Separable Differential Equations

In this chapter, we will consider the following differential equation:

$$y' = g(y)f(x), \tag{1}$$

where f and g are real-valued functions of a real variable. This equation is called a *separable differential equation* (or an *equation with separable variables*). Throughout this chapter, the letters I and J will denote open intervals.

We immediately see that if $g(y_0) = 0$, then $y(x) = y_0$ is a solution to the equation defined on all intervals contained in the domain of f. This solution is called *stationary* (or *equilibrium*). The fundamental theorem that allows us to find non-trivial solutions of a separable differential equation is the following:

Theorem 1 (Solution of a separable equation)**.** *Consider equation* (1)*. Let $f(x)$ be continuous on I, and let $g(y)$ be continuous and non-zero on J. Let $F(x)$ and $G(y)$ be antiderivatives of $f(x)$ and $1/g(y)$ on I and J, respectively.*

Denote by $G^{-1}(z)$ the inverse function of $G(y)$. Let $\tilde{I} \subset I$ and $c \in \mathbb{R}$ be chosen such that $F(x) + c$ belongs to the domain of $G^{-1}(z)$ (i.e., in $G(J)$) for all $x \in \tilde{I}$.

Then the function

$$y(x) = G^{-1}(F(x) + c)\,, \quad x \in \tilde{I}$$

is a solution to equation (1)*.*

Remark. The assumption "$F(x) + c$ lies in the range of G" must be carefully monitored – a formal calculation may lead to a function whose domain is larger than the interval on which this function solves our equation.

Example 1. Find all maximal solutions of the equation $y' = 2\sqrt{|y|}$.

Solution. We immediately see the only stationary solution $y(x) = 0$, $x \in \mathbb{R}$. Furthermore, applying the previous theorem, we obtain the solutions:

1. for $I = \mathbb{R}$, $J = (-\infty, 0)$: $G(y) = -\sqrt{-y}$, $F(x) = x$. Thus $G(J) = (-\infty, 0)$, $\tilde{I} = (-\infty, -c)$. The found solution is $y(x) = -(x + c)^2$, $x \in (-\infty, -c)$. **Caution:** for $x > -c$, the given function is NOT a solution to the equation.

2. for $I = \mathbb{R}$, $J = (0, \infty)$: $G(y) = \sqrt{y}$, $G(J) = (0, \infty)$. The found solution is $y(x) = (x+c)^2$, $x \in (-c, \infty)$. (Again, it is not a solution for $x < -c$).

However, at this point, we do not know whether these solutions are maximal and whether they constitute all solutions.

Sometimes, solutions provided by Theorem 1 may not be maximal. If we have solutions on intervals (a, b) and (b, c), it might be possible to extend the function continuously at the point b and obtain a solution on the larger interval (a, c). This situation is addressed in the following lemma:

Lemma 2 (Gluing lemma)**.** *Let $y_1(x) : (a, x_0) \to \mathbb{R}$ and $y_2(x) : (x_0, b) \to \mathbb{R}$ be solutions to the equation*

$$y' = f(x, y)\,. \tag{2}$$

Let $\lim\limits_{x\to x_0-} y_1(x) = y_0 = \lim\limits_{x\to x_0+} y_2(x)$. Let $f(x, y)$ be continuous at the point $(x_0, y_0) \in \mathbb{R}^2$. Then the function

$$y(x) = \begin{cases} y_1(x), & x \in (a, x_0) \\ y_2(x), & x \in (x_0, b) \\ y_0, & x = x_0 \end{cases}$$

is a solution to equation (2) *on the entire interval (a, b).*

Remark. Lemma 2 essentially resolves a single issue: that the equation is satisfied at the gluing point (x_0, y_0) (elsewhere it is trivially clear from the assumptions) and states that this is guaranteed if we glue the solutions continuously. However, it may also happen that two solutions can be glued together even if f is not continuous at the gluing point.

Solution. (continuation) The function

$$y(x) = \begin{cases} -(x + c)^2, & x < -c \\ 0, & x \geq -c \end{cases}$$

is, by Lemma 2, a solution to the equation $y' = 2\sqrt{|y|}$ on $\mathbb{R}$, as it is obtained by gluing two solutions: $y_1(x) = -(x + c)^2$ on $(-\infty, -c)$ and $y_2(x) = 0$ on $(-c, \infty)$. Similarly, the function

$$y(x) = \begin{cases} (x + c)^2, & x > -c \\ 0, & x \leq -c \end{cases}$$

is a solution to the equation. These solutions are clearly maximal, as they are defined on the entire $\mathbb{R}$. It remains to resolve whether these are all solutions (not forgetting, of course, the zero stationary solution).

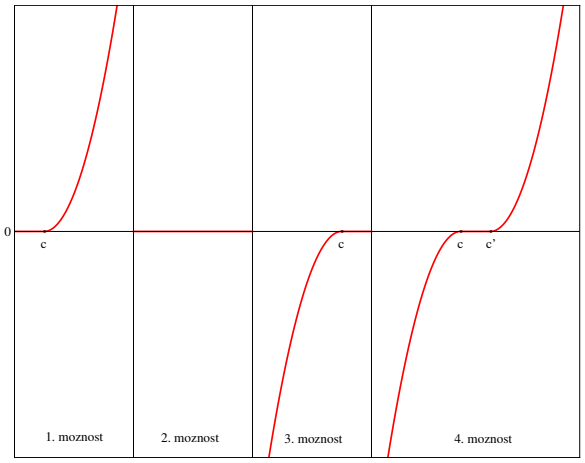
How do we determine whether we have found *all* maximal solutions? Picard's existence and uniqueness theorem gives us the uniqueness of solutions at points in whose neighborhood the functions $(x, y) \mapsto f(x)g(y)$ and $(x, y) \mapsto f(x)g'(y)$ are continuous. Thus, if we have a domain $\Omega \subset \mathbb{R}^2$ in which these functions are continuous, and if we find a family of solutions that covers Ω (at least one passes through each point), then no other solutions exist. Conversely, points where $g'(y)$ does not exist are typically candidates for *branching*. (*Branching* occurs at a point (x_0, y_0) if two solutions pass through it that do not coincide on any punctured neighborhood $P(x_0, \delta)$.)

For separable equations, however, a stronger uniqueness theorem holds:

Proposition 3 (Uniqueness)**.** *If f and g are continuous in Ω, then branching can only occur at points where g' does not exist or is discontinuous, and simultaneously $g = 0$.*

Thus, we do not need the continuity of $(x, y) \mapsto f(x)g'(y)$ as long as g is non-zero. You can prove this proposition as an exercise (see below).

Solution. (completion) Consider again the equation $y' = 2\sqrt{|y|}$. The family of solutions $y_c(x) = (x - c)^2$, $x \in (c, \infty)$ clearly covers the upper half-plane $\Omega = \{(x, y),\ y > 0\}$ (through $(x_0, y_0) \in \Omega$ passes the solution with $c = x_0 - \sqrt{y_0}$). Since both f and g are continuous in Ω and g is additionally non-zero, no other solutions can exist here. Solutions of the form $-(x + c)^2$ similarly cover the lower half-plane. Therefore, branching can only occur at the points $(x_0, 0)$, and we have found all such branchings.



Thus, the complete procedure for finding all maximal solutions of equation (1) is as follows:

1. Determine the maximal open intervals contained in the domain of f. (This defines the maximal intervals on which we can seek solutions.)

2. Find all zeros of g. If $g(c) = 0$, then the function $y \equiv c$ on any interval from Step 1 is a stationary solution of equation (1).

3. Determine the maximal open intervals on which g is non-zero.

4. Take an interval I from Step 1 and an interval J from Step 3. Thus, f is continuous on I, and g is continuous and non-zero on J. We seek solutions of equation (1) that are defined somewhere in I and take values in J. If y is such a solution, then it satisfies

$$\frac{y'(x)}{g(y(x))} = f(x).$$

Let F be an antiderivative of f on I, and let G be an antiderivative of $1/g$ on J. Then there exists a constant $c \in \mathbb{R}$ such that

$$G(y(x)) = F(x) + c$$

on the domain of the solution y, which will be determined in the next step.

5. Now fix c and find the maximal non-empty open intervals contained in the set

$$\{x \in I;\ F(x) + c \in G(J)\}.$$

On each of these intervals, the solution must be of the form

$$y(x) = G^{-1}(F(x) + c),$$

where G^{-1} denotes the inverse function of G. This inverse exists because G is strictly monotone (either strictly increasing or strictly decreasing) on J.

6. From the solutions found in Step 5 and the singular/stationary solutions from Step 2, "glue" all maximal solutions of equation (1).

Example 2. Find all maximal solutions of the equation

$$y' = 2x(1 + y^2)$$

Solution. Step 1: $I = \mathbb{R}$. Step 2: the equation has no stationary solutions. Step 3: $J = \mathbb{R}$. Step 4: dividing by $1 + y^2$ and integrating yields

$$\operatorname{arctg} y = x^2 + c. \tag{3}$$

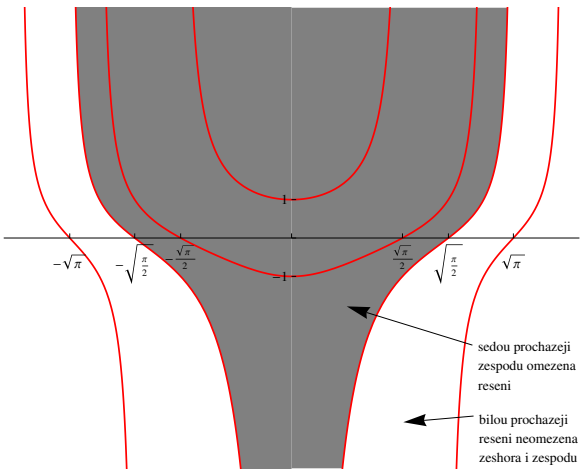
Since we have only one interval for x and one for y, we do not need to distinguish multiple cases, which considerably simplifies the problem. Step 5: since $G = \operatorname{arctg}$ maps $J = \mathbb{R}$ onto $(-\pi/2, \pi/2)$, the right-hand side of equality (3) must lie in $(-\pi/2, \pi/2)$, i.e.,

$$x^2 \in (-\pi/2 - c, \pi/2 - c)\,.$$

If $c \leq -\pi/2$, the solution is defined on the intervals $x \in (-\sqrt{\pi/2 - c}, -\sqrt{-\pi/2 - c})$ and $x \in (\sqrt{-\pi/2 - c}, \sqrt{\pi/2 - c})$. If $c \in (-\pi/2, \pi/2)$, it is defined on the interval $(-\sqrt{\pi/2 - c}, \sqrt{\pi/2 - c})$. If $c \geq \pi/2$, no solution exists. The solution is given by

$$y = \operatorname{tg}(x^2 + c).$$

Step 6: Since at the endpoints of the intervals $(\pm\sqrt{\pm\pi/2 - c})$ we have $\lim y(x) = \pm\infty$, the obtained solutions cannot be extended, and are therefore maximal. From the uniqueness theorem, it follows that we have found all solutions, because every point in the plane is traversed by some solution from our family. Indeed, through a point $[x_0, y_0]$ passes the solution $y = \operatorname{tg}(x^2 + c)$, where $c = \operatorname{arctg} y_0 - x_0^2$, defined on the interval $(-\sqrt{\pi/2 - c}, -\sqrt{-\pi/2 - c})$ if $x_0 < 0$ and $\operatorname{arctg} y_0 - x_0^2 \leq -\pi/2$; on the interval $x \in (\sqrt{-\pi/2 - c}, \sqrt{\pi/2 - c})$ if $x_0 > 0$ and $\operatorname{arctg} y_0 - x_0^2 \leq -\pi/2$; and on the interval $(-\sqrt{\pi/2 - c}, \sqrt{\pi/2 - c})$ if $\operatorname{arctg} y_0 - x_0^2 \in (-\pi/2, \pi/2)$.



Exercises

Find all maximal solutions of the following differential equations:

1. $y' = |y|$
2. $y' = 1 - y^2$
3. $y' = (3/2)\sqrt[3]{y}$

4. $y' = \sqrt{1-y^2}$

5. $(x^2-4x)y' + y = 0$

6. $y' = 3x^2y^2$

7. $y' = \frac{\cos^2 y}{x^2+1}$

8. $y' \sin x = y \ln y$

9. $y' = \sqrt[3]{y^2}$

10. $y' = \exp(-(x+y))$

11. $\frac{yy'}{y^2-1} + \frac{x}{x^2-1} = 0$

12. $y' = 3\sqrt[3]{y^2} \exp x$

13. $y' = y^2 - y$

14. $(x^2+x)y' - (2x+1)y = 0$

15. $y' \cos x + y(y+1) \sin x = 0$

16. $x^2(x^2+4)y' = \cos^2 y$

17. $yy' = \frac{1-2x}{y}$

18. $xy' + y = y^2$

19. $y' = 10^{x+y}$

20. $e^{-y}(1+y') = 1$

21. $y' = \sqrt{\frac{1-y^2}{1-x^2}}$

22. $y' \sin x = y \ln y,\ y(\pi/2) = 1$

23. $y' = \frac{1+y^2}{1+x^2},\ y(0) = 1$

24. $y' = \sqrt{y}e^{-x}$

25. $y' = 20e^{-x}\sqrt[5]{y^4}$

26. $y' = \frac{(1+y^2)x}{1+x^2}$

27. $y' = e^{-y}\cos x$

28. $(2-e^x)y' = -3e^x \operatorname{tg} y \cos^2 y$

29. $y' = (1+y^2)\operatorname{tg} x$

30. $y' = \frac{1-x}{y}$

31. $y' = -\frac{x}{y} \cdot \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$

32. $y' = \frac{\sqrt{1+y^2}}{xy}$

33. $y' = \frac{2xy^2}{1-x^2}$

34. $xy' \cos y + \sin y = \sin^2 y$

35. Find the maximal solution of the initial value problem $y - xy' = b(1 + xy')$, $y(1) = 1$ depending on the parameter $b \in \mathbb{R}$.

36. Find all bounded maximal solutions of the equation $y' = \frac{y^2}{x^2}$.

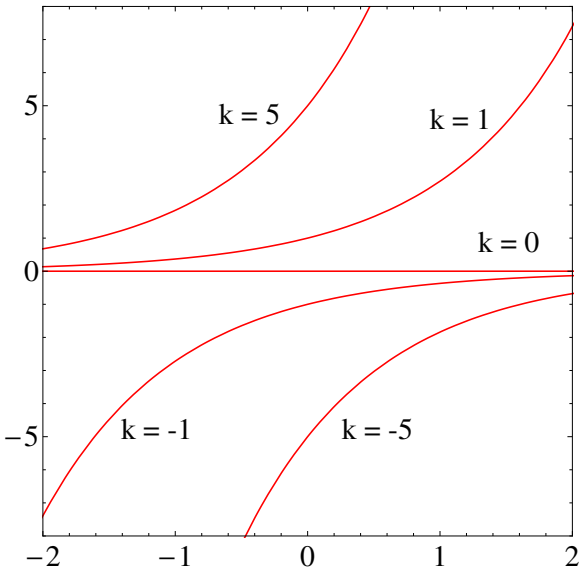
37. Find the set of all points in the plane through which passes exactly one solution of the equation $y' = \frac{\cos x}{e^y}$ defined on the entire $\mathbb{R}$.

38. Find all $A \in \mathbb{R}$ for which there exists a solution of the equation $y'(2 - e^x) = -3e^x \operatorname{tg} y \cos^2 y$ satisfying $\lim_{x\to-\infty} y(x) = \lim_{x\to+\infty} y(x) = A$.

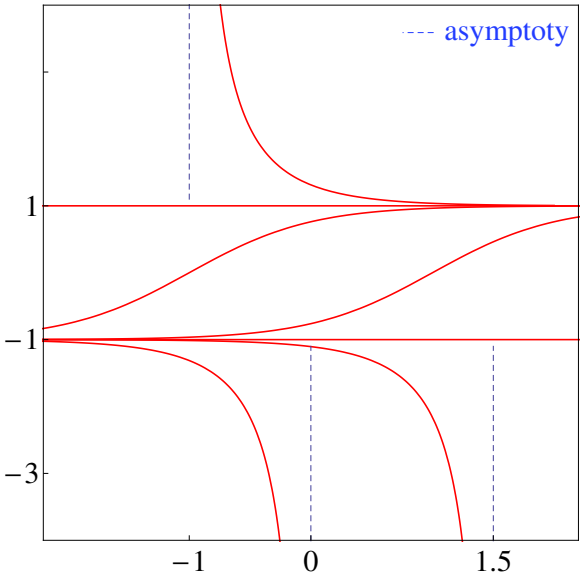
39. Let f be continuous on a neighborhood of x_0 and let g be continuous and non-zero on a neighborhood of y_0. Then branching does not occur at the point (x_0, y_0).

Solutions

1) Defined for $x \in \mathbb{R}$, stationary solution $y = 0$ on $\mathbb{R}$. Solve on intervals $y \in (-\infty; 0)$, $y \in (0; +\infty)$. After integration: $\operatorname{sgn}(y) \ln|y| = x + c$, $c \in \mathbb{R}$. After simplification, for $y < 0$: $y = -\exp(-(x + c))$ and for $y > 0$: $y = \exp(x + c)$. Conclusion: $y(x) = k \exp(x \operatorname{sgn}(k))$, $x \in \mathbb{R}$, $k \in \mathbb{R}$.

