Stability

In this chapter, we will investigate the asymptotic behavior of solutions as time tends to infinity. Consider the autonomous system

$$x' = f(x)\,, \tag{1}$$

where $f \in C^1(\Omega, \mathbb{R}^n)$, $\Omega \subset \mathbb{R}^n$, and $x : \mathbb{R} \to \Omega$. Let x_0 be a *stationary point* (or *equilibrium point*), i.e., $f(x_0) = 0$.

The point x_0 is called *stable* (in the sense of Lyapunov) if

$$\big(\forall \varepsilon > 0\big)\big(\exists \delta > 0\big)\Big[|x(0) - x_0| < \delta \implies |x(t) - x_0| < \varepsilon \quad \forall t \geq 0\Big],$$

where $x(t)$ is the solution of (1) satisfying the initial condition $x(0)$ at time $t = 0$. The point x_0 is called *asymptotically stable* if it is stable and, in addition,

$$\big(\exists \Delta > 0\big)\Big[|x(0) - x_0| < \Delta \implies \lim_{t\to\infty} x(t) = x_0\Big]. \tag{2}$$

The point x_0 is called *unstable* if it is not stable.

Intuitively: stability means that any solution starting "close" to x_0 remains "close" to x_0 for all $t \geq 0$. Asymptotic stability additionally means that any solution starting "close" to x_0 converges to x_0 as $t \to \infty$. Condition (2) alone states that x_0 is a *local attractor*. This condition does not generally imply stability (see Vinogradov's system below).

Linear Systems

For linear systems with a constant matrix

$$x' = Ax, \qquad A \in \mathbb{R}^{n\times n}, \tag{3}$$

the situation is straightforward. If the matrix A is non-singular (invertible), the equation has a unique stationary point $x_0 = 0$, and its stability is characterized by the following theorem. (If A is singular, the equation has an entire subspace of stationary points sharing the same stability type, for which the following theorem also applies.)

Theorem 1 (Lyapunov's Stability Theorem for Linear Systems)**.** *Consider the system* (3) *with a constant matrix* A*.*

(i) The point 0 *is asymptotically stable if and only if all eigenvalues of* A *have strictly negative real parts.*

(ii) The point 0 *is stable if and only if all eigenvalues of* A *have non-positive real parts and the Jordan blocks corresponding to eigenvalues with zero real parts are of size* 1*.*

(iii) The point 0 *is unstable if and only if at least one eigenvalue of* A *has a strictly positive real part, or has a zero real part with an associated Jordan block of size at least* 2*.*

This theorem follows directly from the explicit solution formula; see the chapter on the matrix exponential. To facilitate terminology, recall that the set of eigenvalues of the matrix A is called its spectrum and is denoted by $\sigma(A)$. Furthermore, we introduce the notation $s(A) := \max\{\operatorname{Re}\lambda : \ \lambda \in \sigma(A)\}$. This quantity, which is of fundamental importance for the stability of stationary points, is known as the *spectral bound*.

Recall also the key estimate mentioned in the chapter on the matrix exponential: for every $\beta > s(A)$, there exists $M \geq 1$ such that

$$\|e^{tA}\| \leq Me^{\beta t} \quad \text{for all} \quad t \geq 0. \tag{4}$$

Example 1. Find the solution to the system

$$x' = \begin{pmatrix} -4 & -2 & -1 \\ -3 & -5 & 1 \\ 0 & 2 & -2 \end{pmatrix} x$$

and verify that the result is in accordance with Theorem 1.

Solution. Perform row operations on the λ-matrix:

$$\begin{pmatrix} \lambda+4 & 2 & 1 \\ 3 & \lambda+5 & -1 \\ 0 & -2 & \lambda+2 \end{pmatrix} \sim \begin{pmatrix} 0 & -(\lambda^2+9\lambda+14) & \lambda+7 \\ 3 & \lambda+5 & -1 \\ 0 & -2 & \lambda+2 \end{pmatrix}$$

$$\sim \begin{pmatrix} 0 & -(\lambda^2+9\lambda+12) & 5 \\ 3 & \lambda+5 & -1 \\ 0 & -10 & 5(\lambda+2) \end{pmatrix} \sim \begin{pmatrix} 0 & -(\lambda^2+9\lambda+12) & 5 \\ 3 & \lambda+5 & -1 \\ 0 & (\lambda+7)(\lambda^2+4\lambda+2) & 0 \end{pmatrix}.$$

From this, we obtain $\lambda_1 = -7$, $\lambda_{2,3} = -2 \pm \sqrt{2}$, and

$$x_2(t) = ce^{-7t} + de^{(-2+\sqrt{2})t} + fe^{(-2-\sqrt{2})t}, \quad c, d, f \in \mathbb{R}.$$

Moreover, it is clear that x_1 and x_3 are also linear combinations of these three exponentials. Since all λ_i (which are the eigenvalues of the system

matrix) are strictly negative, we immediately obtain $x_i \to 0$ as $t \to +\infty$ for $i = 1, 2, 3$; thus, the zero equilibrium is asymptotically stable.

Computing x_1 and x_3 explicitly yields

$$\begin{aligned} x_3(t) &= -\frac{2}{5}ce^{-7t} + \sqrt{2}de^{(-2+\sqrt{2})t} - \sqrt{2}fe^{(-2-\sqrt{2})t}, \\ x_1(t) &= \frac{8}{15}ce^{-7t} + \frac{-7+2\sqrt{2}}{3}de^{(-2+\sqrt{2})t} + \frac{-7-2\sqrt{2}}{3}fe^{(-2-\sqrt{2})t}. \end{aligned}$$

Remark. It is important to emphasize that the criteria of the preceding theorem *cannot* be applied in the case of non-autonomous systems, i.e., when A depends on time. This is well illustrated by an example due to L. Markus and H. Yamabe: consider the matrix

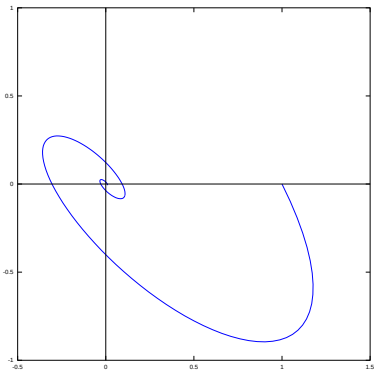
$$A(t) = \tfrac{1}{4}\begin{pmatrix} 3\cos 2t - 1 & 4 - 3\sin 2t \\ -4 - 3\sin 2t & -1 - 3\cos 2t \end{pmatrix}.$$

One easily calculates that for each t, the eigenvalues of $A(t)$ have strictly negative real parts; indeed, $\lambda_{1,2} = \frac{1}{4}(-1 \pm \sqrt{7}i)$ independently of t. However, the function

$$x(t) = e^{t/2}\begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}$$

is a solution to the system $x' = A(t)x$; thus, this system is unstable.

