Universal Algebra Today
Part I

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Outline

- **Today**: Universal Algebra yesterday
- **Tommorow**: Universal Algebra today
- **Thursday**: Universal Algebra tommorow

About the tutorial

- I will give you my (present) opinions on
  - What is Universal Algebra = UA (Part I)
  - Basic concepts and ideas in UA (Part I)
  - More advanced concepts and ideas (Part II)
  - Directions (Part III)
- I will concentrate on concepts and proofs.
  This is not a survey of the most important results.
- Interrupt me!
- Apologies: typos, ugly slides, incorrect theorems and proofs, ...

UA = study of general algebraic structures

- **Algebras in mathematics**
  - Classic algebras – fields, rings, modules (geometry, analysis, number theory)
  - Groups as symmetries (algebra, geometry, combinatorics)
  - Lattices (combinatorics, logic, semantics in CS)
  - Semigroups (combinatorics, automata and languages)
  - Quasigroups, . . . (combinatorics, geometry)
- **GM: What do you do in UA?**
  - UA: Generalize (HSP, iso theorems, decompositions)
  - UA: Organize (Mal'tsev conditions)
  - UA: Study particular classes above – **not this tutorial**
  - UA: Develop complicated, monumental, deep, great theories for large classes (commutator theory, TCT)
- **GM: Why?**

Why?

- **GM: Why do you develop general theories?**
  - UA: To answer complicated questions . . . in UA
    For some reason, we are especially excited about identities = universally quantified equations (GM: "hmm, interesting . . . ")
  - UA: To understand computational complexity
    - GM: Tell me more
    - UA: . . . CSP . . . associated algebra . . . variety . . . BlahBlah . . .
    - GM: Convoluted approach, one of many
    - UA: But we have great results, eg...
    - UA: ... [Bulatov],[Zhuk] solved the dichotomy conjecture
    - UA: ... many consequences, new directions, etc. etc.
- **GM: Why is this approach successful?**
  GM: Where else can it be applied?

Too popular viewpoint

Group theory, Semigroup theory

- group: algebraic structure $\mathbf{G} = (G; \cdot, ^{-1}, 1)$ satisfying ...
- permutation group: when G happens to be a set of bijections, $\cdot$ is composition, ...
- monoid: algebraic structure $\mathbf{M} = (M; \cdot, 1)$ satisfying ...
- transformation monoid: ...

Universal algebra

- algebra: any algebraic structure $\mathbf{Z} = (Z;$ some operations $)$

Rants

- Model theorist: models of purely algebraic signature, why do you avoid relations?
- Algebraist: groups are complicated enough, nothing interesting can be said about general algebras
- All: have you ever seen a 37-ary operation? You shouldn't study such a nonsense

Alternative viewpoint

	concrete	abstract
unary invert. symmetries	permutation group	group
unary symmetries	transformation monoid	monoid
higher arity symmetries	function clone	abstract clone

- permutation group: Subset of $\{f : A \to A\}$ closed under composition and id_A and inverses...

  can be given by a generating unary algebra
- group: Forget concrete mappings, remember composition
- function clone: Subset of $\{f : A^n \to A : n \in \mathbb{N}\}$ closed under composition

  can be given by a generating algebra
- abstract clone: Forget concrete mappings, remember composition

  aka variety, finitary monad over SET, Lawvere theory

What is UA? Traditionally

- UA = study of general algebraic structures, identities
- ie. generalization of classical algebra
- **Concepts:** algebras + homomorphisms, H,S,P, identity, variety, free algebra
- **Insights:**
  - identities $\leftrightarrow$ HSP
  - subdirect representation
  - properties of congruences $\leftrightarrow$ Mal'tsev conditions
- **Low level**: compose operations