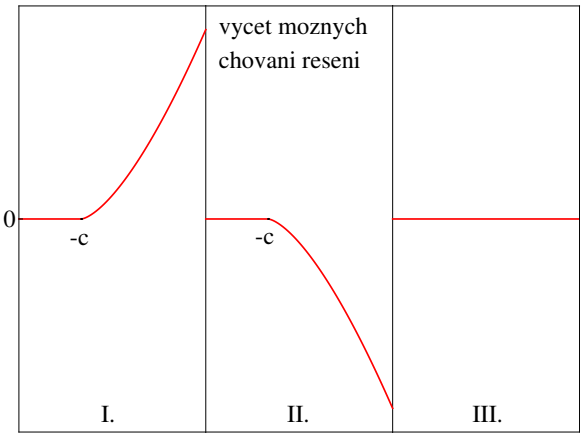


2) Defined for $x \in \mathbb{R}$, stationary solutions $y = \pm 1$ on $\mathbb{R}$. Solve on intervals $y \in (-\infty; -1)$, $y \in (-1; 1)$, $y \in (1; +\infty)$. After integration: $\frac{1}{2} \ln(|\frac{y+1}{y-1}|) = x + c$, $c \in \mathbb{R}$. After simplification: for $y \in (-\infty; -1)$ and for $y \in (1; +\infty)$, we have $y = \operatorname{cotgh}(x + c)$; for $y \in (-1; 1)$, $y = \operatorname{tgh}(x + c)$. Conclusion: $y(x) = \pm 1$, $x \in \mathbb{R}$; $y(x) = \operatorname{tgh}(x + c)$, $x \in \mathbb{R}$, $c \in \mathbb{R}$; $y(x) = \operatorname{cotgh}(x + c)$, $x \in (-\infty; -c)$ or $x \in (-c; +\infty)$, $c \in \mathbb{R}$.



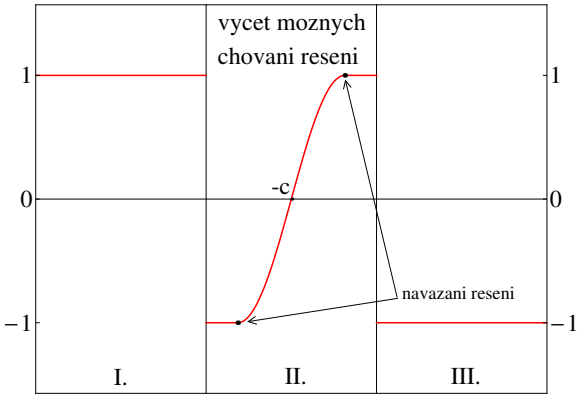
3) Defined for $x \in \mathbb{R}$, stationary solution $y = 0$ on $\mathbb{R}$. Solve on intervals $y \in (-\infty; 0)$, $y \in (0; +\infty)$. After integration: $\sqrt[3]{y^2} = x + c$, $c \in \mathbb{R}$, i.e., $x > -c$. After simplification, for $y < 0$: $y = \sqrt{(x+c)^3}$ and for $y > 0$: $y = -\sqrt{(x+c)^3}$. Gluing is possible at $-c$. Conclusion:

$$y(x) = \begin{cases} 0 & x \in (-\infty; -c), \\ \pm\sqrt{(x+c)^3} & x \in [-c; +\infty), \end{cases} \qquad c \in \mathbb{R}^*$$

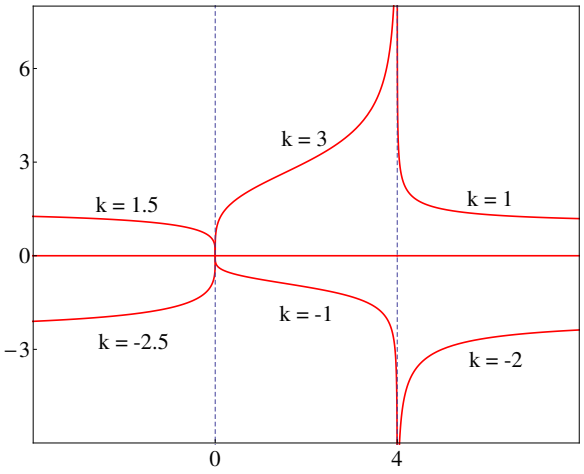


4) Solve for $x \in \mathbb{R}$ and $y \in [-1; 1]$. Stationary solutions $y = \pm 1$ on $\mathbb{R}$. For $y \in (-1; 1)$, we solve $\frac{y'}{\sqrt{1-y^2}} = 1$, which after integration gives $\arcsin y = x + c$, $c \in \mathbb{R}$. Thus $y = \sin(x + c)$, $x \in (-\frac{\pi}{2} - c; \frac{\pi}{2} - c)$. Solutions can be glued at all points where $y = -1$ or $y = 1$. Every maximal solution of the equation is determined by the formula $y(x) = \sin(x + c)$ for $x \in (-\frac{\pi}{2} - c; \frac{\pi}{2} - c)$,

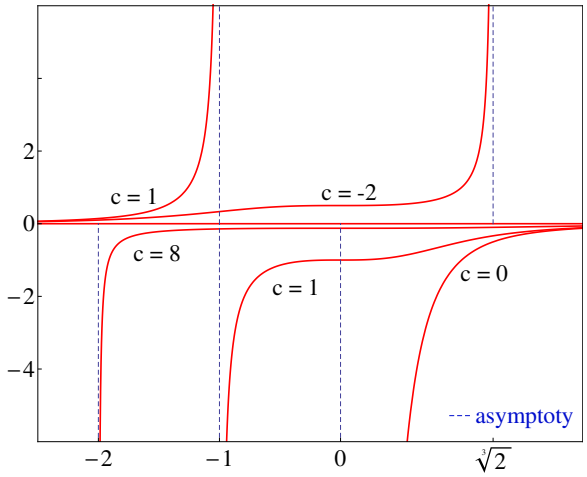
$y(x) = -1$ on $(-\infty; -\frac{\pi}{2} - c]$, and $y(x) = 1$ on $[\frac{\pi}{2} - c; +\infty)$, where c is a real number.



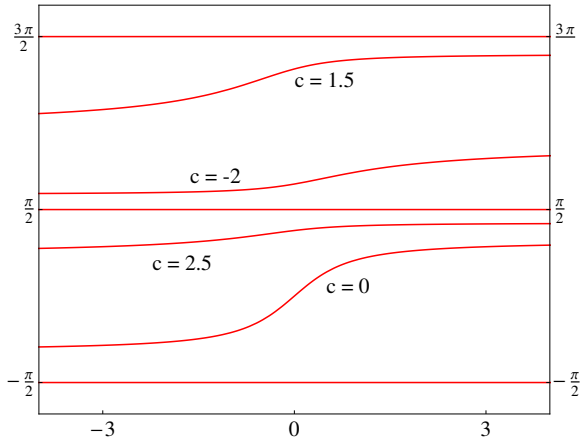
5) Defined for $x \in \mathbb{R}$, stationary solution $y = 0$ on $\mathbb{R}$. Solve on intervals $x \in (-\infty; 0)$, $x \in (0; 4)$, $x \in (4; +\infty)$, $y \in (-\infty; 0)$ or $y \in (0; +\infty)$. After integration: $\ln(|y|) = \frac{1}{4}\ln(|\frac{x}{x-4}|) + c$, $c \in \mathbb{R}$. After simplification, for $y < 0$: $y = -k\sqrt[4]{|\frac{x}{x-4}|}$, $k > 0$, and for $y > 0$: $y = k\sqrt[4]{|\frac{x}{x-4}|}$, $k > 0$. Conclusion: $y(x) = 0$, $x \in \mathbb{R}$; $y(x) = \pm k\sqrt[4]{|\frac{x}{x-4}|}$, $x \in (-\infty; 0)$ or $x \in (0; 4)$ or $x \in (4; +\infty)$, $k \in \mathbb{R}^+$.



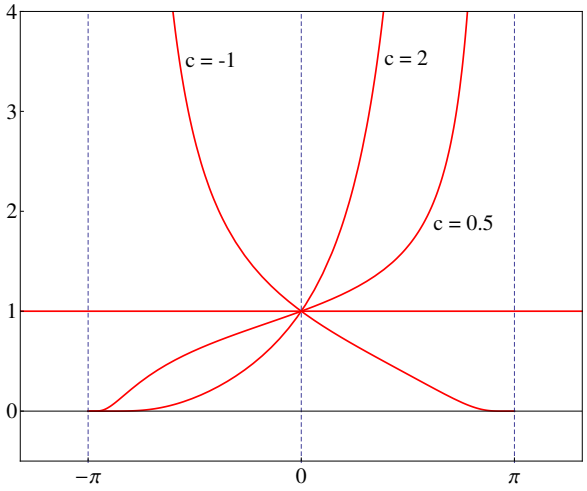
6) Defined for $x \in \mathbb{R}$, stationary solution $y = 0$ on $\mathbb{R}$. Solve on intervals $y \in (-\infty; 0)$ or $y \in (0; +\infty)$. After integration: $-\frac{1}{y} = x^3 + c$, $c \in \mathbb{R}$. After simplification: $y = \frac{-1}{x^3+c}$. Conclusion: $y(x) = 0$, $x \in \mathbb{R}$; $y(x) = \frac{-1}{x^3+c}$, $x \in (-\infty; -\sqrt[3]{c})$ or $x \in (-\sqrt[3]{c}; +\infty)$, $c \in \mathbb{R}$.



7) Defined for $x \in \mathbb{R}$, stationary solutions $y = \frac{\pi}{2} + k\pi$ on $\mathbb{R}$. Solve on intervals $y \in (-\frac{\pi}{2}+k\pi; \frac{\pi}{2}+k\pi)$, $k \in \mathbb{R}$. After integration: $\operatorname{tg}(y) = \operatorname{arctg}(x)+c$, $c \in \mathbb{R}$. After simplification: $y = \operatorname{arctg}(\operatorname{arctg} x + c) + k\pi$. Conclusion: $y(x) = k\pi + \operatorname{arctg}(c + \operatorname{arctg}(x))$, $x \in \mathbb{R}$, $c \in \mathbb{R}$, $k \in \mathbb{Z}$; $y(x) = \frac{\pi}{2} + k\pi$, $x \in \mathbb{R}$, $k \in \mathbb{Z}$.

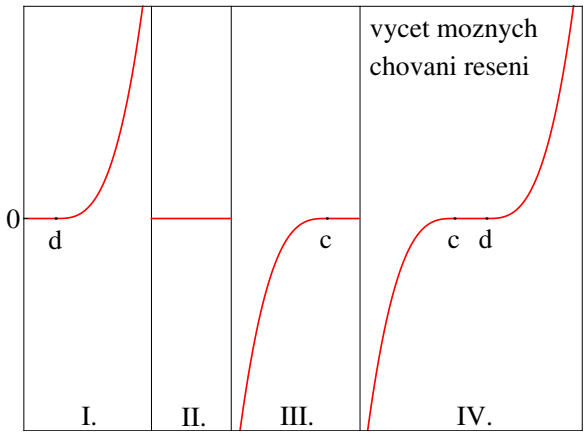


8) Defined for $x \in \mathbb{R}$, stationary solution $y = 1$ on $\mathbb{R}$. Solve on intervals $x \in (k\pi; \pi + k\pi)$, $k \in \mathbb{Z}$, $y \in \mathbb{R}^+$. After integration: $\ln|\ln y| = c - \operatorname{arctgh}(\cos x)$, $c \in \mathbb{R}$. After simplification, for $y < 1$: $y = \exp(-c\sqrt{\frac{1-\cos x}{1+\cos x}}) = \exp(-c|\operatorname{tg}(\frac{x}{2})|)$, $c > 0$ and for $y > 1$: $y = \exp(c\sqrt{\frac{1-\cos x}{1+\cos x}}) = \exp(c|\operatorname{tg}(\frac{x}{2})|)$, $c > 0$. Gluing is possible at $x = 2k\pi$. Conclusion: $y(x) = 1$, $x \in \mathbb{R}$; $y(x) = \exp(c \cdot \operatorname{tg}(\frac{x}{2}))$, $x \in ((2k-1)\pi; (2k+1)\pi)$, $c \in \mathbb{R} \setminus \{0\}$.

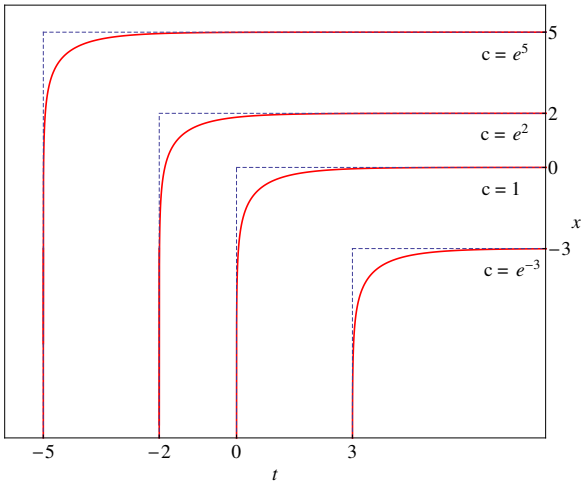


9) Defined for $x \in \mathbb{R}$, stationary solution $y = 0$ on $\mathbb{R}$. Solve on intervals $y \in (-\infty; 0)$ or $y \in (0; +\infty)$. After integration: $3\sqrt[3]{y} = x + c$, $c \in \mathbb{R}$. After simplification: $y = (\frac{x+c}{3})^3$ for $x \in (-\infty; -c)$ or $x \in (-c; +\infty)$. Conclusion:

$$y(x) = \begin{cases} (\frac{x+c}{3})^3; & x \in (-\infty; -c] \\ 0; & x \in (-c; -d) \quad c, d \in \mathbb{R}^*, c \geq d \\ (\frac{x+d}{3})^3; & x \in [-d; +\infty) \end{cases}$$

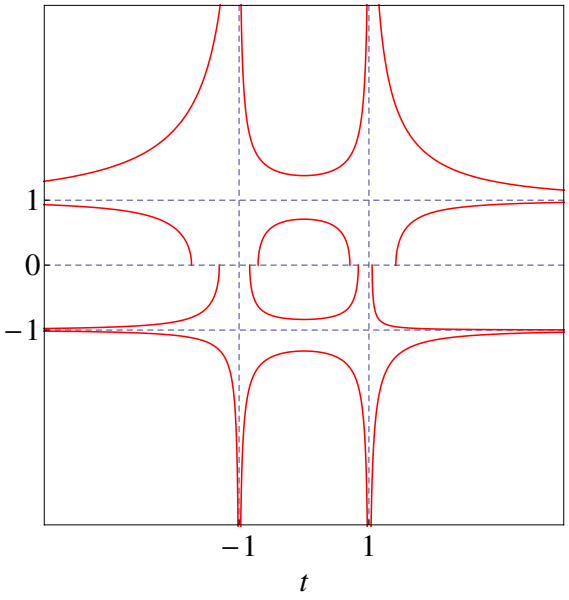


10) Defined for $x, y \in \mathbb{R}$, no stationary solutions on $\mathbb{R}$. After integration: $\exp y = c - \exp(-x)$, $c > 0$. After simplification: $y = \ln(c - \exp(-x))$ for $x \in (-\ln c; +\infty)$. Conclusion: $y(x) = \ln(c - \exp(-x))$; $x \in (-\ln c; +\infty)$, $c > 0$.