The intuition behind this counterexample is not difficult to grasp. It is related to the fact that even in the case of a constant stable matrix, trajectories need not monotonically approach the origin at every instant. For example, solutions of the system $y' = A(0)y$ trace out "elliptical spirals", and near the x-axis, their distance from the origin temporarily increases. Rotating the matrix $A(0)$ at an appropriate angular velocity in the direction of motion of the solution leads to persistent outward displacement and thus instability.



Exercises

1. Find the general solutions to the systems with the following matrices A and verify that their asymptotic behavior agrees with Theorem 1:

$$A = \begin{pmatrix} 0 & 2 & 3 \\ 1 & -1 & -1 \\ 1 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 2 & -3 \\ 0 & -2 & 3 \\ 0 & 4 & -6 \end{pmatrix}, \quad \begin{pmatrix} -1 & 2 & -1 \\ -3 & 1 & 2 \\ 0 & 1 & -1 \end{pmatrix}$$

2. Find a necessary and sufficient condition for all solutions of (3) to be bounded on $\mathbb{R}$.

3. Show that if 0 is an asymptotically stable point of the system (3) and the entries of the matrix $B \in \mathbb{R}^{n\times n}$ are sufficiently small, then 0 is an asymptotically stable point of the system $x' = (A + B)x$.

4. Formulate necessary and sufficient conditions under which a real 2×2 matrix has eigenvalues with strictly negative real parts.

5. Prove or give a counterexample to the following assertion: If there exists a solution to (3) that is unbounded on $(0, +\infty)$, then there exists a non-zero solution x_2 such that $\lim_{t\to-\infty} \|x_2(t)\| = 0$.

6. Show that the zero solution of a linear system is asymptotically stable if and only if it is uniformly exponentially stable, i.e., there exist $M, c > 0$ such that $\|x(t)\| \le Me^{-ct}\|x(0)\|$ for all $t \ge 0$.

Solutions

1) (a) The solutions are linear combinations of the exponentials e^{2t}, e^{-2t}, e^{-t}. A solution is given, for example, by the triple

$$\begin{aligned} x_2(t) &= C(\frac{2}{3}e^{-t} + \frac{1}{12}e^{2t} + \frac{1}{4}e^{-2t}), \\ x_1(t) &= C(-\frac{1}{3}e^{-t} + \frac{7}{12}e^{2t} - \frac{1}{4}e^{-2t}), \\ x_3(t) &= C(-\frac{1}{3}e^{-t} + \frac{1}{3}e^{2t}), \end{aligned}$$

$C \in \mathbb{R}$. Components of this solution tend to infinity, no matter how small the initial condition (C) is chosen. The zero solution is unstable. The eigenvalues of the matrix are 2, -2, -1.

(b) The solutions are

$$\begin{aligned} x_2(t) &= \frac{1}{4}C - \frac{1}{4}Ce^{-8t} + D - \frac{3}{8}F + \frac{3}{8}Fe^{-8t}, \\ x_1(t) &= \frac{1}{2}C - \frac{1}{2}Ce^{-8t} + \frac{1}{4}F + \frac{3}{4}Fe^{-8t}, \\ x_3(t) &= \frac{3}{4}C + \frac{1}{4}Ce^{-8t} + \frac{3}{8}F - \frac{3}{8}Fe^{-8t}, \end{aligned}$$

i.e., the zero solution is stable, but not asymptotically stable. The eigenvalues are 0, 0, -8, and there exist two linearly independent eigenvectors corresponding to the eigenvalue 0 (i.e., two Jordan blocks of size 1).
(c) The solutions are linear combinations of the functions

$$1,\ e^{-t/2}\cos(\sqrt{11}t/2),\ e^{-t/2}\sin(\sqrt{11}t/2),$$

i.e., the zero equilibrium is stable, but not asymptotically stable. The eigenvalues are 0, $-1/2 + i\sqrt{11}/2$, $-1/2 - i\sqrt{11}/2$.

2) All eigenvalues of the matrix A are purely imaginary and have geometric multiplicity equal to algebraic multiplicity (all Jordan blocks have size 1). All solutions are then linear combinations of the functions $\sin \lambda_i t$, $\cos \lambda_i t$.

3) When a matrix C is invertible and $\|B\| < \|C^{-1}\|^{-1}$, then $C + B$ is also invertible, since $C+B = C(I+C^{-1}B)$ and $(C+B)^{-1} = \sum_{i=0}^{\infty}(-C^{-1}B)^i C^{-1}$. Apply this to $C = \lambda I - A$ in combination with the bound $\|(\lambda I - A)^{-1}\|^{-1} \geq k\, d(\lambda, \sigma(A))$, where $k > 0$ depends only on A and not on λ.

4) If and only if $\det A > 0$ and $\mathrm{Tr}(A) < 0$. Hint: a quadratic polynomial $\lambda^2 + p\lambda + q$ has roots with strictly negative real parts if and only if $p > 0$ and $q > 0$.

5) The assertion does not hold; a counterexample is

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and the solution $x_1(t) = (t, 1)^T$. All solutions are linear combinations of 1 and t, none of which decay to zero as $t \to -\infty$.

6) Follows from the construction of e^{tA} using the Jordan canonical form of the matrix, choosing $c = -s(A) - \epsilon > 0$.

Linearized Stability (and Instability)

Consider the nonlinear system (1) and a stationary point x_0. Writing $x = y + x_0$, we obtain

$$y' = f(y + x_0) = f(x_0) + \nabla f(x_0)y + r(y) = Ay + r(y),$$

where $r(y) = o(|y|)$ as $y \to 0$. In a neighborhood of x_0, this represents a linear problem with matrix $A = \nabla f(x_0)$ subject to a small perturbation r. Consequently, the following theorem holds:

Theorem 2 (Linearized (In)stability Theorem / Lyapunov's Indirect Method)**.** *Let $f(x)$ be of class C^1 in a neighborhood of x_0. Define the Jacobian matrix $A := \nabla f(x_0)$.*

(i) If $s(A) < 0$, then x_0 is asymptotically stable.

(ii) If $s(A) > 0$, then x_0 is unstable.

Remark. This theorem is straightforward to apply, but it does not cover all possibilities. The theorem yields no conclusion for critical cases (where some eigenvalues have zero real parts and none have positive real parts), where the higher-order perturbation r can tip the behavior toward stability or instability. In such cases, advanced techniques are required: Lyapunov functions, or LaSalle's Invariance Principle (see the subsequent section "Lyapunov Stability" and the chapter "Dynamical Systems").

Example 2. Find the stationary points of the following system and determine their stability:

$$\begin{aligned} \frac{x'}{z} &= 2xy, \\ y' &= -x - 2z^2 + 8, \\ z' &= z^2 - y^2. \end{aligned}$$

Solution. Multiply the first equation by $z \neq 0$ and solve the system:

$$\begin{aligned} 2xyz &= 0, \\ -x - 2z^2 + 8 &= 0, \\ z^2 - y^2 &= 0. \end{aligned}$$

The first equation requires $x = 0$, $y = 0$, or $z = 0$ (the case $z = 0$ is excluded by the original domain). If $y = 0$, the third equation gives $z = 0$, a contradiction. Thus $x = 0$, which from the second equation implies $z = \pm 2$, and from the third $y = \pm 2$. This yields 4 stationary points: $[0, 2, \pm 2]$ and $[0, -2, \pm 2]$.

