Universal Algebra Today
Part III

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55th Summer School on Algebra and Ordered Sets
Nový Smokovec, 3–8 Sep 2017

Summary of the last talk and an outline

Summary of the last talk:

- properties of congruences $\leftrightarrow$ connectivity properties of relations $\leftrightarrow$ equational conditions
- 3 levels of abstraction
  the last one is trivial for permutation groups!
- theories in today's UA
  - commutator theory: everywhere
  - mostly algebraic: tame congruence theory, Bulatov's theory
  - mostly relational: absorption, Zhuk's theory
- Abelianness, Fundamental Theorem (Abelian + Mal'tsev $\Rightarrow$ module)
- Taylor = idempotent and equationally nontrivial; HSP $\to$ HS

Today:

- Absorption: (1) absorbs connectivity and (2) is everywhere
- Absorption and Abelianness
- Directions

Absorption

Example: Pol($\mathbb{K}_3^c$), part 1/4

$\mathbb{K}_3^c$ (the undirected triangle with singletons):

- domain $A = \{0, 1, 2\}$
- binary relation $R = \{(a, b) \in A^2 : a \neq b\}$ and singletons $C_i = \{i\}$

We will show: $\mathbf{A} := \mathrm{Pol}(\mathbb{K}_3^c) =$ projections

Step 1: $(\forall f \in \mathbf{A}, a_i \in A)\ \ f(a_1, \dots, a_n) \in \{a_1, \dots, a_n\}$

- Each singleton is a subuniverse of $\mathbf{A}$
- Neighbors of 0 are $1, 2 \Rightarrow \{1, 2\} \leq \mathbf{A}$
- Similarly $\{0, 2\}$, $\{0, 1\}$ are subuniverses as well

Example: Pol($\mathbb{K}_3^c$), part 2/4

Step 1: $(\forall f \in \mathbf{A}, a_i \in A)\ f(a_1, \dots, a_n) \in \{a_1, \dots, a_n\}$

Step 2: The only unary and binary operations in **A** are projections

- From Step 1, $f(0,1) \in \{0,1\}$. Assume $f(0,1) = 0$.
- $f(0,1) = 0$, $f(2,2) = 2$ + compatibility with $R \Rightarrow f(1,0) = 1$.
- $f(0,1) = 0$, $f(1,1) = 1$ + compatibility with $R \Rightarrow f(2,0) = 2$.
- ...
- f is the first projection

Step 3: Take n-ary $f \in \mathbf{A}$ and consider $f_i(x,y) := f(x, \dots, x, y, x, \dots, x)$

By Step 2, for each i

- either $(\forall x, y \in A)\ f_i(x,y) = x$
- or $(\forall x, y \in A)\ f_i(x,y) = y$

Example: Pol($\mathbb{K}_3^c$), part 3/4

Step 1: $(\forall f \in \mathbf{A}, a_i \in A)\ f(a_1, \dots, a_n) \in \{a_1, \dots, a_n\}$
Step 2: The only unary and binary operations in **A** are projections
Step 3: Take n-ary $f \in \mathbf{A}$ and consider $f_i(x, y) := f(x, \dots, x, y, x, \dots, x)$
Step 4a: Assume $(\exists i)\ f_i(x, y) = y$, say $i = 1$

- aim: $f = \pi_1$
- $f(0, 1, \dots, 1) = 0$ + compatibility with R + Step 1 $\Rightarrow$
  $f(0, \{0, 2\}, \{0, 2\}, \dots, \{0, 2\}) = \{0\}$
- $f(a, \{a, b\}, \dots, \{a, b\}) = \{a\}$ for each $a, b \in A$
- Similarly $f(a, A, \dots, A) = \{a\}$

Example: Pol($\mathbb{K}_3^c$), part 4/4

Step 1: $(\forall f \in \mathbf{A}, a_i \in A)\ f(a_1, \dots, a_n) \in \{a_1, \dots, a_n\}$
Step 2: The only unary and binary operations in **A** are projections
Step 3: Take n-ary $f \in \mathbf{A}$ and consider $f_i(x, y) := f(x, \dots, x, y, x, \dots, x)$
Step 4b: Assume $(\forall i)\ f_i(x, y) = x \Rightarrow f$ is a near unanimity operation