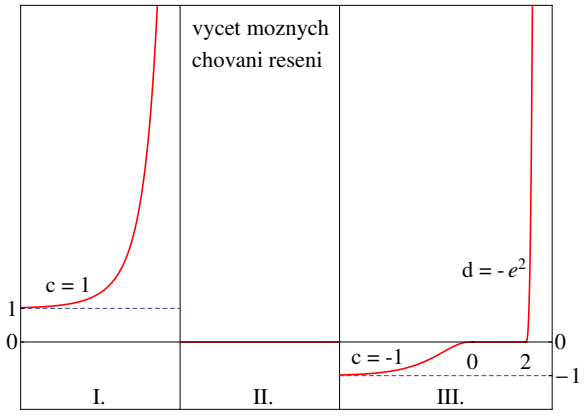
11) Defined for $x \in (-\infty;-1)$ or $x \in (-1;1)$ or $x \in (1;+\infty)$, no stationary solutions. Solve on intervals $y \in (-\infty;-1)$, $y \in (-1;1)$, $y \in (1;+\infty)$. After integration: $\frac{1}{2}\ln|y^2-1| = -\frac{1}{2}\ln|x^2-1| + \frac{1}{2}c$, $c \in \mathbb{R}$. After simplification: $y = \pm\sqrt{1+\frac{k}{x^2-1}}$. Conclusion: $y = \pm\sqrt{1+\frac{k}{1-x^2}}$ for $x \in (-1;1)$ and for $y \in (-1;1)$

$$y = \pm\sqrt{1+\frac{k}{x^2-1}} \begin{cases} x \in (-\infty;-\sqrt{1-k}) \text{ or } x \in (\sqrt{1-k};+\infty) \\ \qquad\qquad \text{for } k \in (-\infty;-1] \\ x \in (-\infty;-\sqrt{1-k}) \text{ or } x \in (-1;1) \text{ or } x \in (\sqrt{1-k};+\infty) \\ \qquad\qquad \text{for } k \in (-1;0) \\ x \in (-\infty;-1) \text{ or } x \in (-\sqrt{1-k};\sqrt{1-k}) \text{ or } x \in (1;+\infty) \\ \qquad\qquad \text{for } k \in (0;1) \\ x \in (-\infty;-1) \text{ or } x \in (1;+\infty) \\ \qquad\qquad \text{for } k \in [1;+\infty) \end{cases}$$



12) Defined for $x \in \mathbb{R}$, stationary solution $y = 0$ on $\mathbb{R}$. Solve on intervals $y \in (-\infty; 0)$ or $y \in (0; +\infty)$. After integration: $\sqrt[3]{y} = \exp(x) + c$, $c \in \mathbb{R}$. After simplification: $y = (c + \exp(x))^3$. For $c < 0$, gluing is possible at $y = 0$. Conclusion: $y(x) = (\exp(x) + c)^3$, $x \in \mathbb{R}$, $c \in \mathbb{R}_0^+$,

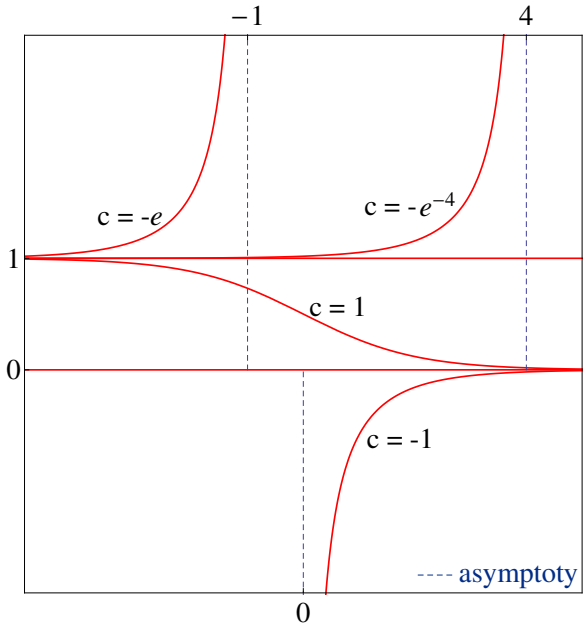
$$y(x) = \begin{cases} (\exp x + c)^3; & x \in (-\infty; \ln(-c)] \\ 0; & x \in (\ln(-c); \ln(-d)) \\ (\exp x + d)^3; & x \in [\ln(-d); +\infty) \end{cases} \qquad c, d \in \mathbb{R}^- \cup \{-\infty\},\ c > d$$



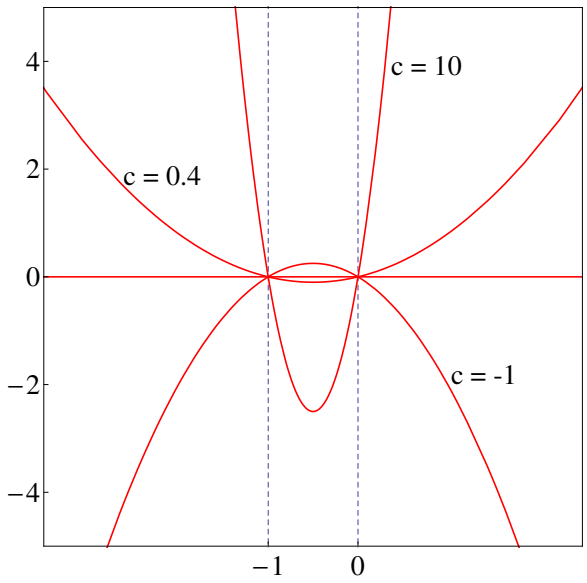
13) Defined for $x \in \mathbb{R}$, stationary solutions $y = 0$ on $\mathbb{R}$ and $y = 1$ on $\mathbb{R}$. Solve on intervals $y \in (-\infty; 0)$, $y \in (0; 1)$, $y \in (1; +\infty)$. After integration: $\ln\left|\frac{y-1}{y}\right| = x + c$, $c \in \mathbb{R}$. After simplification: $y = \frac{1}{1 - c\exp x}$. Conclusion:

$y(x) = 0$, $x \in \mathbb{R}$,

$$y(x) = \frac{1}{1 + c\exp x}, \begin{cases} x \in \mathbb{R} \text{ for } c \in (-\infty; 0] \\ x \in (-\infty; -\ln c) \text{ or } x \in (-\ln c; +\infty) \text{ for } c \in (0; +\infty) \end{cases}$$

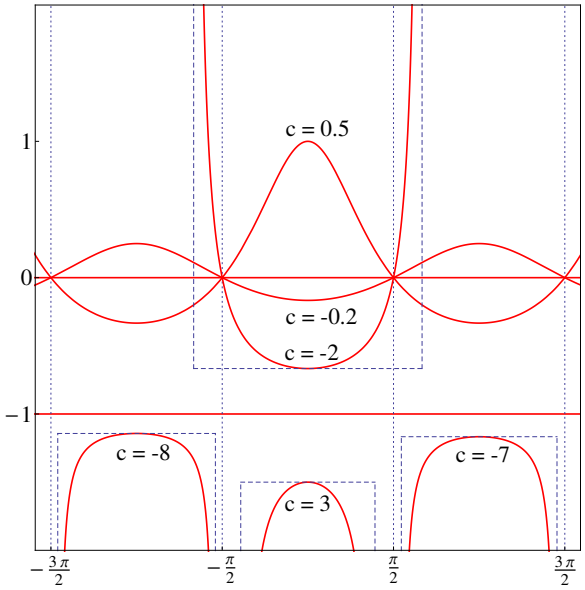


14) Defined for $x \in \mathbb{R}$, stationary solution $y = 0$ on $\mathbb{R}$. Solve on intervals $x \in (-\infty; -1)$, $x \in (-1; 0)$, $x \in (1; +\infty)$, $y \in (-\infty; 0)$, $y \in (0; +\infty)$. After integration: $\ln|y| = c + \ln|x^2 + x|$, $c \in \mathbb{R}$. After simplification: $y = c(x^2 + x)$. Conclusion:



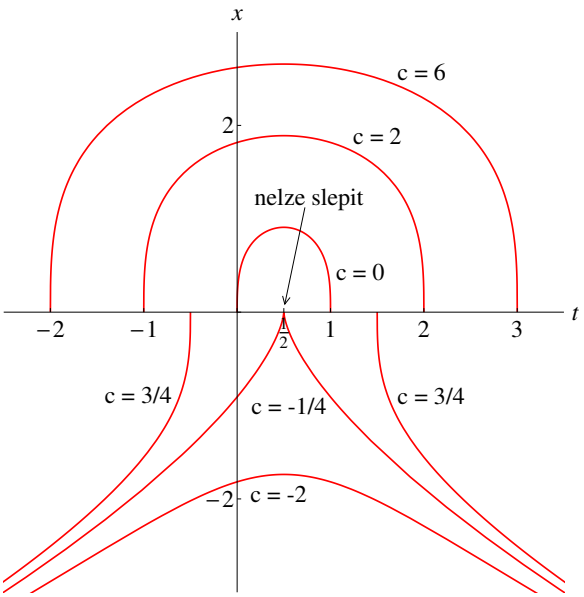
15) Defined for $x \in \mathbb{R}$, stationary solutions $y = 0$ and $y = -1$ on $\mathbb{R}$. Solve on intervals $x \in (-\frac{\pi}{2} + k\pi; \frac{\pi}{2} + k\pi)$, $k \in \mathbb{Z}$, $y \in (-\infty; -1)$, $y \in (-1; 0)$, $y \in (0; +\infty)$. After integration: $\ln|\frac{y}{y+1}| = \ln|\cos x| + c$, $c \in \mathbb{R}$. After simplification: $y = \frac{c \cdot \cos x}{1 - c \cdot \cos x}$. Conclusion: $y(x) = -1$, $x \in \mathbb{R}$ and

$$y(x) = \frac{c \cdot \cos(x)}{1 - c \cdot \cos(x)} \begin{cases} x \in (2k\pi - \arccos(\frac{1}{c}); 2k\pi + \arccos(\frac{1}{c})), \ k \in \mathbb{Z} \\ \qquad\qquad \text{for } c \in (-\infty; -1] \cup [1; +\infty) \\ x \in (2k\pi + \arccos(\frac{1}{c}); 2(k+1)\pi - \arccos(\frac{1}{c})), \\ \qquad\qquad \text{for } c \in (-\infty; -1] \cup [1; +\infty), \ k \in \mathbb{Z} \\ x \in \mathbb{R} \text{ for } c \in (-1; 1) \end{cases}$$

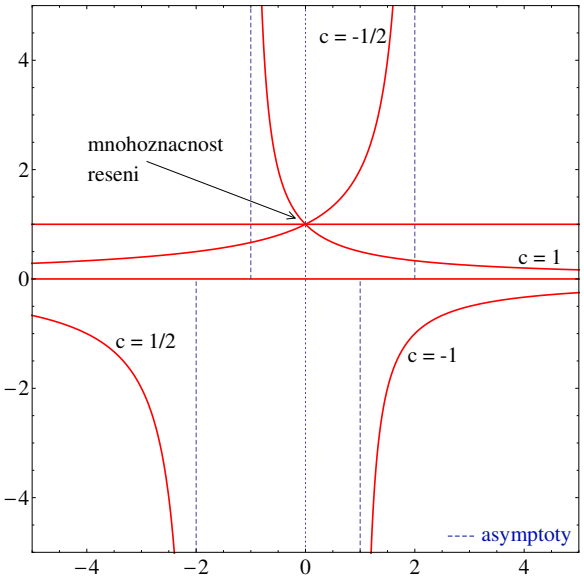


16) Defined for $x \in \mathbb{R}$, stationary solutions $y = \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$. Solve on intervals $x \in (-\infty; 0)$, $x \in (0; +\infty)$, $y \in (-\frac{\pi}{2} + k\pi; \frac{\pi}{2} + k\pi)$, $k \in \mathbb{R}$. After integration: $\operatorname{tg}(y) = \frac{1}{8}\operatorname{arctg}(\frac{x}{2}) - \frac{1}{4x} + c$, $c \in \mathbb{R}$. After simplification: $y = \operatorname{arctg}(\frac{1}{8}\operatorname{arctg}(\frac{x}{2}) - \frac{1}{4x} + c)$. Conclusion: $y(x) = k\pi + \operatorname{arctg}(\frac{1}{8}\operatorname{arctg}(\frac{x}{2}) - \frac{1}{4x} + c)$, $x \in (-\infty; 0)$ or $x \in (0; +\infty)$, $c \in \mathbb{R}$, $k \in \mathbb{Z}$ and $y(x) = k\pi + \frac{\pi}{2}$, $x \in \mathbb{R}$, $k \in \mathbb{Z}$.

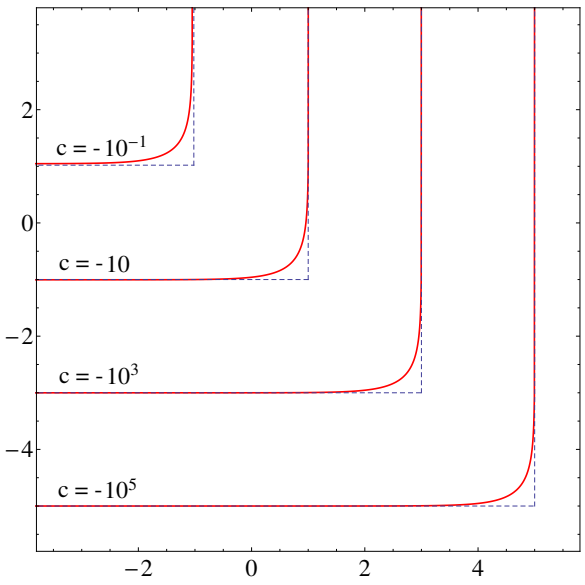
17) Defined for $x \in \mathbb{R}$, no stationary solutions. Solve on intervals $y \in (-\infty; 0)$, $y \in (0; +\infty)$. After integration: $\frac{1}{3}y^3 = x - x^2 + c$, $c \in \mathbb{R}$. After simplification: $y = \sqrt[3]{3(x - x^2 + c)}$. Conclusion: $y(x) = \sqrt[3]{3(x - x^2 + c)}$ on the intervals: If $c \in (-\infty; -\frac{1}{4})$, then $x \in \mathbb{R}$. If $c = -\frac{1}{4}$, $x \in (-\infty; \frac{1}{2})$ or $x \in (\frac{1}{2}; +\infty)$. And if $c \in (-\frac{1}{4}; +\infty)$, then $x \in (-\infty; \frac{1}{2} - \sqrt{\frac{1}{4} + c})$ or $x \in (\frac{1}{2} - \sqrt{\frac{1}{4} + c}; \frac{1}{2} + \sqrt{\frac{1}{4} + c})$ or $x \in (\frac{1}{2} + \sqrt{\frac{1}{4} + c}; +\infty)$.



18) Defined for $x \in \mathbb{R}$, stationary solutions $y = 0$ and $y = 1$ on $\mathbb{R}$. Solve on intervals $x \in (-\infty; 0)$, $x \in (0; +\infty)$, $y \in (-\infty; 0)$, $y \in (0; 1)$, $y \in (1; +\infty)$. After integration: $\ln|\frac{y-1}{y}| = \ln|x| + c$, $c \in \mathbb{R}$. After simplification, for $y < 0$ and $y > 1$: $y = \frac{1}{1+k|x|}$, $k > 0$, and for $0 < y < 1$: $y = \frac{1}{1-k|x|}$, $k > 0$. Conclusion: $y(x) = \frac{1}{1+kx}$, $x \in (-\infty; -\frac{1}{k})$ or $x \in (-\frac{1}{k}; +\infty)$, $k \in \mathbb{R} \setminus \{0\}$, $y(x) = 0$, $y(x) = 1$, $x \in \mathbb{R}$.

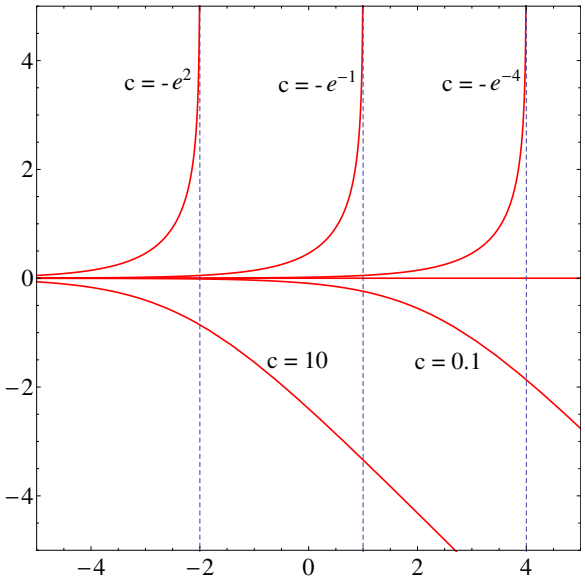


19) Defined for $x \in \mathbb{R}$, no stationary solutions. After integration: $-\frac{10^{-y}}{\ln 10} = \frac{10^x}{\ln 10} + \frac{c}{\ln 10}$, $c < 0$. After simplification: $y = -\log_{10}(-10^x - c)$. Conclusion: $y(x) = -\log_{10}(-10^x - c)$, $x \in (-\infty; \log_{10}(-c))$, $c \in \mathbb{R}^-$.