What is UA? Really

- UA = study of higher arity symmetries
- ie. generalization of group theory from arity 1
- **Concepts:** clones + homomorphisms (**different!**), H,S,P, free clone (name?)
- **Insights**
  - operations ↔ relations
  - homomorphisms ↔ HSP ↔ pp-interpretations
  - subdirect representation
  - properties of relations ↔ Mal'tsev conditions
- **Low level:** pp-define relations, compose operations

operations $\leftrightarrow$ relations

Notation etc

- **Typographical**
  - A ... set (domain, universe, base set, ...)
  - $\mathbf{A}$ ... set of operations on A (will write $f \in \mathbf{A}$)
  - $R \leq \mathbf{A}$ ... subuniverse, $\mathbf{R} \leq \mathbf{A}$ ... subalgebra
  - $\mathbb{A}$ ... set of relations on A
- Operation is $f : A^n \to A$, $n \geq 1$
  - Superposition $f(g_1, \ldots, g_m)$ (f is m-ary, g_i's n-ary)
    $(x_1, \ldots, x_n) \mapsto f(g_1(x_1, \ldots, x_n), \ldots, g_m(x_1, \ldots, x_n))$
- Relation
  - n-ary: $R \subseteq A^n$
  - X-ary: $R \subseteq A^X$ (X will be often finite)
  - pictures for binary
    - list of pairs (rows of a $|R| \times 2$ matrix)
    - subset of the square A^2
    - digraph
    - **bipartite graph**
  - pictures for higher arities

Function clone

Definition

Function clone on A = set of operations on A closed under forming term operations

For each clone **A** on A:

- for each $i \leq n$
$$\pi_i^n : (x_1, \dots, x_n) \mapsto x_i$$
is in **A**
- if f,g are binary operations from **A**, then
$$(x, y, z) \mapsto f(g(f(z, x), y), g(x, x))$$
is in **A**

Notation: For algebra **A**, Clo(**A**) = all term operations of **A**

Compatibility

Definition

$f : A^n \to A$ is compatible with $R \subseteq A^k$

(f is a symmetry of R, f is a polymorphism of R,
R is invariant under f)

if $f(\mathbf{a}_1, \dots \mathbf{a}_n) \in R$ whenever $\mathbf{a}_1, \dots, \mathbf{a}_n \in R$

Notation: For a set of relations $\mathbb{A}$, a set of operations $\mathbf{A}$

- $\mathrm{Pol}(\mathbb{A})$ = all operations compatible with all relations in $\mathbb{A}$
- $\mathrm{Inv}(\mathbf{A})$ = all (finitary) relations invariant under all operations in $\mathbf{A}$

Fact: $\mathrm{Pol}(\mathbb{A})$ is a clone. $\mathrm{Inv}(\mathbf{A})$ is a coclone (TBD)

Compute Pol($\mathbb{A}$)

- $(\{0,1\};\ x \wedge y \rightarrow z, x \wedge y \rightarrow \neg z)$
- $(\{0,1\};\ \leq)$
- $(\{0,1\};$ all binary relations$)$
- $(\{0,1,2\};\ \neq)$
- $(\mathbb{Z}_p;$ vector subspaces of $\mathbb{Z}_p^3)$
- $(\mathbb{Z}_p;$ affine subspaces of $\mathbb{Z}_p^3)$

Compute $\mathrm{Inv}(\mathbf{A})$

- $(\{0,1\}; \vee)$
- $(\{0,1\}; \vee, \wedge)$
- $(\{0,1\}; \mathrm{majority})$
- $(\{0,1\}; \vee, \wedge, \neg)$
- $(\mathbb{Z}_p; x + y)$
- $(\mathbb{Z}_p; x - y + z)$

New clones from old

- $\mathbf{B} \in \mathrm{P}(\mathbf{A})$ (power) if $\mathbf{B} = \mathbf{A}^X$
- $\mathbf{B} \in \mathrm{P}^{\mathrm{fin}}(\mathbf{A})$ (finite power) if $\mathbf{B} = \mathbf{A}^n$ (or $\mathbf{A}^X$ for finite X)
- $\mathbf{B} \in \mathrm{S}(\mathbf{A})$ (subalgebra) if $\mathbf{B} \leq \mathbf{A}$
- $\mathbf{B} \in \mathrm{H}(\mathbf{A})$ (quotient) if $\alpha \in \mathrm{Con}(\mathbf{A})$, $\mathbf{B} \cong \mathbf{A}/\alpha$
- $\mathbf{B} \in \mathrm{E}(\mathbf{A})$ (expansion) if $\mathbf{B} \supseteq \mathbf{A}$

Remarks

- $R \in \mathrm{SP}^{\mathrm{fin}}(\mathbf{A})$ (finite subpower) ie. $R \leq \mathbf{A}^n$ iff $R \in \mathrm{Inv}(\mathbf{A})$
- products of different clones do not make sense for now

FREE CLONE!!!!!!!