The Jacobian matrix is

$$\nabla f(x, y, z) = \begin{pmatrix} 2yz & 2xz & 2xy \\ -1 & 0 & -4z \\ 0 & -2y & 2z \end{pmatrix}.$$

When y and z have the same sign:

$$\nabla f(0, \pm 2, \pm 2) = \begin{pmatrix} 8 & 0 & 0 \\ -1 & 0 & \mp 8 \\ 0 & \mp 4 & \pm 4 \end{pmatrix}.$$

From the first row, 8 is an eigenvalue; hence these points are unstable. Next, for $[0, 2, -2]$:

$$\nabla f(0, 2, -2) = \begin{pmatrix} -8 & 0 & 0 \\ -1 & 0 & 8 \\ 0 & -4 & -4 \end{pmatrix}.$$

The characteristic polynomial is $(\lambda+8)(\lambda(\lambda+4)+32)$, which has roots -8 and $-2 \pm 2i\sqrt{7}$. Since all eigenvalues have negative real parts, this equilibrium is asymptotically stable. Conversely, for $[0, -2, 2]$:

$$\nabla f(0, -2, 2) = \begin{pmatrix} -8 & 0 & 0 \\ -1 & 0 & -8 \\ 0 & 4 & 4 \end{pmatrix},$$

the characteristic polynomial is $(\lambda+8)(\lambda(\lambda-4)+32)$, which has two complex roots with positive real parts $(2 \pm 2i\sqrt{7})$. The point $[0, -2, 2]$ is therefore unstable.

Example 3. By transforming to polar coordinates, show that the origin is an asymptotically stable stationary point of the following system, even though the eigenvalues of $\nabla f(0)$ are zero:

$$\begin{aligned} x' &= -(x^2 + y^2)(2x + 2y + y^2), \\ y' &= -(x^2 + y^2)(x + 2y). \end{aligned}$$

Remark. The exercises include further examples of asymptotically stable and unstable stationary points where the linearized system exhibits zero eigenvalues. In the chapter on Lyapunov stability, you will also find an example of an asymptotically stable equilibrium that becomes unstable upon linearization (Exercise 46).

Solution. Using the transformation $x = r\cos\phi$, $y = r\sin\phi$, we have

$$\begin{aligned} r'\cos\phi - r\sin\phi\phi' &= -r^3(2\cos\phi + 2\sin\phi + r\sin^2\phi), \\ r'\sin\phi + r\cos\phi\phi' &= -r^3(\cos\phi + 2\sin\phi). \end{aligned}$$

Multiplying the first equation by $\cos\phi$, the second by $\sin\phi$, and adding them together gives

$$r' = -r^3(2 + 3\sin\phi\cos\phi + r\sin^2\phi\cos\phi).$$

Since $3\sin\phi\cos\phi \geq -3/2$ and $|r\sin^2\phi\cos\phi| < 1/2$ for sufficiently small r, the bracketed expression is strictly positive $(> 1/2 - 1/2 + \varepsilon > 0)$. Thus $r' \leq -\varepsilon r^3$ in a neighborhood of the origin, establishing asymptotic stability.

The Hartman-Grobman Theorem describes the qualitative dynamics in a neighborhood of a hyperbolic equilibrium point. A stationary point x_0 is called *hyperbolic* if no eigenvalue of $\nabla f(x_0)$ has a zero real part.

Theorem 3 (Hartman-Grobman Theorem)**.** *Let x_0 be a hyperbolic stationary point of* (1) *and denote $A := \nabla f(x_0)$. Then there exist a neighborhood U of* 0*, a neighborhood V of x_0, and a homeomorphism $\Phi : U \to V$ that maps the trajectories of the linear system* (3) *onto the trajectories of the nonlinear system* (1)*, preserving the parameterization by time.*

Example 4. Sketch the phase portrait in a neighborhood of the stationary points for the system:

$$\begin{aligned} x' &= x^2 + y^2 - 1, \\ y' &= e^{x+y} - 1. \end{aligned}$$

Solution. The second equation gives $x+y = 0$, and the first gives $x^2+y^2 = 1$, yielding two equilibria: $[\sqrt{2}/2, -\sqrt{2}/2]$ and $[-\sqrt{2}/2, \sqrt{2}/2]$. The linearized system $Y' = AY = \nabla f(x_0)Y$ at the first point has the matrix

$$A = \begin{pmatrix} \sqrt{2} & -\sqrt{2} \\ 1 & 1 \end{pmatrix}.$$

The eigenvalues are

$$\frac{1}{2}\left(1 + \sqrt{2} \pm \sqrt{(1+\sqrt{2})^2 - 8\sqrt{2}}\right).$$

Since $(1+\sqrt{2})^2 - 8\sqrt{2} = 3 - 6\sqrt{2} < 0$, the eigenvalues are complex with positive real parts; thus, the point $[\sqrt{2}/2, -\sqrt{2}/2]$ behaves as an unstable spiral (focus).

At the second equilibrium $[-\sqrt{2}/2, \sqrt{2}/2]$:

$$A = \begin{pmatrix} -\sqrt{2} & \sqrt{2} \\ 1 & 1 \end{pmatrix}.$$

The eigenvalues are

$$\frac{1}{2}\left(1 - \sqrt{2} \pm \sqrt{(1-\sqrt{2})^2 + 8\sqrt{2}}\right).$$

Since $(1-\sqrt{2})^2 + 8\sqrt{2} > 0$ and $\det(A) = -2\sqrt{2} < 0$, the eigenvalues are real with opposite signs; thus, the point $[-\sqrt{2}/2, \sqrt{2}/2]$ is a saddle point.

Exercises

7. For which values of the parameters are the following systems stable (asymptotically stable)?

$$\begin{aligned} x' &= -x + ay \\ y' &= x - y \end{aligned} \qquad \text{and} \qquad \begin{aligned} x' &= ax + by \\ y' &= bx + ay \end{aligned}$$

8. Investigate the stability of the origin for the system

$$\begin{aligned} x' &= 4(e^x - 1) - 2y - 4z + y^2, \\ y' &= x - 3y - z + (x + y)y^2, \\ z' &= -4z + \sinh x + (x + y)z^2. \end{aligned}$$

9. Investigate the stability of the origin for the system

$$\begin{aligned} x' &= -11x - 48y - 16z + xyz, \\ y' &= x + 3y + 2z + x^2 - yz, \\ z' &= 2y + 2z + \sin x. \end{aligned}$$

10. Find the stationary points of the following system and determine their stability:

$$\begin{aligned} x' &= \sin(xy) - \frac{1}{2}, \\ y' &= -xy - y. \end{aligned}$$

11. Find the stationary points of the following system and determine their stability:

$$\begin{aligned} x' &= 1 - x^2 - y^2, \\ y' &= z^2 - x - y, \\ z' &= z^2 - 1. \end{aligned}$$

12. Find the stationary points of the following system and determine their stability:

$$\begin{aligned} x' &= xy - 2x - y + 2, \\ y' &= xy + yz + xz, \\ z' &= 2y(z + 1). \end{aligned}$$

13. Consider the scalar equation $x' = f(x)$ with continuous f and $f(x_0) = 0$. (a) Formulate and prove sufficient conditions for stability, instability, and asymptotic stability of x_0 based on the sign of f on $(x_0, x_0+\delta)$ and $(x_0-\delta, x_0)$.
(b) Show that when $f'(x_0) \neq 0$, the linearized stability theorem is recovered as a special case.
(c) Let $f(x) = a(x - x_0)^n + g(x)$, where $g(x) = o((x - x_0)^n)$ as $x \to x_0$, with $a \neq 0$ and $n \in \mathbb{N}$. Under what conditions on a and n is x_0 stable?