20) Defined for $x \in \mathbb{R}$, stationary solution $y = 0$ on $\mathbb{R}$. Solve on intervals $y \in (-\infty; 0)$, $y \in (0; +\infty)$. After integration: $\ln|1 - \exp(-y)| = x + c$, $c \in \mathbb{R}$. After simplification, for $y < 0$: $y = -\ln(k \exp x + 1)$ and for $y > 0$: $y = -\ln(1 - k \exp x)$. Conclusion:

$$y(x) = -\ln(1 + k\exp(x)), \begin{cases} x \in (-\infty; -\ln(-k)) & \text{for } k \in (-\infty; 0) \\ x \in \mathbb{R} & \text{for } k \in [0; +\infty) \end{cases}$$



21) Defined for $x \in (-\infty; -1)$ or $x \in (-1; +1)$ or $x \in (1; +\infty)$, stationary solutions $y = \pm 1$ on $\mathbb{R}$. Solve on intervals $x \in (-1; 1)$ and $y \in (-1; 1)$. After

integration: $\arcsin(y) = \arcsin(x) + c$, $c \in (-\pi;\pi)$. After simplification: $y = \sin(c + \arcsin(x))$. On intervals $x \in (-\infty;-1)$ or $x \in (1;+\infty)$ and $y \in (-\infty;-1)$ or $y \in (1;+\infty)$, we obtain after integration $\operatorname{argcosh}(x) = \operatorname{argcosh}(x)+c$, $c \in \mathbb{R}$. Analyzing the various cases and checking how solutions can be glued, we obtain the following conclusion:
stationary solutions:

$$y(x) = \pm 1,\ x \in (-\infty;-1) \text{ or } x \in (-1;+1) \text{ or } x \in (1;+\infty)$$

solutions for $c \in (-\infty;-\pi]$:

$$y(x) = \pm\cosh(c + \operatorname{argcosh}|x|),\ x \in (-\infty;-\cosh c) \text{ or } x \in (\cosh c;+\infty)$$

solutions for $c \in (-\pi;0]$:

$$y(x) = \pm\cosh(c + \operatorname{argcosh}|x|),\ x \in (-\infty;-\cosh c) \text{ or } x \in (\cosh c;+\infty)$$

$$y(x) = \sin(c + \arcsin x),\ x \in (-1;1)$$

$$y(x) = \begin{cases} -1, & x \in (-1;\cos c] \\ \sin(c + \arcsin x), & x \in [\cos c;1) \end{cases}$$

solutions for $c \in [0;\pi)$:

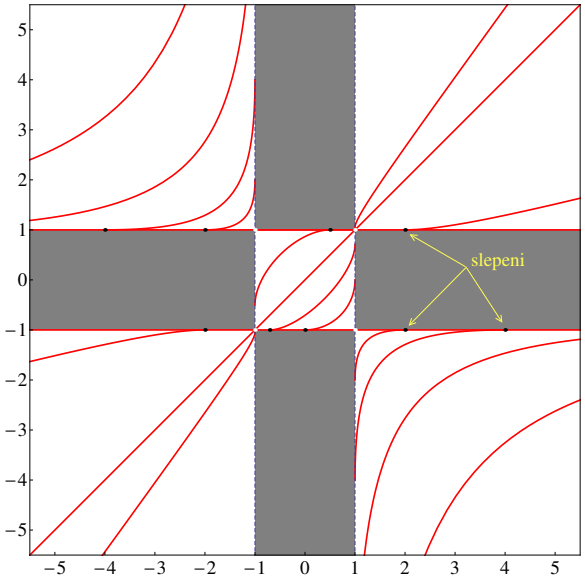
$$y(x) = \pm\cosh(c + \operatorname{argcosh}|x|),\ x \in (-\infty;-1) \text{ or } x \in (1;+\infty)$$

$$y(x) = \sin(c + \arcsin x),\ x \in (-1;1)$$

$$y(x) = \begin{cases} \sin(c + \arcsin x), & x \in (-1;\cos c] \\ 1, & x \in [\cos c;1) \end{cases}$$

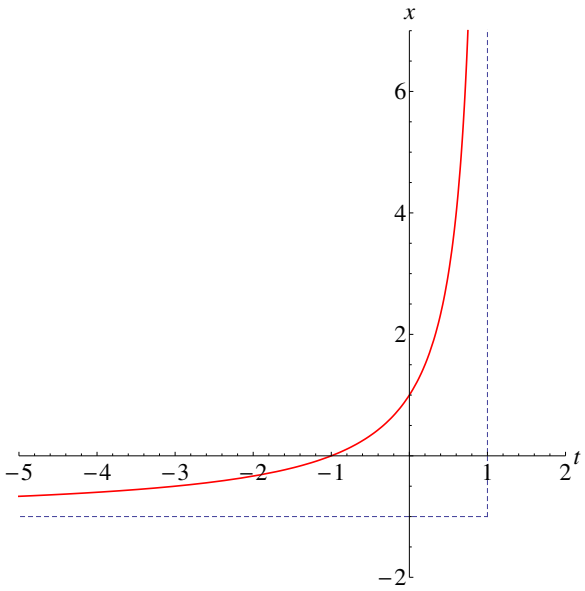
solutions for $c \in [\pi;+\infty)$:

$$y(x) = \pm\cosh(c + \operatorname{argcosh}|x|),\ x \in (-\infty;-1) \text{ or } x \in (1;+\infty)$$



22) Defined for $x \in \mathbb{R}$, stationary solution $y = 1$ on $\mathbb{R}$. After integration: $\ln|\ln|y|| = \frac{1}{2}\ln|\frac{1-\cos(x)}{1+\cos(x)}| + c$, $c \in \mathbb{R}$. After simplification: $y = \exp(k\sqrt{|\frac{1-\cos(x)}{1+\cos(x)}|})$, $k \in \mathbb{R}$. Conclusion: The solution satisfying the initial condition is $y(x) = 1$, $x \in \mathbb{R}$.

23) Defined for $x \in \mathbb{R}$. After integration: $\operatorname{arctg}(y) = \operatorname{arctg}(x) + c$, $c = \frac{\pi}{4}$. After simplification: $y = \exp(k\sqrt{|\frac{1-\cos(x)}{1+\cos(x)}|})$, $k \in \mathbb{R}$. Conclusion: The solution satisfying the initial condition is $y(x) = \operatorname{tg}(\frac{\pi}{4} + \operatorname{arctg} x)$, $x \in (-\infty, \operatorname{tg}\frac{\pi}{4})$.

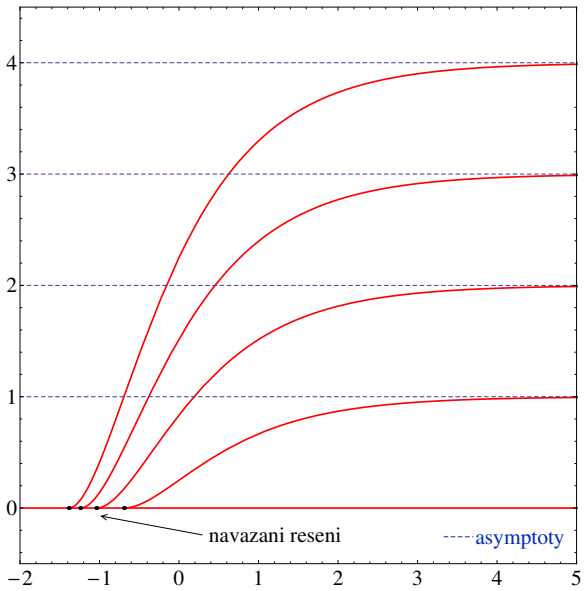


24) Solve for $x \in \mathbb{R}$ and $y \in [0; +\infty)$. Stationary solution $y = 0$ on $\mathbb{R}$.

For $y > 0$, we solve $\frac{y'}{\sqrt{y}} = \exp(-x)$. After integration, we have $2\sqrt{y} = c - \exp(-x)$, $c \in \mathbb{R}^+$, for $c - \exp(-x) > 0$. After simplification: $y = \frac{(c-\exp(-x))^2}{4}$. Conclusion:

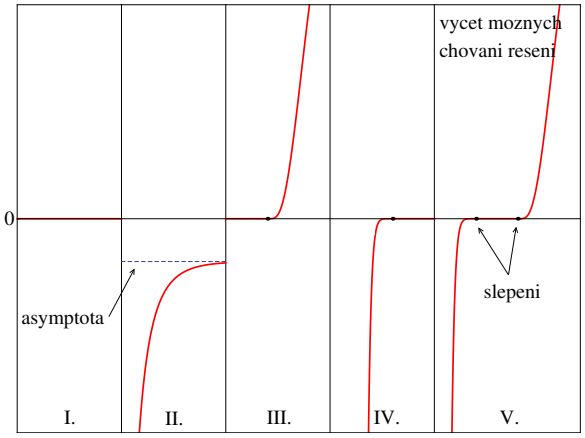
$$y(x) = 0,\ x \in \mathbb{R}$$

$$y(x) = \begin{cases} 0, & x \in (-\infty; -\ln c] \\ \frac{1}{4}(c - \exp(-x))^2, & x \in (-\ln c; +\infty) \text{ for } c \in \mathbb{R}^+ \end{cases}$$

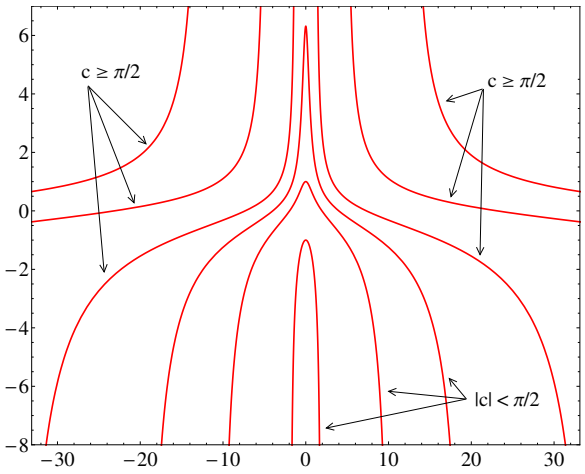


25) Defined for $x \in \mathbb{R}$, stationary solution $y = 0$ on $\mathbb{R}$. Solve on intervals $y \in (-\infty; 0)$, $y \in (0; +\infty)$. After integration: $\sqrt[5]{y} = c - 4\exp(-x)$, $c \in \mathbb{R}$. After simplification: $y = (c - 4\exp x)^5$. Conclusion: $y(x) = (c - 4\exp(x))^5$, $x \in \mathbb{R}$ for $c \in (-\infty; 0]$ and

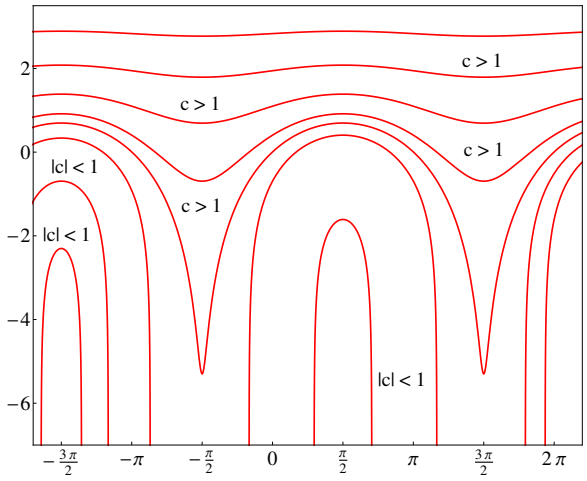
$$y(x) = \begin{cases} (c - 4\exp x)^5, & x \in (-\infty; -\ln(\frac{c}{4})) \\ 0, & x \in (-\ln(\frac{c}{4}); -\ln(\frac{d}{4})) \\ (d - 4\exp x)^5, & x \in (-\ln(\frac{d}{4}; +\infty)) \end{cases} \quad c, d \in [0; +\infty],\ c \geq d$$



26) Defined for $x \in \mathbb{R}$. After integration: $\operatorname{arctg}(y) = c - \frac{1}{2}\ln(1+x^2)$, $c \in (-\frac{\pi}{2}; +\infty)$. After simplification: $y = \operatorname{tg}(c - \frac{1}{2}\ln(x^2+1))$. Conclusion: $y(x) = \operatorname{tg}(c - \frac{1}{2}\ln(x^2+1))$ on the interval $(-\sqrt{\exp(2c+\pi)-1}; \sqrt{\exp(2c+\pi)-1})$ if $c \in (-\frac{\pi}{2}; \frac{\pi}{2})$ and on the intervals $(-\sqrt{\exp(2c+\pi)-1}; \sqrt{\exp(2c-\pi)-1})$ or $(\sqrt{\exp(2c-\pi)-1}; \sqrt{\exp(2c+\pi)-1})$ if $c \in [\frac{\pi}{2}; +\infty)$.



27) Defined for $x \in \mathbb{R}$. After integration: $\exp(y) = \sin(x) + c$, $c \in (-1; +\infty)$. After simplification: $y = \ln(\sin(x) + c)$. Conclusion: $y(x) = \ln(\sin(x) + c)$ on the intervals $(-\arcsin(c) + 2k\pi; -\arcsin(c) + (2k+1)\pi)$, $k \in \mathbb{Z}$ for $c \in (-1; 1)$, on the intervals $(-\frac{\pi}{2} + 2k\pi; \frac{3\pi}{2} + 2k\pi)$, $k \in \mathbb{Z}$ for $c = 1$, and on $\mathbb{R}$ for $c \in (1; +\infty)$.



28) Solve for $x \in \mathbb{R}$ and $y \in (-\frac{\pi}{2}+k\pi; \frac{\pi}{2}+k\pi)$, $k \in \mathbb{Z}$. Stationary solutions $y = k\pi$. For $y \in (-\frac{\pi}{2}+k\pi; k\pi)$ and for $y \in (k\pi; \frac{\pi}{2}+k\pi)$, after integration we have $\ln|\operatorname{tg} y| = 3\ln|2-e^x| + c$, $c \in \mathbb{R}$. Let $d = (\operatorname{sgn}\operatorname{tg} y)e^c$. Then $\operatorname{tg} y = d(2-e^x)^3$, whence $y = k\pi + \operatorname{arctg}(d(e^x-2)^3)$. Conclusion:

$$y(x) = k\pi + \operatorname{arctg}(d(e^x-2)^3),\ x \in \mathbb{R},\ d \in \mathbb{R},\ k \in \mathbb{Z}$$

29) Defined for $x \in (-\frac{\pi}{2}+k\pi; \frac{\pi}{2}+k\pi)$, $k \in \mathbb{Z}$. After integration: $\operatorname{arctg} y = c - \ln|\cos x|$, $c \in (-\infty; \frac{\pi}{2})$, since for $c \geq \frac{\pi}{2}$ we have $c - \ln|\cos x| \geq \frac{\pi}{2}$, but $\operatorname{arctg} y \in (-\frac{\pi}{2}; \frac{\pi}{2})$. Thus:

$$y(x) = \operatorname{tg}\left(c - \ln|\cos x|\right).$$

The solution is defined on the intervals

$$\begin{cases} x \in (k\pi - \arccos e^{c-\frac{\pi}{2}}; k\pi - \arccos e^{c+\frac{\pi}{2}}),\ k \in \mathbb{Z} & c \in (-\infty; -\frac{\pi}{2}] \\ x \in (k\pi + \arccos e^{c+\frac{\pi}{2}}; k\pi + \arccos e^{c-\frac{\pi}{2}}),\ k \in \mathbb{Z} & c \in (-\infty; -\frac{\pi}{2}] \\ x \in (k\pi - \arccos e^{c-\frac{\pi}{2}}; k\pi + \arccos e^{c-\frac{\pi}{2}}),\ k \in \mathbb{Z} & c \in (-\frac{\pi}{2}; \frac{\pi}{2}) \end{cases}$$

30) Defined for $x \in \mathbb{R}$. After integration, we have $\frac{1}{2}y^2 = c - \frac{1}{2}(x-1)^2$, $c > 0$, since for $c \leq 0$ we have $c - \frac{1}{2}(x-1)^2 \leq 0$, but $\frac{1}{2}y^2 \geq 0$. Solution: $y(x) = \pm\sqrt{2c-(x-1)^2}$, $x \in (1-\sqrt{2c}; 1+\sqrt{2c})$ for $c \in (0; +\infty)$.

31) Defined for $x \in (-1; 1)$. Stationary solutions $y = \pm 1$ for $x \in (-1; 1)$. After integration, we have the equation $\sqrt{1-y^2} = c - \sqrt{1-x^2}$. It must hold that $c - \sqrt{1-x^2} \in [0; 1)$, i.e., $\sqrt{1-x^2} \in (c-1; c]$. Hence $x \in (-1; -\sqrt{1-c^2})$ or $x \in (\sqrt{1-c^2}; 1)$ for $c \in (0; 1)$, and $x \in (-\sqrt{1-(c-1)^2}; \sqrt{1-(c-1)^2})$ for $c \in [1; 2)$. For other values of the constant c, the condition $c - \sqrt{1-x^2} \in [0; 1)$ is not satisfied. The condition $y \neq 0 \Leftrightarrow x^2 \neq 1-(c-1)^2$ is satisfied; gluing occurs at points where $y = \pm 1 \Leftrightarrow x^2 = 1-c^2$. Solutions:

for $c \in (0;1)$:

$$y(x) = \begin{cases} -\sqrt{1-(c-\sqrt{1-x^2})^2}, & x \in (-1;-\sqrt{1-c^2}) \\ -1, & x \in [-\sqrt{1-c^2};\sqrt{1-c^2}] \\ -\sqrt{1-(c-\sqrt{1-x^2})^2}, & x \in (\sqrt{1-c^2};1) \end{cases}$$

$$y(x) = \begin{cases} \sqrt{1-(c-\sqrt{1-x^2})^2}, & x \in (-1;-\sqrt{1-c^2}) \\ 1, & x \in [-\sqrt{1-c^2};\sqrt{1-c^2}] \\ \sqrt{1-(c-\sqrt{1-x^2})^2}, & x \in (\sqrt{1-c^2};1) \end{cases}$$

for $c \in (1;2)$:

$$y(x) = \pm\sqrt{1-(c-\sqrt{1-x^2})^2},\ x \in (-\sqrt{1-(c-1)^2};\sqrt{1-(c-1)^2})$$

other solutions:

$$y(x) = \begin{cases} 1, & x \in (-1;0] \\ \sqrt{1-(1-\sqrt{1-x^2})^2}, & x \in (0;1) \end{cases}$$

$$y(x) = \begin{cases} \sqrt{1-(1-\sqrt{1-x^2})^2}, & x \in (-1;0) \\ 1, & x \in [0;1) \end{cases}$$

$$y(x) = \begin{cases} -1, & x \in (-1;0] \\ -\sqrt{1-(1-\sqrt{1-x^2})^2}, & x \in (0;1) \end{cases}$$

$$y(x) = \begin{cases} -\sqrt{1-(1-\sqrt{1-x^2})^2}, & x \in (-1;0) \\ -1, & x \in [0;1) \end{cases}$$

$$y(x) = \pm 1,\ x \in (-1;1)$$

32) Defined for $x \in (-\infty;0)$ or $x \in (0;+\infty)$. After integration and multiplying by two: $\sqrt{y^2+1} = \ln|x| + 1 - c$, $c \in \mathbb{R}$. It must hold that $\ln|x| + 1 - c \geq 1$, i.e., $\ln|x| \geq c$, whence $x \in (-\infty;-e^c)$ or $x \in (e^c;+\infty)$. Solutions on these intervals: $y(x) = \pm\sqrt{(\ln|x| + 1 - c)^2 - 1}$.