- $f(0, 2, \dots, 2) = 2$ + compatibility $\Rightarrow$ $f(\{1, 2\}, \{0, 1\}, \dots \{0, 1\}) \subseteq \{0, 1\}$
- + Step 1 $\Rightarrow f(A, \{0, 1\}, \dots, \{0, 1\}) \subseteq \{0, 1\}$
- Similarly if "A" is at a different coordinate
- Draw R as a bipartite graph
- The following path links 0 and 1 "on the left", within $\{0, 1\}$:

$$f(0, 0, \dots, 0), f(2, 1, \dots, 1),$$
$$f(1, 0, 0, \dots, 0), f(0, 2, 1, \dots, 1),$$
$$f(1, 1, 0, \dots, 0), f(0, 0, 2, 1, \dots, 1),$$
$$\dots$$
$$f(1, 1, \dots, 1)$$

Absorption and Absorption Theorem

Definition

Subuniverse B of $\mathbf{A}$ is absorbing, if $(\exists f \in \mathbf{A})$ such that $f(A, B, B, \dots, B), f(B, A, B, \dots, B), \dots f(B, B, \dots, B, A) \subseteq B$

- Step 4b used that $\{0, 1\}$ was absorbing
- Absorption "absorbs connectivity properties of relations"
- Recall: connectivity properties of relations are important (permutability, congruence distributivity)
- Absorption "behaves nicely" w.r.t. pp-definitions
- Absorption is not rare:

Theorem (Absorption Theorem, [Barto,Kozik])

If $\mathbf{A}$ is finite Taylor, $R \leq_{sd} \mathbf{A}^2$, R is linked, $R \neq A^2$, then $\mathbf{A}$ has a proper absorbing subuniverse.

Loop Lemma

Theorem (Baby Loop Lemma)

If $\mathbf{A}$ *is finite Taylor and* $R \leq_{sd} \mathbf{A}^2$, *R is linked, then* $(\exists a \in A)\ (a, a) \in R$

Proof.

- By induction on $|A|$. Draw R as a bipartite graph and a digraph.
- Absorption Theorem $\Rightarrow$ there exists a proper absorbing subuniverse B
- By walking obtain a proper absorbing C such that there is an infinitely long path within C (in digraph sense)
  - Use the bipartite graph picture of R
  - Start with $B_0 = B$ on the left. Take B_1 – neighbors of B_0 on the right, B_2 – neighbors of B_1 on the left, ...
  - $B_n = A$ for sufficiently large n
  - Say B_i is still proper, WLOG on the left, $B_{i+1} = A$
  - B_i is absorbing (absorption behaves nicely w.r.t pp-definitions)
  - Each element of A on the right (in particular in B_i) has a neighbor in B_i on the left $\Rightarrow$ there is a circle in B_i (in digraph sense)

Loop Lemma

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Proof.

- By induction on $|A|$. Draw R as a bipartite graph and a digraph.
- Absorption Theorem $\Rightarrow$ there exists a proper absorbing subuniverse B
- By walking obtain a proper absorbing C such that there is an infinitely long path within C (in digraph sense)
- Take D – all the elements in infinitely long paths within C
- D is nonempty, absorbing, $S := R \cap (D \times D)$ is subdirect in $D \times D$
- S is linked (absorption absorbs connectivity, see Step 4b).
- Use induction hypothesis for S.

□

Sigger's operation

Theorem (Loop Lemma [Barto,Kozik,Niven])

If $\mathbf{A}$ is finite Taylor, $R \leq_{sd} \mathbf{A}^2$, $R \circ R \dots$ is linked, then $(\exists a \in A)(a,a) \in R$

Theorem ([Kearnes, Marković,McKenzie])

If $\mathbf{A}$ is finite Taylor, then $(\exists s \in \mathbf{A})\ s(r,a,r,e) = s(a,r,e,a)$

Proof.

