example 5. Find a formula for the period of the solutions to the equation

$$x'' + x = 0$$

with initial condition $x(0) = 0$, $x'(0) = c$.

Solution. This equation is linear, and standard linear ODE theory with constant coefficients easily shows that the general solution is $x(t) = a \sin t + b \cos t$, so the period is 2π independently of the initial condition. However, we will demonstrate a more general procedure applicable to equations of the form

$$x'' + f(x) = 0.$$

One easily verifies that

$$V(x, y) = \frac{1}{2}y^2 + F(x)$$

is a first integral of the equation, where $F' = f$ and $y = x'$. Indeed,

$$\frac{d}{dt}V(x(t), x'(t)) = f(x(t))x'(t) + x'(t)x''(t) = 0.$$

In our case, $V(x, y) = \frac{1}{2}(x^2 + y^2)$. Thus,

$$x(t)^2 + (x'(t))^2 = k. \tag{29}$$

If x satisfies the initial condition, we have $0^2 + c^2 = k$, so $k = c^2$. Clearly, as x increases, the velocity x' decreases until the system reaches $x(T) = c$, $x'(T) = 0$. From equation (29), which represents the law of conservation of energy, we obtain

$$x' = \sqrt{c^2 - x^2},$$

which after separation of variables (or using Barrow's formula) yields

$$\int_0^{x(T)} \frac{dx}{\sqrt{c^2 - x^2}} = T - 0.$$

Since we are interested in the value T for which $x(T) = c$, the upper limit is c, and integrating gives

$$\int_0^c \frac{dx}{\sqrt{c^2 - x^2}} = \int_0^1 \frac{dz}{\sqrt{1 - z^2}} = [\arcsin z]_0^1 = \frac{\pi}{2} = T.$$

After reaching maximum displacement, the system returns to its initial state, reaches its most negative value, and returns to the origin again. Because each of these four quarter-phases takes time T, the total period is $4T = 2\pi$. (Note that after these four phases, the system satisfies $x(4T) = 0$ and $x'(4T) = c$; by uniqueness of solutions for autonomous equations, $x(4T + t) = x(t)$, so the solution is strictly periodic with period $4T$.)

Exercises on Second-Order Equations

20. Recall that the equation $x'' + f(x) = 0$, with $f \in C^1(I)$, can be written as $x'' + F'(x) = 0$, where $F' = f$. This represents the motion of a particle in a potential well, where acceleration is governed by the slope of the potential.
(a) Show that if $f(x_0) = 0$ and $f'(x_0) > 0$, then for initial conditions close to $(x_0, 0)$, i.e., $x(0) = x_0 + \varepsilon_1$, $x'(0) = \varepsilon_2$, we obtain periodic solutions.
(b) Show that if $f(x_0) = 0$ and $f'(x_0) > 0$, then the point $(x_0, 0)$ is a stable (though not asymptotically stable) equilibrium point.

21. Using Theorem 1, derive the first integral for the second-order equation $x'' + f(x) = 0$ (i.e., a function V such that $t \mapsto V(x(t), x'(t))$ is constant along solutions).

22. Find a first integral of the equation $x'' + \sin x = 0$, sketch the phase portrait, and use Barrow's formula to find an integral formula for the period.

23. Find a first integral of the equation $x'' + 2x^3 = 0$, sketch the phase portrait, and use Barrow's formula to find an integral formula for the period.

24. How much time does it take for a solution of $x'' = 1/2\ e^x$, $x(0) = 0$, $x'(0) = 0$ to reach the value $x = \ln 5$?

25. How much time does it take for a solution of $x'' = \sinh x \cosh x$, $x(0) = 0$, $x'(0) = 1$ to reach the value $x = 1$?

26. Consider the equation $x'' + f(x) = 0$ and let $V(x, x') := V(x, y) := \frac{y^2}{2} + F(x)$, with $F' = f$. Show that if $\nabla V(x, 0) \neq 0$, then the trajectory crossing the point $(x, 0)$ intersects the x-axis perpendicularly.

27. Consider the motion of a particle in a central gravitational field:

$$X'' + \nabla U(X) = 0, \quad X \in \mathbb{R}^3 \setminus \{0\},$$

where $U(X) = U_0(|X|)$, $U_0 \in C^2((0, +\infty), \mathbb{R})$. Show that the orbit of the particle lies in a plane.

28. Compute a first integral for the equation $x'' + x' + x = 0$.

29. Find a first integral of the equation $x'' + x^2 - x = 0$, sketch the phase portrait, and use Barrow's formula to find an integral formula for the period.

Solutions

20) (a) The first integral is $V(x,y) = y^2/2 + F(x)$. F has a strict local minimum at x_0, being strictly decreasing on $(x_0-\delta, x_0)$ and strictly increasing on $(x_0, x_0+\delta)$, with $F(x_0) = 0$ without loss of generality. Because V is conserved along trajectories, the level sets form closed curves in the phase plane.
(b) Follows directly from (a).