33) Solve for $x \in (-\infty;-1)$, $x \in (-1;1)$, $x \in (1;+\infty)$ and $y \in \mathbb{R}$. Stationary solution $y = 0$ on $\mathbb{R}$. For $y \in (-\infty;0)$ or $y \in (0;+\infty)$, we solve $\frac{-y'}{y^2} = \frac{2x}{x^2-1}$. After integration: $\frac{1}{y} = c + \ln|x^2-1|$, $c \in \mathbb{R}$. Thus $y = \frac{1}{c+\ln|x^2-1|}$ for x such that $x^2 - 1 \neq \exp(-c)$. Conclusion:

$$y(x) = \frac{1}{c + \ln|x^2-1|},$$

The solution is defined for $x \in (-\infty;-\sqrt{1+e^{-c}})$, $x \in (-\sqrt{1+e^{-c}};-1)$, $x \in (-1;1)$, $x \in (1;\sqrt{1+e^{-c}})$, $x \in (\sqrt{1+e^{-c}};+\infty)$ if $c \in (-\infty;0)$, and for $x \in (-\infty;-\sqrt{1+e^{-c}})$, $x \in (-\sqrt{1+e^{-c}};-1)$, $x \in (-1;-\sqrt{1-e^{-c}})$, $x \in (-\sqrt{1-e^{-c}};\sqrt{1-e^{-c}})$, $x \in (\sqrt{1-e^{-c}};1)$, $x \in (1;\sqrt{1+e^{-c}})$, $x \in (\sqrt{1+e^{-c}};+\infty)$ if $c \in [0;+\infty)$.

34) Solve for $x \in \mathbb{R}$ and $y \in \mathbb{R}$. Rewrite the equation as $xy' \cos y = \sin y(\sin y - 1)$. Stationary solutions $y = k\pi$, $k \in \mathbb{Z}$, on $\mathbb{R}$ and $y = \frac{\pi}{2} + 2k\pi$, $k \in \mathbb{Z}$, on $\mathbb{R}$. For $x \neq 0$ and for $y \in (2k\pi;\frac{\pi}{2}+2k\pi)$, $y \in (\frac{\pi}{2}+2k\pi;(2k+1)\pi)$ or $y \in ((2k+1)\pi;(2k+2)\pi)$, we have $\frac{y'\cos y}{\sin y(\sin y-1)} = \frac{1}{x}$. After integration: $\ln|1-\frac{1}{\sin y}| = \ln|x| + \ln c$, $c \in \mathbb{R}^+$, whence $|1-\frac{1}{\sin y}| = c|x|$. For $y \in (2k\pi;\frac{\pi}{2}+2k\pi)$, the solution is $y = \arcsin\frac{1}{1+c|x|} + 2k\pi$. For $y \in (\frac{\pi}{2}+2k\pi;(2k+1)\pi)$, the solution is $y = \pi - \arcsin\frac{1}{1+c|x|} + 2k\pi$. For $y \in ((2k+1)\pi;(2k+2)\pi)$, the solutions are $y = \arcsin\frac{1}{1-c|x|} + (2k+2)\pi$ and $y = \pi - \arcsin\frac{1}{1-c|x|} + (2k+2)\pi$. Solutions can be connected at $x = 0$. Conclusion:
solutions for $y \in (2k\pi;\pi+2k\pi)$, with $c, d \in \mathbb{R}$:

$$y(x) = \begin{cases} \arcsin\frac{1}{1-c|x|} + 2k\pi, & x \in (-\infty;0] \\ \arcsin\frac{1}{1-d|x|} + 2k\pi, & x \in [0;+\infty) \end{cases}$$

$$y(x) = \begin{cases} \arcsin\frac{1}{1-c|x|} + 2k\pi, & x \in (-\infty;0] \\ \pi - \arcsin\frac{1}{1-d|x|} + 2k\pi, & x \in [0;+\infty) \end{cases}$$

$$y(x) = \begin{cases} \pi - \arcsin\frac{1}{1-c|x|} + 2k\pi, & x \in (-\infty;0] \\ \arcsin\frac{1}{1-d|x|} + 2k\pi, & x \in [0;+\infty) \end{cases}$$

$$y(x) = \begin{cases} \pi - \arcsin\frac{1}{1-c|x|} + 2k\pi, & x \in (-\infty;0] \\ \pi - \arcsin\frac{1}{1-d|x|} + 2k\pi, & x \in [0;+\infty) \end{cases}$$

stationary solutions

$$y(x) = k\pi,\ x \in \mathbb{R}$$

solutions for $y \in (\pi + 2k\pi; 2\pi + 2k\pi)$, with $c, d \in \mathbb{R} \setminus \{0\}$:

$$y(x) = \arcsin \frac{1}{1 - c|x|} + (2k+2)\pi,\ x \in (-\infty; -\frac{2}{c})$$

$$y(x) = \arcsin \frac{1}{1 - c|x|} + (2k+2)\pi,\ x \in (\frac{2}{c}; +\infty)$$

$$y(x) = \pi - \arcsin \frac{1}{1 - c|x|} + (2k+2)\pi,\ x \in (-\infty; -\frac{2}{c})$$

$$y(x) = \pi - \arcsin \frac{1}{1 - c|x| + (2k+2)\pi},\ x \in (\frac{2}{c}; +\infty)$$

In all formulas, $k \in \mathbb{Z}$.

35) Defined for $x \in \mathbb{R}$; after rearranging, we solve $(b+1)xy' = y - b$. For $b = -1$, the solution $y = -1$ does not satisfy the initial condition. We solve for $b \neq -1$. The stationary solution $y = b$ satisfies the initial condition for $b = 1$. For $b \neq \pm 1$, solve on intervals $y \in (-\infty; b)$, $y \in (b; +\infty)$. After integration: $(b+1)\ln(|y-b|) = \ln(|x|) + c$, $c \in \mathbb{R}$. After simplification: $|y - b| = k \exp\left(\frac{\ln|x|}{b+1}\right)$, where $k = e^c$. Substituting the initial condition $y(1) = 1$ yields $|1 - b| = k$. Solution for $y < b$: $y = b - |b-1| \exp\left(\frac{\ln(|x|)}{b+1}\right)$, and for $y > b$: $y = b + |b-1| \exp\left(\frac{\ln(|x|)}{b+1}\right)$. Conclusion:

$$\begin{array}{ll} \text{for } b \in (-\infty; -1): & y(x) = b + (1-b)\exp\left(\frac{\ln(x)}{b+1}\right), \qquad x \in (0; +\infty) \\ \text{for } b \in \{-1\}: & \text{no solution} \\ \text{for } b \in (-1; +\infty): & y(x) = \begin{cases} \left\{ \begin{array}{l} b - (b-1)\exp\left(\frac{\ln(-x)}{b+1}\right), \\ b, \\ b + (b-1)\exp\left(\frac{\ln(-x)}{b+1}\right), \end{array} \right\} & x \in (-\infty; 0) \\ b, & x \in \{b\} \\ b + (1-b)\exp\left(\frac{\ln(x)}{b+1}\right), & x \in (0; +\infty) \end{cases} \end{array}$$

36) Defined for $x \in (-\infty; 0)$ and $x \in (0; +\infty)$. The stationary solution $y = 0$ satisfies the conditions. Solve for $y \in (-\infty; 0)$ and $y \in (0; +\infty)$. After integration: $-\frac{1}{y} = -\frac{1}{x} - c$, $c \in \mathbb{R}$. After rearranging: $y = \frac{x}{cx+1}$. The bounded

solutions are:

$$y(x) = \frac{x}{cx+1}, \quad \begin{cases} x \in (-\infty; 0) & \text{for } c \in (-\infty; 0) \\ x \in (0; +\infty) & \text{for } c \in (0; +\infty) \end{cases}$$

$$y(x) = 0, \qquad \begin{cases} x \in (-\infty; 0) \\ x \in (0; +\infty) \end{cases}$$

37) First, we solve the differential equation. It is defined for $x \in \mathbb{R}$ and $y \in \mathbb{R}$. We solve $y' \exp y = \cos x$, which upon integration yields $\exp y = \sin x + c$, $c \in (-\infty; 1)$. The solution is $y(x) = \ln(\sin x + c)$. We determine the constant c such that the domain of this solution is all of $\mathbb{R}$: we require $\sin x + c > 0$, whence $c > -\sin x = \sin(-x)$ for all $x \in \mathbb{R}$. Thus, $c > 1$. Because the right-hand side of the equation is locally Lipschitz with respect to y at every point (being of class $\mathcal{C}^\infty \subseteq \mathcal{C}^1$), exactly one solution passes through each point in $\mathbb{R}^2$. Therefore, the desired set is $M = \{(x, y) \in \mathbb{R}^2; \exp y - \sin x > 1\}$ (these are the points in the plane lying above the graph of $y = \ln(1 + \sin x)$).

38) From Exercise 28, we know that all maximal solutions are of the form $y(x) = k\pi + \operatorname{arctg}(d(e^x - 2)^3)$, where $x \in \mathbb{R}$, $d \in \mathbb{R}$, and $k \in \mathbb{Z}$. All of them are thus defined on a neighborhood of both positive and negative infinity. Evaluating the limits: $\lim\limits_{x \to -\infty} y(x) = k\pi + \operatorname{arctg}(-8d)$ and $\lim\limits_{x \to +\infty} y(x) = k\pi + (\operatorname{sgn} d)\frac{\pi}{2}$. Setting $\lim_{x \to -\infty} y(x) = \lim_{x \to +\infty} y(x)$, we have $k\pi + \operatorname{arctg}(-8d) = k\pi + (\operatorname{sgn} d)\frac{\pi}{2}$. Since for all $d \in \mathbb{R}$ we have $\operatorname{arctg}(-8d) \notin \{\pm\frac{\pi}{2}\}$, it follows that $\operatorname{sgn} d = d = 0$. Thus $\lim_{x \to -\infty} y(x) = \lim_{x \to +\infty} y(x) = k\pi$, $k \in \mathbb{Z}$. Conclusion: $A \in \{k\pi; k \in \mathbb{Z}\}$.

39) If $x \mapsto y(x)$ is a solution passing through (x_0, y_0), then y satisfies the equation from Step 4 of the solution procedure, i.e., $G(y(x)) = F(x) + c$, where $c = G(y_0) - F(x_0)$. Since G, as a map from a neighborhood of y to a neighborhood of x, is injective, it can be inverted, yielding

$$y(x) = G^{-1}(F(x) + G(y_0) - F(x_0)).$$

Thus, the solution is uniquely determined on a neighborhood of the point. (Work out the details of this procedure for yourself).

Homogeneous Equations

Consider the equation

$$y' = f(x, y), \tag{4}$$

where

$$f(\lambda x, \lambda y) = f(x, y)\,, \qquad \forall \lambda \neq 0\,.$$

This equation is called a *homogeneous first-order equation* (or an equation with homogeneous right-hand side of degree zero). We will show that this equation can be transformed into a separable differential equation via substitution.

Solution Procedure. We observe that (for $x \neq 0$)

$$f(x,y) = f(x \cdot 1, x \cdot \frac{y}{x}) = f(1, \frac{y}{x}),$$

i.e., the right-hand side depends only on y/x.

Thus, we define $z(x) = y(x)/x$, so that

$$\begin{aligned} y(x) &= xz(x), \\ y'(x) &= xz'(x) + z(x). \end{aligned}$$

The equation becomes $xz' + z = f(x, xz) = f(1, z)$, i.e.,

$$z' = \frac{1}{x}\big[f(1,z) - z\big], \tag{5}$$

which is a *separable differential equation* (for the unknown function $z = z(x)$). Using standard methods, we find $z(x)$ and consequently $y(x) = xz(x)$.

Remark. Note that if z is a solution to equation (5) on some subinterval of $(0, +\infty)$ or $(-\infty, 0)$, then $y := z \cdot x$ is a solution to equation (4) on the same interval. The converse implication also holds. Because we had to exclude $x = 0$ during the computation, one must check in specific instances whether the obtained solutions can be continuously extended to the origin.

Example 3. Solve the equation $x^2y' + xy = x^2 + y^2$.

Solution. For $x \neq 0$, we rewrite the equation as

$$y' = \frac{-xy + x^2 + y^2}{x^2},$$

which is a homogeneous first-order equation. The substitution $y(x) = xz(x)$ yields

$$\begin{aligned} xz' + z &= -z + 1 + z^2, \\ xz' &= (z-1)^2. \end{aligned}$$

This is a separable equation. Clearly, $z \equiv 1$ is a solution, so $y(x) = x$, $x \in \mathbb{R}$ is a solution of the original equation.

Let us search for solutions under the condition $z \neq 1$, $x \neq 0$. Thus,

$$-\frac{z'}{(z-1)^2} = -\frac{1}{x}$$

and integration yields

$$\frac{1}{z-1} = c - \ln|x|\,.$$

For $I = (0, +\infty)$, $J = (1, +\infty)$, the function on the left-hand side maps J onto $(0, +\infty)$. Thus, $c - \ln x \in (0, +\infty)$ along with $x \in (0, +\infty)$, which gives $x \in (0, e^c)$. For $I = (-\infty, 0)$, we obtain $x \in (-e^c, 0)$. On these intervals, the solution is given by

$$z(x) = \frac{c + 1 - \ln|x|}{c - \ln|x|}\,, \qquad y(x) = x\frac{c + 1 - \ln|x|}{c - \ln|x|}\,. \tag{6}$$

Similarly, for $J = (1, +\infty)$ and $I = (0, +\infty)$ (resp. $I = (-\infty, 0)$), the function G maps J onto $(-\infty, 0)$. We have $c - \ln|x| \in (-\infty, 0)$ along with $x > 0$ (resp. $x < 0$). This yields solutions on $(-\infty, -e^c)$ (resp. $(e^c, +\infty)$) defined again by (6).

Are these solutions maximal? Since the limits of the solutions at $\pm e^c$ from the left and right are not finite, the solutions cannot be extended past these points. It remains to investigate the extension to the origin. Since

$$\lim_{x\to 0} y(x) = 0$$

and defining $y(0) := 0$, we have

$$y'(0) = \lim_{x\to 0} \frac{y(x) - y(0)}{x} = \lim_{x\to 0} \frac{c + 1 - \ln|x|}{c - \ln|x|} = 1\,,$$

we easily verify by substitution that the equation is satisfied at $x = 0$. Thus, the extended function is a solution on the entire interval $(-e^c, e^c)$.

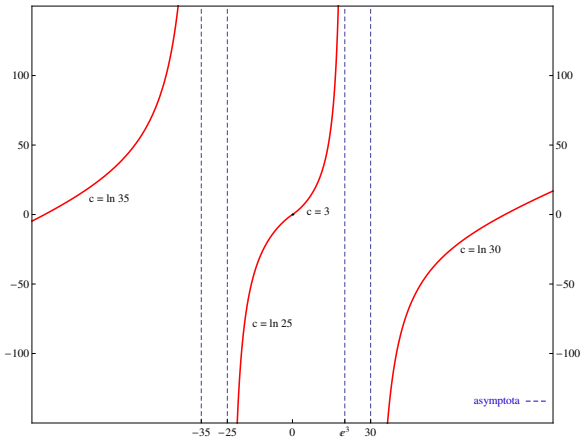
Notice that all these solutions pass through the origin with the same derivative. Therefore, they can be mutually connected (glued together). Such singular behavior is typical for homogeneous equations!

All maximal solutions of the equation are therefore:

$$y(x) = \begin{cases} x\frac{c+1-\ln|x|}{c-\ln|x|}, & x > 0 \\ 0, & x = 0 \\ x\frac{d+1-\ln|x|}{d-\ln|x|}, & x < 0, \end{cases}$$

$c, d \in (-\infty, +\infty]$ (where $c = \infty$ represents the limiting case $y(x) = x \cdot 1$) and

$$y(x) = x\frac{c + 1 - \ln|x|}{c - \ln|x|}, x \in (-\infty, -e^c), \text{ resp. } x \in (e^c, +\infty).$$



Example 4. Solve the equation $y' = \frac{y-x}{y+x}$.