14. By transforming to polar coordinates, show that the origin is an unstable equilibrium of the following system, even though its linearization is neutrally stable (a center):

$$\begin{aligned} x' &= -y + x^3, \\ y' &= x + y^3. \end{aligned}$$

Using transformation to polar coordinates, investigate the stability of the origin:

15.

$$\begin{aligned} x' &= -2y + ax\sqrt{x^2 + y^2}, \\ y' &= 2x + ay\sqrt{x^2 + y^2}. \end{aligned}$$

16.

$$\begin{aligned} x' &= y + ax(x^2 + y^2), \\ y' &= -x + ay(x^2 + y^2). \end{aligned}$$

17.

$$\begin{aligned} x' &= -y - axy^2, \\ y' &= x + ax^2y. \end{aligned}$$

18.

$$\begin{aligned} x' &= -y(x^2 + y^2 - a), \\ y' &= x(x^2 + y^2 - a). \end{aligned}$$

19.

$$\begin{aligned} x' &= y(-a + x^2 + y^2), \\ y' &= -x(-a + x^2 + y^2). \end{aligned}$$

20.

$$x' = 2y + ax(x^2+y^2)^2,$$
$$y' = -2x + ay(x^2+y^2)^2.$$

21.

$$x' = 4y + ax\sqrt{x^2+y^2},$$
$$y' = -4x + ay\sqrt{x^2+y^2}.$$

22. Investigate the stability of the origin assuming f is monotonic in a neighborhood of zero:

$$x' = y + xf(x^2+y^2)\sqrt{x^2+y^2},$$
$$y' = -x + yf(x^2+y^2)\sqrt{x^2+y^2}.$$

23. (Vinogradov's System.) Show that the origin is NOT stable, even though it is a local (and globally attracting) attractor for the system

$$x' = x^2(y-x) + y^5, \qquad y' = y^2(y-2x).$$

24. Investigate the behavior of solutions of the system

$$\begin{aligned} x' &= -y + xr^2\sin(\pi/r), \\ y' &= x + yr^2\sin(\pi/r) \end{aligned}$$

near the origin ($r = \sqrt{x^2+y^2}$). Show that there exists a sequence of circles to which trajectories converge as $t \to \pm\infty$.

25. Let the origin be a non-asymptotically stable equilibrium of a system which in polar coordinates takes the form $r' = f_1(r,\phi)$, $\phi' = f_2(r,\phi)$. Is it true that the origin is unstable for the perturbed system
(a) $r' = f_1(r,\phi) + g(r,\phi)$, $\phi' = f_2(r,\phi)$
(b) $r' = f_1(r,\phi) + g(r,\phi)$, $\phi' = f_3(r,\phi)$
for all $g > 0$ and f_3?

26. Let x_0 be a stable, non-asymptotically stable stationary point of $x' = f(x)$. Can there exist a non-zero function $g : \mathbb{R}^n \to \mathbb{R}^n$ with $g(x_0) = 0$ such that x_0 is
(a) stable
(b) asymptotically stable
for both $x' = f(x) + g(x)$ and $x' = f(x) - g(x)$ simultaneously?

27. Show that the following statement is false: If 0 is an asymptotically stable equilibrium of $x' = f(x)$ and the entries of the matrix B are sufficiently small, then 0 is an asymptotically stable point of $x' = f(x) + Bx$. (Compare with Exercise 3)

28. Show that for nonlinear systems, asymptotic stability does not imply uniform exponential stability. (Compare with Exercise 6)

29. Prove the following non-autonomous version of the linearized stability theorem: Let 0 be an equilibrium of $x' = Ax + g(t, x)$, i.e., $g(t, 0) = 0$ for all $t \in (T, +\infty)$.
(a) If $s(A) < 0$ and $g(t, x) = o(|x|)$ uniformly in t as $x \to 0$, then 0 is (exponentially) asymptotically stable.
(b) If $s(A) > 0$ and $g(t, x) = o(|x|)$ uniformly in t as $x \to 0$, then 0 is unstable.
(c) If $s(A) < 0$ and $|g(t, x) - g(t, y)| \leq L|x - y|$ with L sufficiently small, then 0 is (exponentially) asymptotically stable.

30. Prove that if 0 is a uniformly exponentially stable stationary point of $x' = f(x)$, then:
(a) $s(\nabla f(0)) < 0$.
(b) 0 is also uniformly exponentially stable for $x' = f(x) + Bx$ provided the matrix B is sufficiently small.

Sketch the local phase portraits near the stationary points (using the Hartman-Grobman Theorem):

31.

$$\begin{aligned} x' &= \exp(2x + 2y) + x, \\ y' &= \arccos(x - x^3) - \pi/2. \end{aligned}$$

32.

$$\begin{aligned} x' &= \ln(1 - y), \\ y' &= \sqrt[3]{x - 4y} + x - 2. \end{aligned}$$

33.

$$\begin{aligned} x' &= \ln(5 - 2x - 2y), \\ y' &= \exp(xy) - 1. \end{aligned}$$

34.

$$\begin{aligned} x' &= \sinh(y - x^2 - x), \\ y' &= 3x - x^2 - y. \end{aligned}$$

35.

$$x' = 2x + y^2 - 1,$$
$$y' = \sin x - y^2 + 1.$$

36.

$$x' = \ln(x + y),$$
$$y' = x^3 + y^3 - 1.$$

Solutions

7) (a) Asymptotically stable for $a < 1$, stable for $a \leq 1$.
(b) Asymptotically stable for $a + |b| < 0$, stable for $a + |b| \leq 0$.

8) Eigenvalues of the Jacobian are -3 and $\pm\sqrt{10}$; since $\sqrt{10} > 0$, the origin is unstable.

9) Eigenvalues of the Jacobian are -3, -2, -1; all have negative real parts, so the origin is asymptotically stable.

10) Stationary points are $[-1, 2k\pi - \pi/6]$ (unstable for $k > 0$ and asymptotically stable for $k \leq 0$) and $[-1, 2k\pi - 5\pi/6]$ (unstable for $k \leq 0$ and asymptotically stable for $k > 0$), $k \in \mathbb{Z}$.

11) Asymptotically stable equilibrium: $[1, 0, -1]$. Unstable equilibria: $[1, 0, 1]$ and $[0, 1, \pm 1]$.