- denote $a := \pi^3_1$, $e := \pi^3_2$, $r := \pi^3_3$
- Consider subuniverse R of $\mathbf{Free_A}(3) \times \mathbf{Free_A}(3)$ generated by $(r,a),(a,r),(r,e),(e,a)$
- $R = \{(f(r,a,r,e), f(a,r,e,a)) : f \in \mathbf{A} \text{ 4-ary}\}$
- $\mathbf{Free_A}(3)$ is finite Taylor, $R \circ R$ is linked (because of generators)
- By the Loop Lemma, R contains a loop $\Rightarrow f(r,a,r,e) = f(a,r,e,a)$ for some f

Sigger's operation

Theorem (Loop Lemma [Barto,Kozik,Niven])

If $\mathbf{A}$ *is finite Taylor,* $R \leq_{sd} \mathbf{A}^2$, $R \circ R \dots$ *is linked, then* $(\exists a \in A)(a, a) \in R$

Theorem ([Kearnes, Marković,McKenzie])

If $\mathbf{A}$ *is finite Taylor, then* $(\exists s \in \mathbf{A})\ s(r, a, r, e) = s(a, r, e, a)$

- The same proof idea as for eg. "rectangularity $\Rightarrow$ Mal'tsev"
- **The weakest nontrivial idempotent identity for finite algebras**!

Absorption and Abelianness

Fundamental Theorem on Abelian Algebras, 2nd version

Theorem (Hobby, McKenzie?)

If $\mathbf{A}$ *is a finite Taylor Abelian clone,*
then $\mathbf{A} \subseteq \mathrm{Clo}(\mathbf{M} + consts)$ *for some module* $\mathbf{M}$ *(*$M = A$*) over a ring* $\mathbf{R}$

- finiteness necessary in this version
- original proof very complicated
- proof via absorption:
  - By the 1st version of Fundamental Theorem, enough to show that $\mathbf{A}$ has a Mal'tsev operation
  - Abelian $\Rightarrow$ avoids absorption
  - Avoids absorption $\Rightarrow$ avoids absorption in a stronger sense
  - $\Rightarrow$ avoids absorption in free algebra + Absorption Theorem $\Rightarrow$ Mal'tsev

Abelianness prevents absorption

Proposition

If $\mathbf{A}$ *is a finite Taylor Abelian clone, then no* $\mathbf{B} \leq \mathbf{A}$ *has a proper absorbing subuniverse*

Proof.

Short □

Proposition

If $\mathbf{A}$ *is idempotent and no* $\mathbf{B} \leq \mathbf{A}$ *has a proper absorbing subuniverse, then no* $\mathbf{B} \leq \mathbf{A}^n$ *has a proper absorbing subuniverse*

Proof.

Similar to the Bulatov's "getting rid of powers" proposition □

No absorption implies Mal'tsev

Proposition

If **A** *is a finite Taylor Abelian clone, then* **A** *has a Mal'tsev operation*

Proof.

- Previous proposition $\Rightarrow$ $\mathbf{Free}_{\mathbf{A}}(2)$ is absorption free
- Consider (again) the subuniverse R of $\mathbf{Free}_{\mathbf{A}}(2) \times \mathbf{Free}_{\mathbf{A}}(2)$ generated by $(\pi_2, \pi_1), (\pi_1, \pi_1), (\pi_1, \pi_2)$
- It is linked
- By Absorption Theorem, $R = (\mathrm{Free}_{\mathbf{A}}(2))^2$. In particular $(\pi_2, \pi_2) \in R$
- Then m has a Mal'tsev term (as in the "rectangularity $\Rightarrow$ Mal'tsev", again)

$\square$

Directions

Absorption, Bulatov, Zhuk

- Connection to tame congruence theory?
- Concepts are different, some results are almost the same
- Bulatov's theory is very technical
- Bulatov (and partly Zhuk) doesn't apply to all Taylor clones, taking reducts is necessary

14 days with Bulatov, Kozik, and Zhuk this summer:

- There are very tight links among the 3 theories for Taylor minimal clones
  - **Def:** **A** is Taylor minimal if it is Taylor and no reduct is Taylor
  - **Fact:** Each finite Taylor clone has a finite Taylor minimal reduct
  - Inspiration from the work of [Zarathustra Brady]
- (At least parts of) Bulatov can be simplified if we take a more relational approach