Solution. The substitution $y = xz$ leads to

$$xz' + z = \frac{z-1}{z+1}$$
$$z' = -\frac{1}{x} \cdot \frac{z^2+1}{z+1}$$

We see that there are no stationary solutions, and we seek solutions on the intervals $I_1 = (-\infty, 0)$, $I_2 = (0, +\infty)$ and for z: $J_1 = (-\infty, -1)$, $J_2 = (-1, +\infty)$. Dividing by $g(z)$ and integrating yields

$$\ln\sqrt{z^2+1} + \operatorname{arctg} z = c - \ln|x|. \tag{7}$$

Let G denote the function on the left-hand side. We cannot explicitly invert this function (we cannot solve for z), but we know that G is strictly decreasing for $z < -1$ and strictly increasing for $z > -1$, mapping $(-\infty, -1)$ injectively onto $(\ln\sqrt{2} - \pi/4, +\infty)$ and $(-1, +\infty)$ injectively onto the same set. Thus,

$$\ln|x| \in (-\infty, c + \pi/4 - \ln\sqrt{2})$$

or

$$x \in (0, e^{c+\pi/4-\ln\sqrt{2}}), \quad \text{resp. } x \in (-e^{c+\pi/4-\ln\sqrt{2}}, 0).$$

The solution z is thus implicitly defined by (7) on one of the above intervals. Specifically, the solution on the first interval satisfies

$$\ln\sqrt{z^2+1}+\operatorname{arctg} z = c - \ln x\,, \quad \text{i.e.,} \quad \sqrt{z^2+1}\cdot e^{\operatorname{arctg} z} = \frac{e^c}{x}$$

and the solution on the second interval satisfies

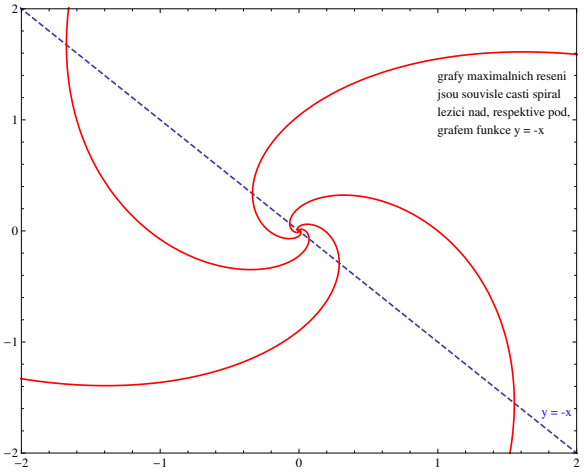
$$\ln\sqrt{z^2+1}+\operatorname{arctg} z = c - \ln(-x)\,, \quad \text{i.e.,} \quad \sqrt{z^2+1}\cdot e^{\operatorname{arctg} z} = -\frac{e^c}{x}$$

Substituting $z = y/x$ and multiplying both sides by $|x|$, we obtain the implicitly defined solutions y

$$\sqrt{x^2+y^2}\cdot e^{\operatorname{arctg}(y/x)} = e^c\,, \quad c\in\mathbb{R}$$

defined on the above intervals.

Are these solutions maximal? As $x\to 0+$ with $y>0$, we have $y\to e^{c_1-\pi/2}$, and as $x\to 0-$ with $y>0$, we have $y\to e^{c_2+\pi/2}$. Thus, we can glue them whenever $c_1 = c_2+\pi$, obtaining a solution on $(-e^{c_2+\pi+\pi/4-\ln\sqrt{2}}, e^{c_2+\pi/4-\ln\sqrt{2}})$. At the endpoints of this interval, we have $z\to -1$, i.e., $y\to -x$. After extending by the limit, the original equation is not defined. Thus, the found solutions are maximal. That they are all solutions follows again from the uniqueness theorem and the fact that every point in the plane is traversed by some solution from the family (as seen from the following remark).



Remark. The solution of the previous problem can be elegantly expressed in polar coordinates $x = r\cos\phi$, $y = r\sin\phi$ ($r = \sqrt{x^2+y^2}$, $\phi = \operatorname{arctg}(y/x) + k\pi$). The solutions are segments of the logarithmic spirals $r = de^{-\phi}$. More on substitutions in differential equations can be found in the chapter on nonlinear systems.

Example 5. Find all maximal solutions of the differential equation

$$y' = e^{y/x} + y/x.$$

Solution. Using the substitution $y = x \cdot z$, we transform the equation into

$$e^{-z} z' = 1/x.$$

For $x \in (-\infty, 0)$ (resp. $(0, +\infty)$) and $z \in \mathbb{R}$, integration yields

$$-e^{-z} = \ln|x| + c.$$

The function on the left maps the interval $\mathbb{R}$ onto $(-\infty, 0)$, i.e.,

$$\ln|x| + c < 0.$$

Hence,

$$|x| < e^{-c},$$

and therefore

$$x \in (0, e^{-c}), \quad \text{resp.} \quad x \in (-e^{-c}, 0).$$

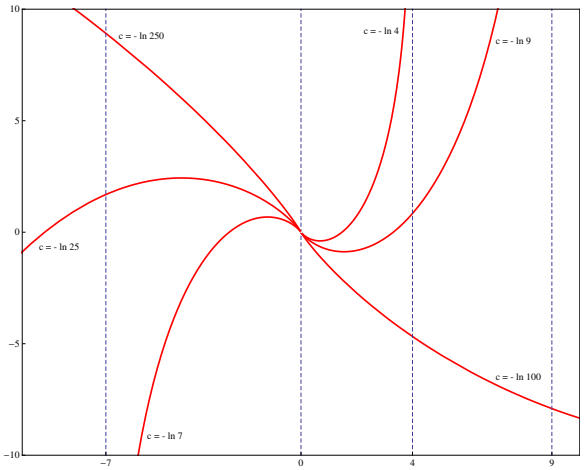
From the relation between x and z, we obtain

$$z(x) = -\ln(-\ln|x| - c)$$

and

$$y(x) = -x \ln(-\ln|x| - c)$$

on the above intervals. These solutions are maximal because for $x = 0$ the original equation is undefined, and at $\pm e^{-c}$ the solutions have infinite limits. These solutions constitute all solutions, as they fill the entire set $\{(x, y) \in \mathbb{R}^2,\ x \neq 0\}$.



Exercises

40. $(x^2+y^2)y' = 2xy$

41. $x^2y' + xy = 2y^2$

42. $x^2y' = y^2 + 2xy$

43. $xy' = y(1 + \ln \frac{y}{x})$

44. $xy + x^2 = x^2y'$

45. $2xyy' + x^2 - y^2 = 0$

46. $xyy' = y^2 - x^2$

47. $xy' = x - y$

48. $xy' = y - x$

49. $xy' = -(x + y)$

50. $x^2y' = y(x - y)$

51. $2xyy' = x^2 + y^2$

52. $xy' = y + \sqrt{x^2 + y^2}$

53. $2xyy' = 3y^2 - x^2$

54. $2\sqrt{xy} - y = -xy'$

55. * $y^2 + x^2y' = xyy'$

56. * $(x^2 - y^2)y' = 2xy$

57. * $(4y^2 + 3xy + x^2)y' = -(y^2 + 3xy + 4x^2)$

58. * $(y^2 - 3x^2)y' + 2xy = 0$

59. * $xy' = \frac{x^2+y^2}{x+y}$

60. * $x + 2y + yy' = 0$

61. * $(x^3 + y^3)y' = x^2y$

62. * $y' = \frac{y-2x}{x+2y}$

63. * $y' = \frac{x+2y}{x}$

64. * $y' = \frac{y}{x} \cos\left(\ln \frac{y}{x}\right)$

65. * $y' = \frac{y+\sqrt{xy}}{x}$

66. * $xy' = y + \sqrt{x^2 - y^2}$

Solutions

40) Solve for $x \in \mathbb{R}$, $y \in \mathbb{R}$. After substitution and simplification, we solve $xz' = \frac{z-z^3}{1+z^2}$. Stationary solutions $z = 0$ and $z = 1$. For $z \notin \{0; 1\}$ and $x \neq 0$, we have $z'\left(\frac{1}{z} - \frac{1}{z-1} - \frac{1}{z+1}\right) = \frac{1}{x}$. After integration: $\ln(|\frac{z}{z^2-1}|) = \ln(|x|) + c$,

$c \in \mathbb{R}$. After simplification, we obtain $z = \frac{1 \pm \sqrt{1+4k^2x^2}}{2kx}$, where $k = \pm e^c$. Since $y = xz$, we obtain the conclusion:

$$y(x) = 0, x \in \mathbb{R}$$
$$y(x) = x, x \in \mathbb{R}$$
$$y(x) = \tfrac{1}{2c}\left(1 \pm \sqrt{1+4c^2x^2}\right), x \in \mathbb{R}, c \in \mathbb{R} \setminus \{0\}$$

41) Solve for $x \in \mathbb{R}$, $y \in \mathbb{R}$. After substitution and simplification, we solve $xz' = 2(z^2 - z)$. Stationary solutions $z = 0$ and $z = 1$. For $z \notin \{0; 1\}$ and $x \neq 0$, we have $z'\left(\frac{1}{z} - \frac{1}{z-1}\right) = \frac{1}{x}$. After integration and simplification: $\ln(|\frac{z}{z-1}|) = 2\ln(|x|) + c$, $c \in \mathbb{R}$. From this, $z = \frac{kx^2}{kx^2-1}$, where $k = \pm e^c$. Since $y = xz$, we obtain the conclusion:

$$y(x) = 0, x \in \mathbb{R}$$
$$y(x) = x, x \in \mathbb{R}$$
$$y(x) = \frac{cx^3}{cx^2-1}, \begin{cases} x \in \mathbb{R}, & c \in (-\infty; 0) \\ x \in (-\infty; -\frac{1}{\sqrt{c}}), & c \in (0; +\infty) \\ x \in (-\frac{1}{\sqrt{c}}; \frac{1}{\sqrt{c}}), & c \in (0; +\infty) \\ x \in (\frac{1}{\sqrt{c}}; +\infty), & c \in (0; +\infty) \end{cases}$$

42) Solve for $x \in \mathbb{R}$, $y \in \mathbb{R}$. After substitution and simplification, we solve $xz' = z(z+1)$. Stationary solutions $z = 0$ and $z = -1$. For $z \notin \{-1; 0\}$ and $x \neq 0$, we have $z'\left(\frac{1}{z} - \frac{1}{z+1}\right) = \frac{1}{x}$. After integration and simplification, we get $z = (z+1)cx$, $c \in \mathbb{R} \setminus \{0\}$. Multiplying by x and substituting $y = xz$ gives $y = cxy + cx^2$. Thus $y = \frac{x^2}{d-x}$, where $d = \frac{1}{c}$ for $c \neq 0$, resp. $y \equiv 0$ for $c = 0$. Conclusion after gluing at $x = 0$:
first solution

$$y(x) = -x, y(x) = 0, x \in \mathbb{R}$$

second solution

$$y(x) = \frac{x^2}{c-x}, x \in (-\infty; c) \text{ for } c \in (-\infty; 0) \text{ and } x \in (c; +\infty) \text{ for } c \in (0; +\infty)$$

third solution

$$y(x) = \begin{cases} \dfrac{x^2}{c-x}, x \in (-\infty; 0] \text{ for } c \in (0; +\infty) \text{ or } x \in (c; 0] \text{ for } c \in (-\infty; 0) \\ \dfrac{x^2}{d-x}, x \in [0; +\infty) \text{ for } d \in (-\infty; 0) \text{ or } x \in [0; d) \text{ for } d \in (0; +\infty) \end{cases}$$

43) Solve for $x \in \mathbb{R} \setminus \{0\}$, $y \in \mathbb{R} \setminus \{0\}$, $x \cdot y > 0$. After substitution and simplification, we solve $xz' = z\ln(z)$ for $z \in (0; +\infty)$. Stationary solutions $z = 0$ and $z = 1$. After integration: $\ln\big(|\ln(z)|\big) = \ln(|x|) + c$, $c \in \mathbb{R}$. Thus $z = \exp(dx)$, where $d = \pm e^c$. Solutions cannot be glued. In total, we obtain:

$$y(x) = x\exp(cx), x \in (-\infty; 0) \text{ or } x \in (0; +\infty), c \in \mathbb{R}$$

44) Solve for $x \in \mathbb{R}$ and $y \in \mathbb{R}$. After substitution and simplification, we solve $xz' = 1$. After integration: $z = c + \ln|x|$, $c \in \mathbb{R}$. Hence $y = x(c + \ln|x|)$. Since $\lim\limits_{x\to 0} y(x) = 0$, but $y'(0) = \lim\limits_{x\to 0} \frac{y(x)-0}{x-0} = \lim\limits_{x\to 0}(c + \ln|x|) = -\infty$, the solutions cannot be connected at the origin. Conclusion: $y(x) = x(\ln|x| + c)$, $x \in (-\infty; 0)$ and $x \in (0; +\infty)$, $c \in \mathbb{R}$. (Can also be solved as a linear equation.)

45) Solve for $x \in \mathbb{R}$, $y \in \mathbb{R}$. After substitution and simplification, we solve $2xzz' = -(z^2 + 1)$. After integration: $\ln(1 + z^2) = c - \ln(|x|)$, $c \in \mathbb{R}$. Thus $x^2 + y^2 = 2kx$, where $k = \pm e^c$. Conclusion:

$$y(x) = \pm\sqrt{c^2 - (x - c)^2}, x \in (2c; 0) \text{ for } c \in (-\infty; 0) \text{ and } x \in (0; 2c) \text{ for } c \in (0; +\infty)$$

(Can also be solved as a Bernoulli equation.)

46) Solve for $x \in \mathbb{R}$ and $y \in \mathbb{R}$. After substitution and simplification for $x \neq 0$, we get $2zz' = -\frac{2}{x}$, which upon integration yields $z^2 = c - \ln(x^2)$, $c \in \mathbb{R}$. Cannot be glued. Conclusion:

$$y(x) = \pm x\sqrt{c - \ln(x^2)}, x \in (-\exp(\tfrac{c}{2}); 0) \text{ or } x \in (0; \exp(\frac{c}{2})), c \in \mathbb{R}$$

(Can also be solved as a Bernoulli equation.)

47) Solve for $x \in \mathbb{R}$ and $y \in \mathbb{R}$. After substitution and simplification: $x^2z' = x(1 - 2z)$. Stationary solution $z = \frac{1}{2}$. For $z \neq \frac{1}{2}$ and $x \neq 0$, solve $\frac{z'}{1-2z} = \frac{1}{x}$. After integration: $-\frac{1}{2}\ln(|1 - 2z|) = \ln(|x|) + c$, $c \in \mathbb{R}$. After rearranging: $y = \frac{x}{2} + \frac{k}{x}$, $k \in \mathbb{R}$. Conclusion:

$$y(x) = \frac{x}{2} + \frac{c}{x}, x \in (-\infty; 0) \text{ or } x \in (0; +\infty) \text{ for } c \in \mathbb{R} \setminus \{0\}$$

$$y(x) = \frac{x}{2}, x \in \mathbb{R}$$

(Can also be solved as a linear equation.)

48) Solve for $x \in \mathbb{R}$ and $y \in \mathbb{R}$. After substitution and simplification: $x^2 z' = -x$. For $x \neq 0$, solve $z' = -\frac{1}{x}$, which integrates to $z = -\ln(|x|) + c$, $c \in \mathbb{R}$. Conclusion:

$$y(x) = -x \ln|x| + cx, x \in (-\infty; 0) \text{ or } x \in (0; +\infty), c \in \mathbb{R}$$

(Can also be solved as a linear equation.)