12) Stationary points are $[1, 0, 0]$ and $[2, 2, -1]$; both are unstable.

13) (a) By direct phase-line analysis:
(i) $f \leq 0$ on $(x_0, x_0 + \delta)$ and $f \geq 0$ on $(x_0 - \delta, x_0)$ $\implies$ stability.
(ii) The same with strict inequalities $\implies$ asymptotic stability.
(iii) $f > 0$ on $(x_0, x_0 + \delta)$ or $f < 0$ on $(x_0 - \delta, x_0)$ $\implies$ instability.
(b) If $f'(x_0) < 0$, then $f > 0$ to the left and $f < 0$ to the right of x_0, giving asymptotic stability. If $f'(x_0) > 0$, it gives instability.
(c) Asymptotically stable if $a < 0$ and n is odd; otherwise unstable.

14) In polar coordinates, $r' = r^3(\cos^4\phi + \sin^4\phi) \geq \frac{1}{2}r^3 > 0$, so $r(t)$ grows unboundedly; the origin is unstable.

15) Writing $x(t) = r(t)\cos\phi(t)$, $y(t) = r(t)\sin\phi(t)$, we obtain $r' = ar^2$ and $\phi' = 2$. The origin is stable for $a \leq 0$, and asymptotically stable for $a < 0$; unstable for $a > 0$.

16) $r' = ar^3$; stable for $a \leq 0$, asymptotically stable for $a < 0$; unstable for $a > 0$.

17) $r' = 0$ and $\phi' = 1 + ar^2 \sin\phi\cos\phi$; trajectories are circles $r(t) = \text{const}$, so the origin is stable (center).

18) $r' = 0$; trajectories are concentric circles, so the origin is stable.

19) $r' = 0$; stable.

20) $r' = ar^5$; stable for $a \leq 0$, asymptotically stable for $a < 0$; unstable for $a > 0$.

21) $r' = ar^2$; stable for $a \leq 0$, asymptotically stable for $a < 0$; unstable for $a > 0$.

22) $r' = r^2 f(r^2)$. If $f < 0$ on $(0, \delta)$, the origin is asymptotically stable; if $f > 0$, it is unstable; if $f \equiv 0$, it is stable (neutrally).

23) See the detailed proof at the end of the chapter.

24) In polar coordinates, $r' = r^3 \sin(\pi/r)$ and $\phi' = 1$. The circles $r = 1/k$ ($k \in \mathbb{N}$) are invariant limit cycles, alternating between stable and unstable cycles.

25) (a), (b) False. A counterexample is obtained from the previous exercise with $r' = r^3 \sin(\pi/r)$ by adding a sufficiently small perturbation $g(r, \phi) = \varepsilon r^4$, which still leaves invariant circles near the origin.

26) (a) Yes; for instance, $x' = -y, y' = 2x$ with $g(x, y) = (0, x)^T$.
(b) Yes; for instance, using a system with concentric invariant barriers or limit cycles as in Exercise 20 and a perturbation that contracts radii from both directions.

27) Consider $x' = y - x^3$, $y' = -x - y^3$, for which the origin is asymptotically stable, but the linear part is a center ($\sigma(A) = \{\pm i\}$). Taking $B = \epsilon I$ with $\epsilon > 0$ makes the origin unstable for any $\epsilon > 0$.

28) Consider $x' = -x^3$. The solution is $x(t) = \pm(2t + x(0)^{-2})^{-1/2}$, which decays algebraically as $t^{-1/2}$, not exponentially.

29) (a) Follows from (c) or standard Gronwall estimates using $\|e^{tA}\| \leq Me^{\beta t}$ with $\beta < 0$.
(b) Proved using variation of constants and the unstable manifold projection.
(c) Using estimate (4) and variation of constants, show that the integral operator is a contraction on the exponentially weighted Banach space $C_\gamma([T, \infty))$ with $\|x\|_\gamma = \sup_{t \geq T} e^{\gamma t}|x(t)|$ for $0 < \gamma < -s(A)$.

30) (a) Let $x' = Ax + g(x)$ with $g(x) = o(|x|)$. If A had an eigenvalue with $\operatorname{Re}\lambda \geq 0$, consider $y(t) = x(t)e^{at}$ with $a > 0$. The system for y is $y' = (A + aI)y + e^{at} g(e^{-at} y)$, where $A + aI$ has strictly positive eigenvalues, contradicting uniform exponential decay.
(b) Follows from (a) and Exercise 3, since $s(\nabla f(0) + B) < 0$ for sufficiently small B.

31) Point $(-1,1)$: $\lambda_{1,2} = -1,4$ (saddle), eigenvectors $v_1 = (-1,2)^T$, $v_2 = (2,1)^T$.

32) Point $(1,0)$: $\lambda = -2/3 \pm 2\sqrt{2}i/3$ (stable spiral/focus – counterclockwise).

33) Point $(0,2)$: $\lambda_{1,2} = -1 \pm i\sqrt{3}$ (stable spiral – counterclockwise); point $(2,0)$: $\lambda_{1,2} = \pm 2$ (saddle), $v_1 = (1,-2)^T$, $v_2 = (1,0)^T$.

34) Point $(0,0)$: $\lambda_{1,2} = -1 \pm \sqrt{3}$ (saddle), $v_1 = (1,\sqrt{3})^T$, $v_2 = (1,-\sqrt{3})^T$; point $(1,2)$: $\lambda_{1,2} = -2 \pm \sqrt{2}$ (stable node), $v_1 = (1,1+\sqrt{2})^T$, $v_2 = (1,1-\sqrt{2})^T$.

35) Point $(0,-1)$: $\lambda_{1,2} = 2 \pm i\sqrt{2}$ (unstable spiral – counterclockwise); point $(0,1)$: $\lambda_{1,2} = \pm\sqrt{6}$ (saddle), $v_1 = (1,\sqrt{6}/2-1)^T$, $v_2 = (1,-\sqrt{6}/2-1)^T$.

36) Point $(0,1)$: $\lambda_{1,2} = 1,3$ (unstable node), $v_1 = (1,0)^T$, $v_2 = (1,2)^T$; point $(1,0)$: $\lambda_{1,2} = (1 \pm \sqrt{13})/2$ (saddle), $v_1 = (2,\sqrt{13}-1)^T$, $v_2 = (-1,1+\sqrt{13})^T$.

Lyapunov Stability

In this section, we introduce Lyapunov's direct method for establishing stability and asymptotic stability of an equilibrium a for the autonomous system

$$x' = f(x)\,, \tag{5}$$

as well as for non-autonomous systems

$$x' = f(t,x), \tag{6}$$

where f is continuous on $I \times \Omega$, $I = (T,+\infty)$, and $\Omega \subset \mathbb{R}^n$ is an open set. In the non-autonomous setting, a point a is called an *equilibrium point* (stationary point) if $f(t,a) = 0$ for all $t \in I$. The point a is called *stable* if

$$\big(\forall t_0 > T\big)\big(\forall \varepsilon > 0\big)\big(\exists \delta > 0\big)\Big[|x_0 - a| < \delta \implies |x(t) - a| < \varepsilon \quad \forall t \geq t_0\Big], \tag{7}$$

where $x(t)$ is the solution of (6) with initial condition $x(t_0) = x_0$. The point a is called *asymptotically stable* if it is stable and, in addition,

$$\big(\forall t_0 > T\big)\big(\exists \Delta > 0\big)\Big[|x_0 - a| < \Delta \implies \lim_{t\to\infty} x(t) = a\Big]. \tag{8}$$

The point a is called *unstable* if it is not stable.