21) From the system $x' = y, y' = -f(x)$, we obtain $y'(x) = -f(x)/y$, which gives $y^2/2 = -F(x)+c$. The first integral is thus $V(x,y) = y^2/2+F(x)$.

22) $V(x,y) = 1 - \cos x + y^2/2$, with $T = 4\int_0^{\arccos(1-c^2/2)} \frac{1}{\sqrt{2c^2+4(\cos x-1)}}\,dx$. Substituting $z = \cos x$ and $y = (1-z)/c^2$ yields $T = 4\int_0^{1/2} \frac{1}{\sqrt{y(2-cy)(2-4y)}}\,dy$, showing that larger initial amplitudes result in longer periods.

23) $V(x,y) = x^4/2 + y^2/2$, with $T = 4\int_0^{\sqrt[4]{2}c} \frac{1}{\sqrt{2c-x^4}}\,dx$. After substitution, $T = 4(2c)^{-1/4}\int_0^1 (1-z^4)^{-1/2}\,dz$, showing that larger initial amplitudes result in shorter periods.

24) $V(x,y) = y^2/2 - e^x/2 = -1/2$ (determined by initial conditions). Then $t = \int_0^{\ln 5}(e^x-1)^{-1/2}\,dx$. After substitution, $t = \operatorname{arctg} 2$.

25) $V(x,y) = y^2/2 - \cosh^2(x)/2 = 0$ (determined by initial conditions). Then $t = \int_0^1 (\cosh x)^{-1}\,dx = 2\operatorname{arctg} e - \pi/2$.

26) Since $0 = dV/dt = yy' + f(x)x'$ and $y = 0$, $f(x) \neq 0$, it follows that $x' = 0$ relative to y, meaning the tangent vector (x', y') is vertical.

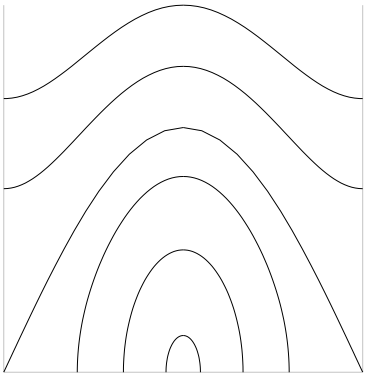
27) Choose the coordinate system such that $X(0) = (x_0, 0, 0)$ and $X'(0) = (v_1, v_2, 0)$. Let $X(t) = (x(t), y(t), z(t))$. Then $X'' = -U_0'(|X|)\frac{1}{|X|}X$, which in the third coordinate gives $z'' = g(t)z$ with $z(0) = 0, z'(0) = 0$, implying $z(t) \equiv 0$ by uniqueness.

28) $V(x,y) = 1/2\ln|y^2 - yx - x^2| - 1/\sqrt{3}\operatorname{arctgh}((2y/x-1)/\sqrt{3})$ is a local first integral (for $x \neq 0$).

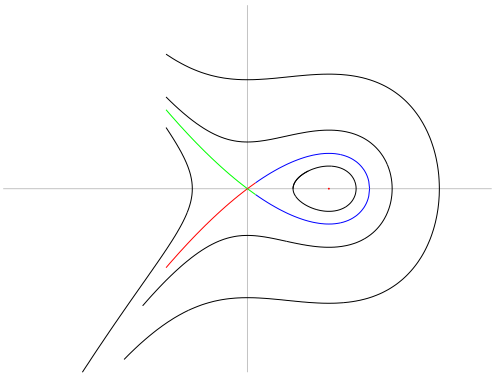
29) $V(x,y) = y^2/2 + x^3/3 - x^2/2$; stationary points at $(0,0)$ and $(1,0)$; reflection symmetry across the x-axis ($y = 0$); homoclinic orbit $y = \pm x\sqrt{1-2x/3}$, $x \in [0, 3/2]$; periodic solutions around $(1,0)$ inside the homoclinic loop, unbounded solutions outside. The period of the solution with initial condition $x(0) = x_0 \in (0,1)$, $x'(0) = 0$ is

$$2\int_{x_0}^{1} \frac{1}{\sqrt{x^2 - x_0^2 - \frac{2}{3}(x^3 - x_0^3)}}\,dx + 2\int_{1}^{x_1} \frac{1}{\sqrt{x^2 - x_0^2 - \frac{2}{3}(x^3 - x_0^3)}}\,dx,$$

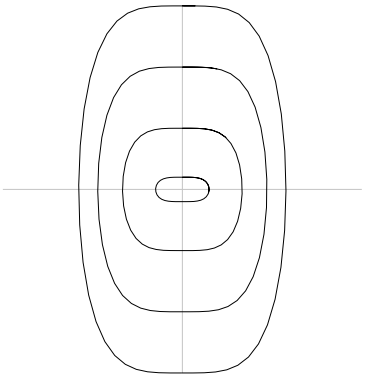
where $x_1 = \frac{3}{4} - \frac{1}{2}x_0 + \frac{1}{2}\sqrt{\frac{9}{4} + 3x_0 - 3x_0^2}$.



(a) Exercise 22



(b) Exercise 29



(a) Exercise 23