49) Solve for $x \in \mathbb{R}$ and $y \in \mathbb{R}$. After substitution and simplification: $x^2 z' = -x(2z+1)$. For $x \neq 0$, solve $\frac{-z'}{2z+1} = \frac{1}{x}$. After integration: $-\frac{1}{2}\ln(|2z+1|) = \ln(|x|) + c$, $c \in \mathbb{R}$. After rearranging: $y = -\frac{x}{2} + \frac{k}{x}$, $k \in \mathbb{R}$. Conclusion:

$$y(x) = \frac{c}{x} - \frac{x}{2}, x \in (-\infty; 0) \text{ or } x \in (0; +\infty), c \in \mathbb{R} \setminus \{0\}$$

$$y(x) = -\frac{x}{2}, x \in \mathbb{R}$$

50) Solve for $x \in \mathbb{R}$ and $y \in \mathbb{R}$. After substitution and simplification for $x \neq 0$, we solve $xz' = -z^2$. Stationary solution $z = 0$. For $z \neq 0$, we have $\frac{-z'}{z^2} = \frac{1}{x}$. After integration: $\frac{1}{z} = \ln(|x|) + c$, $c \in \mathbb{R}$. Thus $z = \frac{1}{\ln(|kx|)}$, where $c = \ln(k)$, $k > 0$, and further $y = xz = \frac{x}{\ln(|kx|)}$. Solutions can be connected at $x = 0$. Conclusion:

$$y(x) = 0, x \in \mathbb{R}$$

$$y(x) = \frac{x}{\ln(|cx|)}, x \in (-\infty; -\tfrac{1}{c}) \text{ or } x \in (\tfrac{1}{c}; +\infty), c \in (0; +\infty)$$

$$y(x) = \begin{cases} \dfrac{x}{\ln|cx|}, & x \in (-\frac{1}{c}; 0], c \in (0; +\infty) \\ \dfrac{x}{\ln(|dx|)}, & x \in [0; \frac{1}{d}), d \in (0; +\infty) \end{cases}$$

$$y(x) = \begin{cases} 0, & x \in (-\infty; 0] \\ \dfrac{x}{\ln(|cx|)}, & x \in [0; \frac{1}{c}), c \in (0; +\infty) \end{cases}$$

$$y(x) = \begin{cases} \dfrac{x}{\ln|cx|}, & x \in (-\frac{1}{c}; 0], c \in (0; +\infty) \\ 0 \end{cases}$$

51) Solve for $x \in \mathbb{R}$ and $y \in \mathbb{R}$. After substitution and simplification: $2x^3zz' = x^2(1-z^2)$. Stationary solutions $z = \pm 1$. For $z \notin \{-1;1\}$ and $x \neq 0$, we have $-z'\left(\frac{1}{z-1} + \frac{1}{z+1}\right) = \frac{1}{x} + c$. After integration and simplification: $\ln(|1-z^2|) = -\ln(|x|) - c$, $c \in \mathbb{R}$. After rearranging and using $y = xz$, we obtain the conclusion:

$$y(x) = \pm x, x \in \mathbb{R}$$

$$y(x) = \pm x\sqrt{1 + \frac{c}{x}}, x \in (-\infty; \min(0; -c)) \text{ or } x \in (\max(0; -c); +\infty), c \in \mathbb{R}\backslash\{0\}$$

52) Solve for $x \in \mathbb{R}$ and $y \in \mathbb{R}$. After substitution and simplification: $x^2z' = |x|\sqrt{1+z^2}$. For $x \neq 0$, we have $\frac{z'}{\sqrt{1+z^2}} = \frac{\operatorname{sgn}(x)}{x}$. After integration: $\operatorname{sgn}(z)\ln\left(|z| + \sqrt{z^2+1}\right) = \operatorname{sgn}(x)\ln|x| + c$, $c \in \mathbb{R}$. Setting $k = e^c > 0$, for $x > 0$ we get $z = \frac{k^2x^2-1}{2kx}$ and for $x < 0$ we get $z = \frac{x^2-k^2}{2kx} = \frac{p^2x^2-1}{2px}$, where $p = \frac{1}{k}$. Solutions can be connected at $x = 0$. Conclusion:

$$y(x) = \frac{c^2x^2 - 1}{2c} = \frac{c}{2}x^2 - \frac{1}{2c}, x \in \mathbb{R}, c \in (0; +\infty)$$

53) Solve for $x \in \mathbb{R}$ and $y \in \mathbb{R}$. After substitution and simplification for $x \neq 0$, we get $2xzz' = z^2 - 1$. Stationary solutions $z = \pm 1$. For $z \notin \{-1;1\}$ and $x \neq 0$, the equation integrates to $\ln(|z^2-1|) = \ln(|x|) + c$, $c \in \mathbb{R}$. Solutions can be connected at $x = 0$. Conclusion:

$$y(x) = \pm x\sqrt{cx+1}, \begin{cases} x \in (-\infty; -\frac{1}{c}) & \text{for } c \in (-\infty; 0) \\ x \in \mathbb{R} & \text{for } c \in \{0\} \\ x \in (-\frac{1}{c}; +\infty) & \text{for } c \in (0; +\infty) \end{cases}$$

54) Solve for $x, y \in \mathbb{R}$ such that $x \cdot y \geq 0$. Stationary solution $y = 0$. After substitution: $2|x|\sqrt{z} = -x^2z'$. For $z \neq 0$ and $x \neq 0$, solve $\frac{z'}{2\sqrt{z}} = -\frac{\operatorname{sgn}(x)}{x}$. After integration: $\sqrt{z} = -\operatorname{sgn}(x)\ln(|x|) + c$, $c \in \mathbb{R}$. Thus $z = (c - \operatorname{sgn}(x)\ln(|x|))^2$, since $x \cdot y \geq 0 \Rightarrow z > 0$. Let $c\operatorname{sgn}(x) = \ln(k)$ for $k > 0$. Then $z = (\ln(\frac{k}{|x|}))^2$. Solutions can be glued to the stationary solution at $|x| = k$. Conclusion:

$$y(x) = \begin{cases} x\ln^2(cx), & x \in (-\infty; \frac{1}{c}] \\ 0, & x \in [\frac{1}{c}; +\infty) \end{cases}$$

for $c \in (-\infty; 0)$

$$y(x) = \begin{cases} x\ln^2(cx), & x \in (0; \frac{1}{c}] \\ 0, & x \in [\frac{1}{c}; +\infty) \end{cases}$$

for $c \in (0; +\infty)$

55) Solve for $x \in \mathbb{R}$ and $y \in \mathbb{R}$. Substituting $y = xz$, $y' = z + xz'$ leads after simplification to $x^3(z-1)z' = x^2 z$. Stationary solution $z = 0$ on $\mathbb{R}$. Solve for $x \neq 0$ and $z \neq 0$. After integration: $z - \ln|z| = c + \ln|x|$, $c \in \mathbb{R}$. For $z \in (-\infty; 0)$, we have $z - \ln|z| \in (-\infty; +\infty)$, so $x \in (-\infty; 0)$ or $x \in (0; +\infty)$; for $z \in (0; +\infty)$, we have $z - \ln|z| \in [1; +\infty)$, so $x \in (-\infty; -e^{1-c})$ or $x \in (e^{1-c}; +\infty)$. Back-substituting $z = \frac{y}{x}$ yields the implicit equation $y = x(c + \ln|y|)$, $c \in \mathbb{R}$, which defines solutions for $x \in (-\infty; 0)$, $x \in (0; +\infty)$, $x \in (-\infty; -e^{1-c})$, and $x \in (e^{1-c}; +\infty)$. Additionally, we have the stationary solution $y(x) = 0$ for $x \in (-\infty; 0)$ and $x \in (0; +\infty)$. The graph of the solution on $(-\infty; 0)$ lies in the second quadrant since $z \in (-\infty; 0)$, so $0 < xz < +\infty$. The solution on $(0; +\infty)$ lies in the fourth quadrant ($-\infty < xz < 0$). The solution on $(-\infty; -e^{1-c})$ lies in the third quadrant ($-\infty < xz < 0$). The solution on $(e^{1-c}; +\infty)$ lies in the first quadrant ($0 < xz < +\infty$). From $z - \ln|z| = c + \ln|x|$, as $x \to 0$ we have $z - \ln|z| \to -\infty$, whence $z \to -\infty$. For solutions on $(-\infty; 0)$ or $(0; +\infty)$, from $z = c + \ln|y|$ as $x \to 0$ we obtain $\lim\limits_{x\to 0} y(x) = 0$, and $y'(0) = \lim\limits_{x\to 0} \frac{y(x)-y(0)}{x-0} = \lim\limits_{x\to 0} \frac{y(x)}{x} = \lim\limits_{x\to 0} z(x) = -\infty$, so the solution in the second quadrant cannot be connected with that in the fourth quadrant (the derivative at $x = 0$ would not be finite). For solutions on $(-\infty; -e^{1-c})$ (resp. $(e^{1-c}; +\infty)$), substituting $x = \pm e^{1-z}$ gives $z = 1$ for $z > 0$. Thus $\lim\limits_{x\to \pm e^{1-c}} y(x) = \pm e^{1-c}$, and $|y'(\pm e^{1-c})| = +\infty$, so these cannot be extended past $\pm e^{1-c}$. We obtain the maximal solutions defined implicitly on $(-\infty; 0)$, $(-\infty; -e^{1-c})$, $(0; +\infty)$, and $(e^{1-c}; +\infty)$, along with the stationary solution $y(x) = 0$, $x \in \mathbb{R}$.

56) Solve for $x \in \mathbb{R}$ and $y \in \mathbb{R}$. Substituting $y = xz$, $y' = z + xz'$ leads after simplification to $x^3(1 - z^2)z' = z^3 + z$. Solve for $x \neq 0$. Stationary solution $z = 0$ on $\mathbb{R}$. For $z \neq 0$, we have $z'\left(\frac{1}{z} - \frac{2z}{1+z^2}\right) = \frac{1}{x}$. After integration: $\ln|z| - \ln(z^2+1) = \ln|x| + c$, $c \in \mathbb{R}$. Hence $z = \frac{1 \pm \sqrt{1-4k^2x^2}}{2kx}$, where $k = e^c > 0$, and so $y = xz = \frac{1 \pm \sqrt{1-4k^2x^2}}{2k}$. Conclusion:

$$y(x) = \frac{1 \pm \sqrt{1 - 4k^2x^2}}{2k},\ x \in \left(-\frac{1}{2k}; \frac{1}{2k}\right)$$

57) Solve for $x \in \mathbb{R}$ and $y \in \mathbb{R}$. Substituting $y = xz$, $y' = z + xz'$ leads to $x^3(4z^2 + 3z + 1)z' = -4x^2(z^3 + z^2 + z + 1)$. Solve for $x \neq 0$. Stationary solution $z = -1$ on $\mathbb{R}$. For $z \neq -1$, we obtain $z'\left(\frac{3z}{z^2+1} + \frac{1}{z+1}\right) = -\frac{4}{x}$. After integration: $\ln\sqrt{(z^2+1)^3(z+1)^2} = c - \ln(x^4)$, $c \in \mathbb{R}$. For $z \in (-\infty; 0)$ as well as $z \in (0; +\infty)$, the left-hand side covers $(-\infty; +\infty)$, so $x \in (-\infty; 0)$ or $x \in (0; +\infty)$. Setting $e^c = k > 0$, we obtain $(z^2+1)^3(z+1)^2 = \frac{k^2}{x^8}$, which transforms to $(x^2z^2 + x^2)^3(xz + x)^2 = k^2$. Re-substituting $y = xz$ and writing $c = k^2$ yields the implicit relation $(y^2 + x^2)^3(y + x)^2 = c$, $c \in [0; +\infty)$, for

$x \in (-\infty;0)$ and $x \in (0;+\infty)$. As $x \to 0$, we find $\lim\limits_{x\to 0} y(x) = \pm\sqrt[8]{c}$ and $y'(0) = -\frac{1}{4}$. Thus, the relation $(y^2+x^2)^3(y+x)^2 = c$ defines a maximal solution on all of $\mathbb{R}$ for every $c \in [0;+\infty)$.

58) Solve for $x \in \mathbb{R}$ and $y \in \mathbb{R}$. Substituting $y = xz$, $y' = z + xz'$ leads to $x^3z'(z^2-3) = x^2z(1-z^2)$. Solve for $x \neq 0$. Stationary solutions $z = -1$, $z = 0$, $z = 1$ on $\mathbb{R}$. For $z \notin \{-1;0;1\}$, we get $z'\left(\frac{-3}{z} + \frac{-1}{1-z} + \frac{1}{1+z}\right) = \frac{1}{x}$. After integration: $\ln\left|\frac{z^2-1}{z^3}\right| = \ln|x| + c$, $c \in \mathbb{R}$, or $\left|\frac{z^2-1}{z^3}\right| = k|x|$ with $k = e^c > 0$. For $z \in (-\infty;-1)$ or $z \in (1;+\infty)$, we have $\left|\frac{z^2-1}{z^3}\right| \in (0;\frac{2\sqrt{3}}{9}]$, so $x \in (-\frac{2\sqrt{3}}{9k};0)$ or $x \in (0;\frac{2\sqrt{3}}{9k})$; for $z \in (-1;0)$ or $z \in (0;1)$, the expression covers $(0;+\infty)$, so $x \in (-\infty;0)$ or $x \in (0;+\infty)$. Substituting $y = xz$ gives $\left|\frac{y^2-x^2}{y^3}\right| = c$, $c \in [0;+\infty)$. Stationary solution $y(x) = 0$ on $(-\infty;0)$ and $(0;+\infty)$. Three types of solutions arise:
1) Solutions tending to 0 as $x \to 0$ can be glued at $x = 0$ by setting $y(0) = 0$ (with $y'(0) = \pm 1$), yielding maximal solutions on $(-\infty;\frac{2\sqrt{3}}{9c})$, $(-\frac{2\sqrt{3}}{9c};+\infty)$, $(-\infty;+\infty)$, and $(-\frac{2\sqrt{3}}{9c};\frac{2\sqrt{3}}{9c})$.
2) Solutions on $(-\frac{2\sqrt{3}}{9c};0)$ and $(0;\frac{2\sqrt{3}}{9c})$ tending to $\pm\frac{1}{c}$ as $x \to 0$ can be glued by setting $y(0) = \pm\frac{1}{c}$ (with $y'(0) = 0$), yielding solutions on $(-\frac{2\sqrt{3}}{9c};\frac{2\sqrt{3}}{9c})$.
3) Stationary solution $y(x) = 0$ on $\mathbb{R}$.
Note: The points $x = \pm\frac{2\sqrt{3}}{9c}$ correspond to $y = \pm\frac{2}{3c}$ on the curve $y^2 - 3x^2 = 0$, where $|y'(x)| = +\infty$.

59) Solve on $\{(x,y) \in \mathbb{R}^2; x+y \neq 0\}$. Substituting $y = xz$, $z \neq -1$, $y' = z + xz'$ leads to $x^2z'(z+1) = x(1-z)$. Stationary solution $z = 1$ on $\mathbb{R}$. For $z \neq 1$, we have $z'\left(1 + \frac{2}{z-1}\right) = -\frac{1}{x}$. After integration: $z + 2\ln|z-1| = c - \ln|x|$, $c \in \mathbb{R}$. For $z \in (-\infty;1) \setminus \{-1\}$, we have $x \in (-\infty;-\frac{1}{4}e^{c+1})$ or $x \in (\frac{1}{4}e^{c+1};+\infty)$; for $z \in (1;+\infty)$, we have $x \in (-\infty;0)$ or $x \in (0;+\infty)$. Back-substitution yields the implicit equation $y + 2x\ln|y-x| = x\ln|x| + cx$, along with $y(x) = x$ for $x \in (-\infty;0)$ and $x \in (0;+\infty)$. No gluing at $x = 0$ is possible because the derivative would become infinite or $z \to -1$ (which is excluded by $x + y \neq 0$). Maximal solutions are defined on $(-\infty;0)$, $(-\infty;-\frac{1}{4}e^{c+1})$, $(\frac{1}{4}e^{c+1};+\infty)$, and $(0;+\infty)$, plus $y(x) = x$ on $(-\infty;0)$ and $(0;+\infty)$.

60) Solve for $x \in \mathbb{R}$ and $y \in \mathbb{R}$. Substituting $y = xz$, $y' = z + xz'$ yields $x^2zz' = -x(z+1)^2$. Stationary solution $z = -1$ on $\mathbb{R}$. For $z \neq -1$, we get $z'\left(\frac{1}{z+1} - \frac{1}{(z+1)^2}\right) = -\frac{1}{x}$. After integration: $\frac{1}{z+1} + \ln|z+1| = c - \ln|x|$, $c \in \mathbb{R}$. For $z \in (-\infty;-1)$, $x \in (-\infty;0)$ or $x \in (0;+\infty)$; for $z \in (-1;+\infty)$, $x \in (-e^{c-1};0)$ or $x \in (0;e^{c-1})$. The implicit solution is $\frac{x}{x+y} + \ln|x+y| = c$, along with $y(x) = -x$. Two types of gluings at $x = 0$ occur:

1) Connecting solutions with $y(0) = 0$ (derivative $y'(0) = -1$), yielding solutions on $(-\infty; e^{c-1})$, $(-e^{c-1}; +\infty)$, and $(-e^{c-1}, e^{c-1})$.
2) Connecting solutions with $y(0) = \pm e^c$ (derivative $y'(0) = \mp 2$), yielding maximal solutions on $(-\infty; e^{c-1})$ and $(-e^{c-1}; +\infty)$. At $\pm e^{c-1}$, $|y'| \to +\infty$.

61) Solve for $x \in \mathbb{R}$ and $y \in \mathbb{R}$. Substituting $y = xz$, $y' = z + xz'$ yields $x^4(1 + z^3)z' = -x^3z^4$. Stationary solution $z = 0$ on $\mathbb{R}$. For $z \neq 0$, we have $z'\left(\frac{1}{z} + \frac{1}{z^4}\right) = -\frac{1}{x}$. After integration: $\ln|z| - \frac{1}{3z^3} = c - \ln|x|$, $c \in \mathbb{R}$. This yields the implicit equation $\ln|y| - \frac{x^3}{3y^3} = c$, $c \in \mathbb{R}$, along with $y(x) = 0$. Two types of gluings at $x = 0$ occur:
1) Connecting solutions with $y(0) = 0$ ($y'(0) = 0$), yielding solutions on $(-\infty; e^{c-\frac{1}{3}})$, $(-e^{c-\frac{1}{3}}; +\infty)$, $(-\infty; +\infty)$, and $(-e^{c-\frac{1}{3}}; e^{c-\frac{1}{3}})$.
2) Connecting solutions with $y(0) = \pm e^c$ ($y'(0) = 0$), yielding maximal solutions on $(-e^{c-\frac{1}{3}}; +\infty)$ and $(-\infty; e^{c-\frac{1}{3}})$.
Note: The points $x = \pm e^{c-\frac{1}{3}}$ lie on $x^3 + y^3 = 0$, where $|y'(x)| = +\infty$.