Without loss of generality, we assume $a = 0$ in the following.

Definition 1. Let $\mathcal{U}$ be a neighborhood of the origin in $\mathbb{R}^n$. A function $V : I \times \mathcal{U} \to \mathbb{R}$ is called a *Lyapunov function* of equation (6) on $\mathcal{U}$ if it is continuous and satisfies:

(i) $V(t, 0) = 0$ for all $t \in I$,

(ii) there exists a continuous positive definite function $\omega : \mathcal{U} \to \mathbb{R}$ such that $V(t, x) \geq \omega(x) > 0$ for all $x \in \mathcal{U} \setminus \{0\}$,

(iii) the function $t \mapsto V(t, x(t))$ is non-increasing for every solution $x(t)$ of (6) in $\mathcal{U}$.

Theorem 4 (Lyapunov's Stability Theorem)**.** *If equation* (6) *admits a Lyapunov function in a neighborhood of* 0*, then the origin is stable.*

Definition 2. If V is of class C^1, we define the *orbital derivative* (or *Lie derivative*) with respect to f as

$$\dot{V}_f(t, x) = \frac{\partial V(t, x)}{\partial t} + \nabla_x V(t, x) \cdot f(t, x) = \frac{\partial V(t, x)}{\partial t} + \sum_{j=1}^{n} \frac{\partial V(t, x)}{\partial x_j} f_j(t, x).$$

Key Observation: If $x(t)$ is a solution of (6), then

$$\frac{d}{dt} V(t, x(t)) = \dot{V}_f(t, x(t));$$

in particular, the condition $\dot{V}_f(t, x) \leq 0$ on $\mathcal{U}$ guarantees condition (iii) in the definition of a Lyapunov function.

Strengthening this assumption yields asymptotic stability:

Theorem 5. *Suppose equation* (6) *admits a Lyapunov function V in a neighborhood $\mathcal{U}$ of the origin, and assume additionally that:*

(i) there exists a continuous function $\lambda : \mathcal{U} \to \mathbb{R}$ with $\lambda(0) = 0$ such that $V(t, x) \leq \lambda(x)$ for all $x \in \mathcal{U} \setminus \{0\}$,

(ii) there exists a continuous function $\eta : \mathcal{U} \to \mathbb{R}$ with $\eta(0) = 0$ such that $\dot{V}_f(t, x) \leq -\eta(x) < 0$ for all $x \in \mathcal{U} \setminus \{0\}$.

Then the origin is asymptotically stable.

In the autonomous case, these conditions simplify to the following classical result:

Theorem 6. *If equation* (5) *admits a time-independent Lyapunov function* $V(x)$ *in a neighborhood* $\mathcal{U}$ *of* 0 *such that* $\dot{V}_f(x) < 0$ *for all* $x \in \mathcal{U} \setminus \{0\}$*, then the origin is asymptotically stable.*

Remark. By Massera's Theorem, this sufficient condition is also necessary for autonomous systems: if 0 is an asymptotically stable equilibrium of (5), then there exists a smooth, time-independent Lyapunov function $V(x)$ satisfying $\dot{V}_f(x) < 0$ for $x \neq 0$.

Exercises

37. Let $q(y)$ be a continuous function preserving the sign of y (i.e., $q(y)y \geq 0$ for $y \neq 0$). Show that the zero solution of

$$x'' + q(x') + x = 0$$

is stable. Is it asymptotically stable?

38. For the equation $x' = (t^{-2} - 1)x + x^3$, show that:
(a) the zero solution is uniformly stable on (η, ∞), where $\eta > 0$ is fixed.
(b) the zero solution is not uniformly stable on $(0, \infty)$.
(Uniform stability means that δ *in* (7) *can be chosen independently of* t_0*).*

39. Consider the delay differential equation ($a, b, r > 0$ constants)

$$x'(t) = ax(t) + bx(t - r). \tag{9}$$

(a) Let $\phi(\tau) : [-r, 0] \to \mathbb{R}$ be continuous. Then there exists a unique solution $x : [-r, \infty) \to \mathbb{R}$ such that $x(\tau) = \phi(\tau)$ on $[-r, 0]$ and $x(t)$ satisfies (9) for all $t > 0$.
(b) If $a < 0$ and $|b| \leq |a|$, then the zero solution of (9) is stable.
(c) If $a < 0$ and $|b| < |a|$, then the zero solution of (9) is asymptotically stable.

Investigate the stability of the origin for the following autonomous systems using Lyapunov functions:

40.

$$\begin{aligned} x' &= -2y - x^3, \\ y' &= x - y^3. \end{aligned}$$

41.

$$\begin{aligned} x' &= y - 2x^3, \\ y' &= -2x - y^3. \end{aligned}$$

42.

$$x' = -x - y^2,$$
$$y' = xy - x^2 y.$$

43.

$$x' = -xy^2,$$
$$y' = -y - 2x^2 y.$$

44.

$$x' = 2y + x^3,$$
$$y' = -x + y^3.$$

45.

$$x' = -y + 2x^3,$$
$$y' = 2x + y^3.$$

46. Investigate the stability of the origin for the system

$$x' = -2y^3\,, \qquad y' = x.$$

Compare with the stability of the linearized system!

47. Find an example of a system whose origin is asymptotically stable, but whose linearization is unstable.

48. Let x_0 be an equilibrium of $x' = f(x)$. Prove or disprove: There exist $\delta > 0$ and a continuous positive function $G : (0, \delta) \to (0, +\infty)$ such that the origin is an asymptotically stable equilibrium of $x' = f(x) + g(x)$ for every perturbation g satisfying $|g(x)| < G(|x|)$ on the punctured δ-neighborhood.

49. If there exists a Lyapunov function satisfying $\dot{V}(x) \leq -aV(x)$ and $c_1|x|^b \leq V(x) \leq c_2|x|^b$ on a neighborhood of the origin for constants $a, b, c_1, c_2 > 0$, show that the origin is uniformly exponentially stable, i.e., $|x(t)| \leq (c_2/c_1)^{1/b} e^{-at/b}|x(0)|$.

Solutions

37) Use the Lyapunov function $V(x, y) = x^2 + y^2$, where $y = x'$. Then $\dot{V} = 2xx' + 2yy' = 2xy + 2y(-q(y) - x) = -2yq(y) \leq 0$. In general, it is not asymptotically stable (e.g., if $q \equiv 0$). It is asymptotically stable if $q(y)y > 0$ for $y \neq 0$ (by LaSalle's Invariance Principle).