Definition

A . . . clone

$\mathrm{Free}_{\mathbf{A}}(n) = \{$all n-ary members of **A**$\}$

- It is a subset of A^{A^n}
- ie. A^n-ary relation on A [the list picture]
- It is invariant under all operations of **A**:
  - Consider m-ary $f \in \mathbf{A}$ and $g_1, \dots, g_m \in \mathrm{Free}_{\mathbf{A}}(n)$
  - What is $f(g_1, \dots, g_m)$ applied component-wise?

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  - What is $f(g_1, \dots, g_m)$ applied component-wise?
  - It is $f(g_1, \dots, g_m)$ – the superposition!
- ie. subuniverse of $\mathbf{A}^{A^n}$ (subpower of **A**), ie. $\mathbf{Free}_{\mathbf{A}}(n)$ makes sense
- It is generated by $\pi_1^n, \dots, \pi_n^n$

Remark: If **A** is an algebra, $\mathbf{Free}_{\mathbf{A}}(n)$ is the clone of the n-generated free algebra in the variety generated by **A**.

Clones determined by relations

Theorem ([Geiger],[Bodnarčuk, Kalužnin, Kotov, Romov])

$\mathbf{A}$ *clone,* A *finite,* $f : A^n \to A$

If f *is compatible with each* $R \in \mathrm{Inv}(\mathbf{A})$*, then* $f \in \mathbf{A}$

Proof.

- In particular, f is compatible with $\mathrm{Free}_{\mathbf{A}}(n)$
- $\pi_1^n, \dots, \pi_n^n \in \mathrm{Free}_{\mathbf{A}}(n)$, thus $f(\pi_1^n, \dots, \pi_n^n)$ $(= f)$ $\in \mathrm{Free}_{\mathbf{A}}(n)$

□

Coclone

Fact: Inv($\mathbf{A}$) is a coclone

Definition

Coclone on A = set of (nonempty) relations on A closed under
pp-definitions = 1st order definitions using $\exists, =,$ *and*

Example: If binary R, S in Inv($\mathbf{A}$), then T in Inv($\mathbf{A}$)

$$T = \{(x, y, z) : (\exists u)(\exists v)\ R(x, u) \text{ and } S(u, v) \text{ and } R(y, y)\}$$

What can we do with pp-definitions

- intersect, eg. $T(x, y, z) := R(x, y, z)$ and $S(x, y, z)$
- introduce dummy variables, permute coordinates, glue coordinates eg. $T(x, y, z) := R(y, x, x)$
- project onto some coordinates, eg. $T(x, y) := (\exists z)\ R(x, y, z)$
- compose binary relations, eg. $T(x, y) := (\exists z)\ R(x, z)$ and $S(z, y)$

Coclones determined by operations

Theorem ([Geiger],[Bodnarčuk, Kalužnin, Kotov, Romov])

$\mathbb{A}$ coclone, A finite, $R \subseteq A^m$

If R is invariant under every $f \in \mathrm{Pol}(\mathbb{A})$, then $R \in \mathbb{A}$

Proof.

- $\mathbf{A} := \mathrm{Pol}(\mathbb{A})$
- Say $A = \{1, \dots, k\}$, $m = 2$, $n = |R|$, $R = \{(a_1, b_1), \dots, (a_n, b_n)\}$
- $\mathrm{Free}_{\mathbf{A}}(n)$ is pp-definable from $\mathbb{A}$ (without $\exists$)
  $\mathrm{Free}_{\mathbf{A}}(n)(x_{11\dots1}, x_{11\dots2}, \dots x_{kk\dots k}) = \dots$
- Existentially quantify all variables but $x_{a_1 a_2 \dots a_n}$ and $x_{b_1 b_2 \dots b_n}$
- Call the relation S
- $R \subseteq S$ because of projections
- $R \supseteq S$ because of compatibility

□

Clones ↔ Coclones

Theorem ([Geiger]; [Bodnarčuk, Kalužnin, Kotov, Romov])

For finite A, Pol, Inv *are (mutually inverse) bijections*

$$\textit{Clones on } A \quad \leftrightarrow \quad \textit{Coclones on } A$$

Remarks:

- $\mathrm{Clo}(\mathbf{A}) = \mathrm{Pol}(\mathrm{Inv}(\mathbf{A}))$, $\mathrm{Coclo}(\mathbb{A}) = \mathrm{Inv}(\mathrm{Pol}(\mathbb{A}))$
- Clones determined by invariant relations
- Coclones determined by polymorphisms (symmetries)
- Understanding clones = understanding coclones

Birkhoff's HSP

Clone homomorphism

Definition

$\mathbf{A}, \mathbf{B}$ ... clones. Mapping $\xi : \mathbf{A} \to \mathbf{B}$ is a clone homomorphism if it preserves arities and terms.