62) Solve on $\{(x, y) \in \mathbb{R}^2; x + 2y \neq 0\}$. Substituting $y = xz$, $z \neq -\frac{1}{2}$, $y' = z + xz'$ gives $x(\frac{2z}{z^2+1} + \frac{1}{z^2+1})z' = -2$. After integration: $\ln(z^2 + 1) - \operatorname{arctg} z = c - 2\ln|x|$, $c \in \mathbb{R}$. The implicit relation is $\ln(x^2 + y^2) + \operatorname{arctg}\frac{y}{x} = c$. As $x \to 0$, $y \to \pm e^{\frac{c}{2} - \frac{\pi}{4}}$ and $y' \to \frac{1}{2}$, allowing connection across $x = 0$. The endpoints correspond to the line $x + 2y = 0$.

63) Solve for $x \neq 0$. Substituting $y = xz$, $y' = z + xz'$ yields $xz' = 1 + z$. Stationary solution $z = -1$. For $z \neq -1$, $\frac{z'}{z+1} = \frac{1}{x}$, which integrates to $\ln|z + 1| = \ln|x| + c$, giving $z = cx - 1$, $c \in \mathbb{R}$. Since $y = xz$, we obtain:

$$y(x) = cx^2 - x,\ x \in (-\infty; 0) \text{ or } x \in (0; +\infty),\ c \in \mathbb{R}$$

64) Solve on $\{(x, y) \in \mathbb{R}^2; x \neq 0\ \&\ \frac{y}{x} > 0\}$. Substituting $y = xz$ ($z > 0$), $y' = z + xz'$ yields $xz' = z(\cos(\ln z) - 1)$. Stationary solutions $z = e^{2k\pi}$, $k \in \mathbb{Z}$. Integrating gives $\operatorname{cotg}\frac{\ln z}{2} = \ln|x| + c$, $c \in \mathbb{R}$. Thus $y = x\exp\big(2k\pi + 2\operatorname{arccotg}(c + \ln|x|)\big)$, $x \in (-\infty; 0)$ or $x \in (0; +\infty)$, $c \in \mathbb{R}$, $k \in \mathbb{Z}$.

65) Solve on $\{(x, y) \in \mathbb{R}^2; x \neq 0\ \&\ x \cdot y > 0\}$. Substituting $y = xz$ ($z > 0$), $y' = z + xz'$ yields $|x|z' = \sqrt{z}$, i.e., $\frac{z'}{\sqrt{z}} = \frac{\operatorname{sgn} x}{x}$. Integrating yields $2\sqrt{z} = \operatorname{sgn}(x)\ln|x| + c$, $c \in \mathbb{R}$. Thus:

$$y(x) = x\left(\ln\sqrt{|x|} + \frac{c}{2}\right)^2,\ x \in (-e^{-c}; 0) \text{ or } x \in (e^{-c}; +\infty),\ c \in \mathbb{R}$$

66) Solve on $\{(x, y) \in \mathbb{R}^2; |x| \geq |y|\}$. Substituting $y = xz$ ($|z| \leq 1$), $y' = z + xz'$ yields $|x|z' = \sqrt{1 - z^2}$. Stationary solutions $z = \pm 1$. For $z \in (-1; 1)$, integration yields $\arcsin z = \operatorname{sgn}(x)\ln|x| + c$, $c \in \mathbb{R}$. Solutions

can be glued at $x = \pm\exp(-c - \frac{\pi}{2})$. Conclusion:

$$y(x) = \begin{cases} -x, & x \in (-\infty; -e^{c+\frac{\pi}{2}}) \\ x\sin(c - \ln(-x)), & x \in [-e^{c+\frac{\pi}{2}}, -e^{c-\frac{\pi}{2}}) \\ x, & x \in [-e^{c-\frac{\pi}{2}}; +\infty) \end{cases}$$

$$c \in \mathbb{R} \cup \{-\infty\}$$

$$y(x) = \begin{cases} -x, & x \in (-\infty; e^{-c-\frac{\pi}{2}}) \\ x\sin(c + \ln x), & x \in [e^{-c-\frac{\pi}{2}}; e^{-c+\frac{\pi}{2}}) \\ x, & x \in [e^{-c+\frac{\pi}{2}}; +\infty) \end{cases}$$

$$c \in \mathbb{R} \cup \{+\infty\}$$

Equations of the form $y' = \frac{ay+bx+c}{dy+ex+f}$

Equations of the form

$$y' = \frac{ay + bx + c}{dy + ex + f}$$

can be reduced to homogeneous equations via the translation substitution $X := x + A$, $Y(X) := y(x) + B$ for appropriate constants A and B, as shown in the following example.

Example 6. Find all maximal solutions of the differential equation

$$y' = \frac{2x - y + 1}{x - 2y + 1}.$$

Solution. Using the substitution $Y(X) = y(x) + B$, $X = x + A$, we obtain

$$Y'(X) = y'(x) = \frac{2x - y(x) + 1}{x - 2y(x) + 1} = \frac{2X - 2A - Y(X) + B + 1}{X - A - 2Y(X) + 2B + 1} = \frac{2X - Y(X)}{X - 2Y(X)},$$

where the last equality holds if $-2A + B + 1 = 0$ and $-A + 2B + 1 = 0$, i.e., $A = 1/3$, $B = -1/3$.

The equation

$$Y' = \frac{2X - Y}{X - 2Y}$$

is now homogeneous; we solve it by substitution $Y = X \cdot Z$, reducing it to

$$Z' = 2\frac{Z^2 - Z + 1}{1 - 2Z} \cdot \frac{1}{X}.$$

For $X \in (-\infty, 0)$ (resp. $(0, +\infty)$) and $Z \in (-\infty, 1/2)$ (resp. $(1/2, +\infty)$), integration yields

$$\ln(Z^2 - Z + 1) = -2 \ln |X| + C.$$

The function on the left maps each interval onto $(\ln(3/4), +\infty)$, so $-2 \ln |X| + C > \ln(3/4)$. Thus,

$$X \in \left(0, \frac{2e^{C/2}}{\sqrt{3}}\right), \quad \text{resp.} \quad X \in \left(-\frac{2e^{C/2}}{\sqrt{3}}, 0\right).$$

From the relation between X and Z,

$$Z = \frac{1}{2}(1 \pm \sqrt{4e^C X^{-2} - 3}).$$

Back-substituting gives

$$Y(X) = \frac{1}{2}(X \pm \sqrt{4e^C - 3X^2})$$

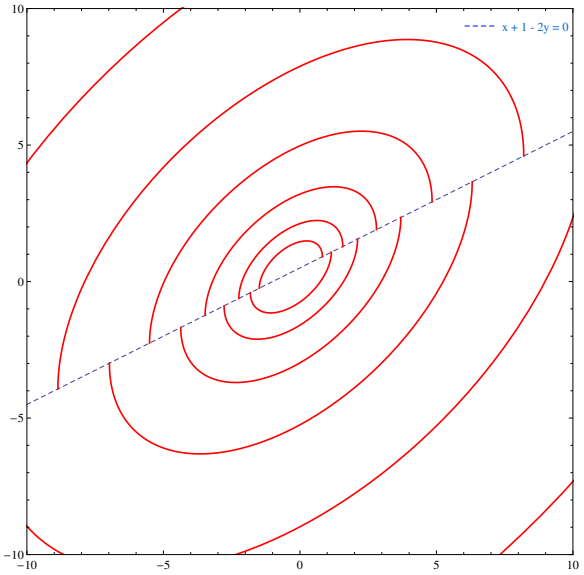
which can be extended to $X = 0$. Hence,

$$y(x) = \frac{1}{2}\left(x + 1 \pm \sqrt{4e^C - 3(x + 1/3)^2}\right)$$

on the interval

$$x \in \left(-\frac{2e^{C/2}}{\sqrt{3}} - \frac{1}{3}, \frac{2e^{C/2}}{\sqrt{3}} - \frac{1}{3}\right).$$

This interval is maximal because at its endpoints $y = 1/2(x + 1)$, where the right-hand side is undefined. These constitute all solutions, as every point outside $x + 1 - 2y = 0$ is covered.



Exercises

67. $y' = \frac{2y-x-5}{2x-y+4}$.

68. $y' = \frac{2(y+2)^2}{(x+y-1)^2}$

69. $y' = \frac{2x+y+2}{4x+2y+9}$

70. $y' = \frac{4-x-2y}{2(1+x+2y)}$

71. $y' = \frac{2y-x+1}{4y-2x+6}$

72. $y' = \frac{x+y+1}{x-y+3}$

Solutions

67) Solve on $\{(x,y) \in \mathbb{R}^2; 2x - y + 4 \neq 0\}$. Substituting $x = X - 1$ and $y = Y + 2$ yields the homogeneous equation $Y' = \frac{2Y-X}{2X-Y}$ for $Y \neq 2X$. Using $Y = X \cdot z$ $(z \neq 2)$, the equation becomes $(2-z)Xz' = z^2 - 1$ with stationary solutions $z = \pm 1$. Integrating yields $\ln\left(\left|\frac{z-1}{(z+1)^3}\right|\right) = \ln(X^2) + 2c$, $c \in \mathbb{R}$. The resulting implicit equation is $|x-y+3| = k|x+y-1|^3$, $k \in [0; +\infty]$. Solutions can be glued across $x = -1$, yielding maximal solutions on $(-\infty; -1 + \frac{1}{27k})$, $(-1 - \frac{1}{27k}; +\infty)$, $(-\infty; -1)$, $(-1; +\infty)$, $(-1 - \frac{1}{27k}; -1)$, and $(-1; -1 + \frac{1}{27k})$.

68) Solve on $\{(x,y) \in \mathbb{R}^2; x + y - 1 \neq 0\}$. Substituting $x = X + 3$ and $y = Y - 2$ gives $Y' = \frac{2Y^2}{(X+Y)^2}$. With $Y = Xz$, we get $(z+1)^2 Xz' = -z(1+z^2)$. Integrating yields the implicit relation $\ln|y+2| + 2\operatorname{arctg}(\frac{y+2}{x-3}) = c$, $c \in \mathbb{R}$, along with the stationary solution $y(x) = -2$. Solutions cannot be glued as the boundary points lie on the excluded line $x + y - 1 = 0$.

69) Solve on $\{(x,y) \in \mathbb{R}^2; 4x + 2y + 9 \neq 0\}$. Substituting $z = 2x + y$ and $z' = 2 + y'$ gives $(2z + 9)z' = 5z + 20$ with stationary solution $z = -4$. Integration yields $2z + \ln(|z + 4|) = 5x + c$, leading to the implicit relation $2y + \ln(|y + 2x + 4|) = x + c$, $c \in \mathbb{R}$, on $(-\infty; -\frac{9}{5} - \frac{1}{5}\ln 2 - \frac{c}{5})$ and $(-\infty; +\infty)$, alongside $y(x) = -2x - 4$ on $\mathbb{R}$. These are maximal solutions.

70) Solve on $\{(x,y) \in \mathbb{R}^2; x + 2y + 1 \neq 0\}$. Substituting $z = x + 2y$ and $z' = 1 + 2y'$ gives $z' - 1 = \frac{4-z}{1+z}$. Integrating gives $z + \frac{1}{2}z^2 = 5x + c$, whence $z = -1 \pm \sqrt{1 + 2c + 10x}$. Conclusion:

$$y(x) = -\frac{1}{2}x - \frac{1}{2} \pm \frac{1}{2}\sqrt{1 + 2c + 10x}, x \in (-\frac{2c+1}{10}; +\infty), c \in \mathbb{R}$$

71) Solve on $\{(x,y) \in \mathbb{R}^2; 4y - 2x + 6 \neq 0\}$. Substituting $z = 2y - x$ and $z' = 2y' - 1$ gives $z' + 1 = \frac{z+1}{z+3}$. Integrating gives $\frac{1}{2}z^2 + 3z = -2x + c$, whence $z + 3 = \pm\sqrt{9 + 2c - 4x}$. Conclusion:

$$y(x) = \frac{1}{2}x - \frac{3}{2} \pm \frac{1}{2}\sqrt{9 + 2c - 4x}, x \in (-\infty; \frac{2c+9}{4}), c \in \mathbb{R}$$

72) Solve on $\{(x,y) \in \mathbb{R}^2; x - y + 3 \neq 0\}$. Substituting $x = X - 2$ and $y = Y + 1$ yields $Y' = \frac{X+Y}{X-Y}$. Setting $Y = Xz$ yields the implicit relation $\operatorname{arctg}(\frac{y-1}{x+2}) - \ln(\sqrt{(x+2)^2 + (y-1)^2}) = c$, $c \in \mathbb{R}$. Connecting solutions across $x = -2$ produces maximal solutions on $(-2 - \frac{\exp(\frac{\pi}{4} - c_1)}{\sqrt{2}}; -2 + \frac{\exp(\frac{\pi}{4} - c_2)}{\sqrt{2}})$ for $c_1 \pm \pi = c_2$.

Other Equations Reducible to Separable Variables

Example 7. Solve the equation $y' = \sin(x + y)$.

Solution. The substitution $z = x + y$ yields

$$z' = y' + 1 = \sin z + 1,$$

which is a separable differential equation. It has stationary solutions $z = 2k\pi - \frac{\pi}{2}$. On intervals between these zeros, dividing by $\sin z + 1$ and integrating yields

$$\frac{\cos z}{\sin z + 1} = x + c. \tag{8}$$

Because the function on the left-hand side maps each interval $(2k\pi - \pi/2, 2k\pi + 3\pi/2)$ monotonically onto $\mathbb{R}$, equation (8) gives an implicitly defined function z with domain $\mathbb{R}$. These solutions are maximal and cover the entire plane, so by uniqueness, they constitute all solutions. Substituting back gives all maximal solutions of the original equation:

$$\frac{\cos(x+y)}{\sin(x+y)+1} = x + c \quad x \in \mathbb{R}\,.$$

Exercises

73. $y' = \cos(x + 2y)$

74. $y' = \sin(x + y + 3) - 1$

75. $y' = \sqrt{(x+y)^2 + 1}$

76. $y' - \sin x = \cos(y + \cos x)$

Solutions

73) Solve for $x \in \mathbb{R}$ and $y \in \mathbb{R}$. Substitute $z = x + 2y$, $z' = 1 + 2y'$, yielding $z' = 1 + 2\cos z$. Stationary solutions $z = \frac{2\pi}{3} + 2k\pi$ and $z = \frac{4\pi}{3} + 2k\pi$, $k \in \mathbb{Z}$. Integrating yields the conclusion:

$$y(x) = \operatorname{arctg}\left(\sqrt{3}\operatorname{tgh}\left(\frac{\sqrt{3}}{2}(x+c)\right)\right) + k\pi - \frac{x}{2},\ x \in \mathbb{R}$$

$$y(x)=\begin{cases}\operatorname{arctg}\left(\sqrt{3}\operatorname{cotgh}\left(\dfrac{\sqrt{3}}{2}(x+c)\right)\right)+\pi+k\pi-\dfrac{x}{2}, & x\in(-\infty;-c)\\ \dfrac{\pi}{2}+k\pi+\dfrac{c}{2}, & x\in\{-c\}\\ \operatorname{arctg}\left(\sqrt{3}\operatorname{cotgh}\left(\dfrac{\sqrt{3}}{2}(x+c)\right)\right)+k\pi-\dfrac{x}{2}, & x\in(-c;+\infty)\end{cases}$$

$$y(x)=\frac{\pi}{3}+k\pi-\frac{x}{2},\ x\in\mathbb{R}$$

$$y(x)=\frac{2\pi}{3}+k\pi-\frac{x}{2},\ x\in\mathbb{R}$$

$c\in\mathbb{R}$, $k\in\mathbb{Z}$

74) Solve for $x\in\mathbb{R}$ and $y\in\mathbb{R}$. Substitute $z=x+y+3$, $z'=1+y'$ to get $z'=\sin z$. Stationary solutions $z=k\pi$, $k\in\mathbb{Z}$. After integration: $-\operatorname{argtgh}(\cos z)=x+c$, $c\in\mathbb{R}$. Conclusion:

$$y(x)=\arccos\left(\operatorname{tgh}(-x-c)\right)+k\pi-x-3,\ x\in\mathbb{R}$$

$$y(x)=k\pi-x-3,\ x\in\mathbb{R}$$

$c\in\mathbb{R}$, $k\in\mathbb{Z}$

75) Solve for $x\in\mathbb{R}$ and $y\in\mathbb{R}$. Substitute $z=x+y$, $z'=1+y'$ to obtain $\frac{z'}{1+\sqrt{1+z^2}}=1$. Integrating yields the implicit relation $(x+y)\operatorname{argsinh}(x+y)-\sqrt{1+(x+y)^2}-(x+c)(x+y)+1=0$, which defines solutions for $x\in\mathbb{R}$ and $c\in\mathbb{R}$.

76) Solve for $x\in\mathbb{R}$ and $y\in\mathbb{R}$. Substitute $z=y+\cos x$, $z'=y'-\sin x$ to get $z'=\cos z$. Stationary solutions $z=\frac{\pi}{2}+k\pi$, $k\in\mathbb{Z}$. Integrating gives $\operatorname{argtgh}(\sin z)=x+c$, $c\in\mathbb{R}$. Conclusion:

$$y(x)=\arcsin\left(\operatorname{tgh}(x+c)\right)-\cos x+k\pi,x\in\mathbb{R}$$

$$y(x)=\frac{\pi}{2}-\cos x+k\pi,x\in\mathbb{R}$$

$c\in\mathbb{R}$, $k\in\mathbb{Z}$