38) (a) Use the Lyapunov function $V(t,x) = x^2 e^{2/t}$. (b) Choose $\delta > 0$ such that $t^{-2} - 1 \geq t^{-2}/2$ for $t \in (0, 2\delta)$. Then $x' \geq x/(8\delta^2)$ for $t \in (\delta, 2\delta)$; solutions with initial condition $x(\delta) = \delta$ become large at $t = 2\delta$.

39) (a) Method of steps across intervals of length r. (b) Show that the Lyapunov-Krasovskii functional

$$V(t) := x^2(t) + |a| \int_{t-r}^{t} x^2(s)\, ds$$

satisfies $\dot{V}(t) \leq 0$ and $x^2(t) \leq V(t)$. (c) Consider $y(t) = x(t)e^{\gamma t}$ and apply the same functional method for small $\gamma > 0$.

40) Choose $V(x,y) = x^2 + 2y^2$. Then $\dot{V}_f = 2x(-2y - x^3) + 4y(x - y^3) = -2x^4 - 4y^4 < 0$ for $(x,y) \neq (0,0)$. The origin is asymptotically stable.

41) Choose $V = 2x^2 + y^2$. Then $\dot{V}_f = -8x^4 - 2y^4 < 0$; asymptotically stable.

42) Choose $V = x^2 + y^2$. Then $\dot{V}_f = 2x(-x - y^2) + 2y(xy - x^2 y) = -2x^2 - 2x^2 y^2 \leq 0$. The origin is stable, but not asymptotically stable (any point $(0, y_0)$ with $x = 0$ has $x' = -y^2$).

43) Choose $V = 2x^2 + y^2$. Then $\dot{V}_f = -2y^2(1 + 3x^2) \leq 0$. The origin is stable, but not asymptotically stable because $(x_0, 0)$ are stationary solutions for all x_0.

44) Choose $V = x^2 + 2y^2$. Then $\dot{V}_f = 2x^4 + 4y^4 > 0$ for $(x,y) \neq (0,0)$. This implies asymptotic stability backwards in time, and hence instability forwards in time.

45) Choose $V = 2x^2 + y^2$. Then $\dot{V}_f = 8x^4 + 2y^4 > 0$; unstable.

46) Using the Lyapunov function $V(x,y) = x^2 + y^4$, we obtain $\dot{V}_f = 2x(-2y^3) + 4y^3(x) = 0$; trajectories are closed curves $x^2 + y^4 = C$, so the origin is stable. However, the linearized system $x' = 0, y' = x$ has a double zero eigenvalue with a single eigenvector (Jordan block of size 2), and is therefore unstable.

47) $x' = -2y^3 - x^3$, $y' = x - y^3$. Lyapunov function $V = x^2 + y^4$ yields $\dot{V} = -2x^4 - 4y^6 < 0$. Linearization is $x' = 0, y' = x$, which is unstable.

48) True. By Massera's Theorem, there exists a smooth Lyapunov function V with $\dot{V}_f(x) < 0$. On each sphere $|x| = r$, the maximum of $\nabla V(x) \cdot f(x)$ is strictly negative (say, $-m(r) < 0$). Setting $G(r) < m(r)/\max_{|x|=r} |\nabla V(x)|$ ensures that $\nabla V(x) \cdot (f(x) + g(x)) < 0$.

49) From $\dot{V}(x(t)) \leq -aV(x(t))$, we have $V(x(t)) \leq e^{-at} V(x(0))$. Thus $c_1 |x(t)|^b \leq V(x(t)) \leq e^{-at} V(x(0)) \leq c_2 e^{-at} |x(0)|^b$, which gives $|x(t)| \leq (c_2/c_1)^{1/b} e^{-at/b} |x(0)|$.

Hurwitz's Theorem

In stability analysis, it is frequently necessary to determine the signs of the real parts of the eigenvalues of a matrix. This reduces to determining the real parts of the roots of its characteristic polynomial. The utility of the following theorem in addressing this question is self-evident.

Definition 3. A polynomial $p(\lambda)$ is called *Hurwitz* (or *stable*) if all of its roots have strictly negative real parts.

Theorem 7 (Hurwitz's Criterion)**.** *Let $a_0, \ldots, a_n$ be real numbers with $a_0 > 0$. Then the polynomial*

$$p(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$$

is Hurwitz if and only if all leading principal minors (Hurwitz determinants) of the $n \times n$ matrix

$$H = \begin{pmatrix} a_1 & a_0 & 0 & 0 & \ldots & 0 \\ a_3 & a_2 & a_1 & a_0 & \ldots & 0 \\ a_5 & a_4 & a_3 & a_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{2n-1} & a_{2n-2} & a_{2n-3} & a_{2n-4} & \ldots & a_n \end{pmatrix}$$

are strictly positive, where $a_k = 0$ for $k \notin \{0, 1, \ldots, n\}$.

Corollary. *Let $a_0 > 0$. Then the cubic polynomial*

$$a_0\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3$$

is Hurwitz if and only if

$$a_1 > 0, \quad a_1a_2 - a_0a_3 > 0, \quad a_3 > 0.$$

Corollary. *Let $a_0 > 0$. Then the quartic polynomial*

$$a_0\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4$$

is Hurwitz if and only if

$$a_1 > 0, \quad a_1a_2 - a_0a_3 > 0, \quad a_1(a_2a_3 - a_1a_4) - a_0a_3^2 > 0, \quad a_4 > 0.$$

Remark. Let $a_0 > 0$. For a quadratic polynomial

$$a_0\lambda^2 + a_1\lambda + a_2,$$

it is Hurwitz if and only if $a_1 > 0$ and $a_2 > 0$. From this (by factoring into linear and quadratic terms), we obtain a straightforward *necessary condition* for a real polynomial to be Hurwitz: if $a_0 > 0$, then all other coefficients $a_1, \ldots, a_n$ must be strictly positive.

Analysis of Vinogradov's System

Vinogradov's system

$$x' = x^2(y - x) + y^5,$$
$$y' = y^2(y - 2x),$$

is an important explicit example of a system for which the origin is unstable, even though it is a local (and global) attractor, i.e., it attracts all trajectories as $t \to \infty$.

We sketch the proof in several steps. First, we show that the origin is an attractor.

STEP 0. On the line $y = 0$, the system reduces to $x' = -x^3$, so $x(t) \to 0$ as $t \to \infty$. The system is invariant under the point reflection $(x, y) \mapsto (-x, -y)$; thus, it suffices to restrict our analysis to the half-plane $y > 0$.

STEP 1. Let us examine the vector field by elementary geometric arguments. We have $y' = 0$ if and only if $y = 2x$, while $x' = 0$ if and only if

$$x^2y - x^3 + y^5 = 0.$$

This equation has no solutions for $x < 0$ and a unique solution $y = \phi(x)$ for $x \geq 0$ (with $y > 0$). The function $\phi(x)$ is smooth (by the implicit function theorem), satisfies $\phi(0) = 0$, is strictly increasing with $\phi(x) < x$ (since $x^2y - x^3 \leq 0$), and behaves asymptotically as $\phi(x) \sim x^{3/5}$ for large x.