Finite → Infinite

Surprisingly, some results from finite UA generalize to infinite

- commutator theory [Kearnes, Kiss]
- stronger Malt'sev condition for CD and CM

Theorem ([Kazda, Kozik, McKenzie, Moore] Classic formulation)

A variety is congruence distributive iff it has directed Jónsson terms $p_1, \dots, p_n$ ie.

$$p_1(x,x,y) = x, p_n(x,y,y) = y, (\forall i)\ p_i(x,y,x) = x \quad \textit{like Jónsson}$$

$$(\forall i)\ p_i(x,y,y) = p_{i+1}(x,x,y)$$

- there is a weakest nontrivial idempotent equational condition

Theorem ([Olšák])

A clone is Taylor iff it has a 6-ary operation t such that

$$t(y,x,x,x,y,y) = t(x,y,x,y,x,y) = t(x,x,y,y,y,x)$$

Generalizations and many more levels of abstraction

- **"reasonably" infinite clones** [Bodirsky, Pinsker]
  - they capture complexity of many decision problems (including undecidable)
  - todo: use, incorporate, and algebraize techniques in topology, model theory, Ramsey theory
  - eg. topological Birkhoff theorem [Bodirsky, Pinsker]
  - eg. Ramsey theory via group actions [Kechris, Pestov, Todorcevic]
  - eg. cores for infinite structures via Fraissé argument on the algebraic side [Barto, Kompatscher, Van Pham, Pinsker]
- **weighted clones** [Cohen, Cooper, Creed, Jeavons, Živný]
  - they capture complexity of optimization problems
  - success – full complexity classification of "valued CSPs" [Kolmogorov, Krokhin, Rolínek]
  - use and incorporate probabilistic and analytical techniques
  - eg. correlation decay $\to$ some form of Absorption Theorem [Brown-Cohen, Raghavendra]
- **minor closed sets** [Pöschel], [Aichinger], [Brakensiek,Guruswami]
  - they capture complexity of promise problems
  - linear Birkhoff theorem [Barto, Opršal, Pinsker]

Approximation and Label Cover

δ-approximation of 3-SAT:

INPUT: 3-SAT instance, eg. $(x_1 \vee \neg x_5 \vee x_3) \wedge (\neg x_3 \vee x_2 \vee \neg x_9) \wedge \dots$
OUTPUT: assignment satisfying at least δ-fraction of clauses
for satisfiable instances

- $(\exists \delta < 1)$ this problem is NP-hard ... the PCP theorem
  Gödel Prize [Arora, Feige, Goldwasser, Lund, Lovász, Motwani, Safra, Sudan, Szegedy]
- $(\forall \delta > 7/8)$ this problem is NP-hard ... tight!
  Gödel Prize [Hastad]
  based on hardness of approximating "Label Cover"
- *"Label Cover is the mother of strong inapproximability results"* [Guruswami]

Label Cover and others

Label Cover is a UA problem about linear identities! [Bulín]

INPUT: finite set of linear identities
OUTPUT: are they trivial (satisfiable in **Proj**) or nontrivial?

Approximate version (depending on δ):

OUTPUT: are they trivial or δ-robustly nontrivial?

Other fun facts in complexity:

- essential tool: Long code ... code $i \in [n]$ by π_i^n
- complexity of approximation depends on "approximate polymorphisms" assuming the Unique Games Conjecture [Khot] (Nevanlinna Prize for this conjecture)
- convex programming is "based on" polymorphisms

Summary

Summary

This is the most exciting period of time for universal algebra

- 1st substantial application outside (in computational complexity)
- many new directions (topology, analysis), connections and potential applications (in the CS mainstream)
  - semigroups? Some questions in automata are UA, they look different [Bojanczyk]
- new theories (Absorption, Bulatov, Zhuk, higher commutators): calls for simplification, unification, improvements
- new fundamental concepts and basic results (eg. minor closed set, topological clone, weighted clone, variants of Birkhoff)
- older parts are getting stronger (eg. CD, Olšák) and simpler (eg. Fundamental Theorem on Abelian Clones)