Examples of preserving terms:

- if $f, g \in \mathbf{A}$, $h(a,b,c) := f(a, g(b,c))$, then $\xi h(a,b,c) = \xi f(a, \xi g(b,c))$
- sends π_i^n (on A) to π_i^n (on B)

Note: preserves terms = preserves identities

Examples of homomorphisms $\xi : \mathbf{A} \to \mathbf{B}$

- $\mathbf{B} \in \mathrm{P}(\mathbf{A})$ ie. $\mathbf{B} = \mathbf{A}^X$, $\xi(f) = f^X$ (componentwise)
- $\mathbf{B} \in \mathrm{S}(\mathbf{A})$ ie. $\mathbf{B} \leq \mathbf{A}$, $\xi(f) = f_{|X}$
- $\mathbf{B} \in \mathrm{H}(\mathbf{A})$ ie. $\alpha \in \mathrm{Con}(\mathbf{A})$, $\mathbf{B} \cong \mathbf{A}/\alpha$, $\xi(f) = f/\alpha$
- $\mathbf{B} \in \mathrm{E}(\mathbf{A})$ ie. $\mathbf{B} \supseteq \mathbf{A}$, $\xi(f) = f$

Birkhoff

Theorem (Birkhoff)

$\mathbf{A}, \mathbf{B}$ ... *clones*

If $\exists \xi : \mathbf{A} \to \mathbf{B}$, *then* $\mathbf{B} \in \mathrm{EHSP}(\mathbf{A})$

Proof.

- Say $B = \{1, \dots, n\}$
- Take $\mathbf{Free_A}(n) \in \mathrm{SP}(\mathbf{A})$
- Define α: $(f, g) \in \alpha$ iff $\xi f(1, 2, \dots, n) = \xi g(1, 2, \dots, n)$
- $\alpha \in \mathrm{Con}(\mathbf{Free_A}(n))$ since ξ is a homomorphism
- $\mathbf{Free_A}(n)/\alpha \cong$ image of ξ

$\square$

Relational Birkhoff

Recall: If $\mathbb{A}, \mathbb{B}$ relational structures with the same domain $A = B$, then $\mathbb{B}$ is pp-definable from $\mathbb{A}$ iff $\mathrm{Pol}(\mathbb{B}) \in \mathrm{E}(\mathrm{Pol}(\mathbb{A}))$

Theorem ([Birkhoff])

$\mathbb{A}, \mathbb{B}$ *... relational structures, A, B finite,* $\mathbf{A} = \mathrm{Pol}(\mathbb{A})$, $\mathbf{B} = \mathrm{Pol}(\mathbb{B})$

TFAE

- $\exists \xi : \mathbf{A} \to \mathbf{B}$
- $\mathbf{B} \in \mathrm{EHSP}(\mathbf{A})$
- $\mathbb{B}$ *is pp-interpretable in* $\mathbb{A}$

Remarks:

- first two items don't require finiteness
- classic Birkhoff $\approx$ this Birkhoff
- variant with onto homomorphism (remove E in the second item)

Abstract clones

Abstract clone:

- To decide whether $\xi : \mathbf{A} \to \mathbf{B}$ is a homomorphism, we do not need all information about the clone...
- ...we only need to know how the operations compose (the identities) $\to$ (abstract) clone
- Formalization is not important

Algebras in a variety $\sim$ clone actions

- group $\mathbf{Z}$ acting on a set $A \to$ permutation group on A ($\xi : \mathbf{Z} \to$ full permutation group on A) $\to \xi(\mathbf{Z})$
- clone $\mathbf{Z}$ acting on a set $A \to$ clone on A ($\xi : \mathbf{Z} \to$ full clone on A) $\to \xi(\mathbf{Z})$
- product of two clone actions of $\mathbf{Z}$ is defined naturally
- usually we work with clone actions of a single clone $\mathbf{Z}$ – we have products of clones