The curves $y = 2x$ and $y = \phi(x)$ divide the upper half-plane $\{y > 0\}$ into three sectors:

$$\Omega_1 = \{(x, y) : \ x < 0 \text{ or } 0 \leq 2x < y\},$$
$$\Omega_2 = \{(x, y) : \ x > 0 \text{ and } \phi(x) < y < 2x\},$$
$$\Omega_3 = \{(x, y) : \ x > 0 \text{ and } y < \phi(x)\}.$$

The equations show that $y(t)$ increases in Ω_1 and decreases in $\Omega_{2,3}$; while $x(t)$ increases in $\Omega_{1,2}$ and decreases in Ω_3.

Trajectories in Ω_3 are forced to enter the origin as $t \to \infty$, trajectories in Ω_2 must eventually enter Ω_3, and trajectories in Ω_1 must cross the line $y = 2x$ into Ω_2 because $x' > 0$ is bounded away from zero.

This proves that the origin is a global attractor.

STEP 2. It remains to show that the origin is unstable. Linearized stability cannot be applied because the Jacobian at the origin is identically zero.

We construct an explicit triangular trapping region $\mathcal{K}$ with vertices $(0,0)$, $(0,\eta)$, and $(\Delta\eta,\eta)$, where $\Delta,\eta>0$ are small parameters to be chosen. Denote the boundary segments by

$$\begin{aligned}\Gamma_h &= \{(x,\eta):\ 0<x<\Delta\eta\},\\ \Gamma_l &= \{(0,y):\ 0<y\le\eta\},\\ \Gamma_r &= \{(x,y):\ 0<y\le\eta,\ x=\Delta y\}.\end{aligned}$$

We show that every solution starting strictly inside $\mathcal{K}$ must escape $\mathcal{K}$ through the top boundary Γ_h. This implies instability, since initial points can be chosen arbitrarily close to $(0,0)$.

Since solutions cannot cross through the stationary origin (by uniqueness) and cannot cross Γ_l outward (where $x'=y^5>0$, so the vector field points strictly inward), they can only escape through Γ_h or Γ_r.

On Γ_r, where $x=\Delta y$, the slope of the trajectory satisfies

$$\frac{dy}{dx}=\frac{y'}{x'}=\frac{y^2(y-2x)}{x^2(y-x)+y^5}=\frac{1-2\Delta}{\Delta^2(1-\Delta)+y^2}.$$

For sufficiently small $\Delta>0$ and $\eta>0$, this slope satisfies $dy/dx>\Delta^{-1}$, which means trajectories on Γ_r point inward into $\mathcal{K}$. Therefore, solutions can only escape through the top edge Γ_h at height $y=\eta$, completing the proof of instability.

Step 3. (Alternative proof using blow-up and polar coordinates.) In polar coordinates:

$$\begin{aligned}r' &= r^3\big[3\sin\varphi\cos^3\varphi-\sin 2\varphi+1-2\cos^2\varphi\big]+\mathcal{O}(r^4),\\ \varphi' &= r^2\big[\sin\varphi\cos\varphi-3\cos^3\varphi+3\cos^4\varphi\big]+\mathcal{O}(r^3).\end{aligned}\tag{10}$$

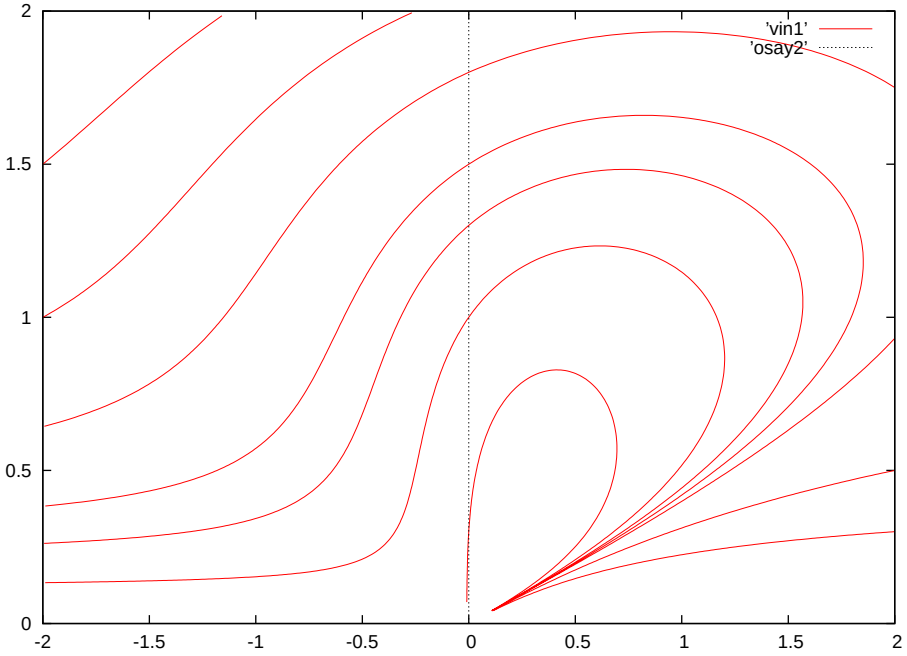
Dividing the vector field by the scalar factor $r^2>0$ (which reparameterizes time without altering the orbit geometry or direction) gives

$$\begin{aligned}\frac{dr}{d\tau} &= r\big[3\sin\varphi\cos^3\varphi-\sin 2\varphi+1-2\cos^2\varphi\big]+\mathcal{O}(r^2),\\ \frac{d\varphi}{d\tau} &= \big[\sin\varphi\cos\varphi-3\cos^3\varphi+3\cos^4\varphi\big]+\mathcal{O}(r).\end{aligned}\tag{11}$$

The point $(r,\varphi)=(0,\pi/2)$ is a stationary point for this desingularized system. The Jacobian matrix at $(0,\pi/2)$ is

$$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

This has a positive eigenvalue $+1$ corresponding to the r-direction. By the Unstable Manifold Theorem, there exists a trajectory along which $r(\tau)$ grows exponentially away from 0 along $\varphi \approx \pi/2$, which corresponds in the original Cartesian system to a solution moving away from the origin along the positive y-axis.



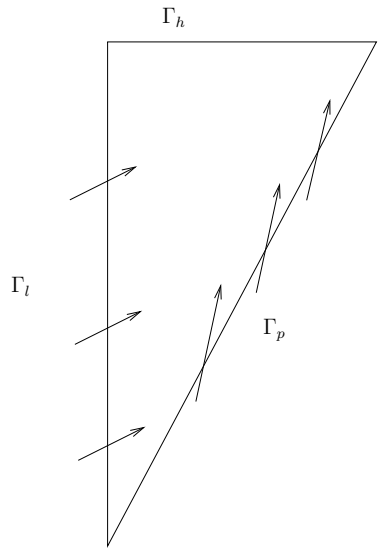


Figure 1: Step 2: Region $\mathcal{K}$