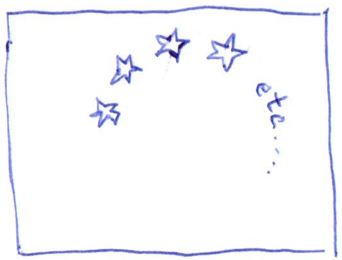
Baby PCP Theorem

&

Reductions between Promise CSPs

Libor Barto, Charles University

Marcin Kozik, Jagellonian University



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PCSP

Template

$\mathbb{A} = (\underbrace{D}_{\text{domain}};\ \underbrace{R, S, \ldots}_{\text{relations on } D})$

$\mathbb{A}_w = (D_w;\ R_w, S_w, \ldots)$

$\mathbb{A} \to \mathbb{A}_w$, both finite

	strong	weak
domain	D	D_w
relations	$R \subseteq D \times D$	$R_w \subseteq D_w \times D_w$
	$S \subseteq \ldots$	$S_w \subseteq \ldots$

$\mathrm{PCSP}(\mathbb{A}, \mathbb{A}_w)$

decision version:
- INPUT: $\mathbb{X}$
- YES: $\exists$ homo $\mathbb{X} \to \mathbb{A}$
- NO: $\neg \exists$ homo $\mathbb{X} \not\to \mathbb{B}$

search version:
- GIVEN: $\mathbb{X}$ such that $\exists$ homo $\mathbb{X} \to \mathbb{A}$
- FIND: $\mathbb{X} \to \mathbb{A}_w$

input:

$$\begin{tikzcd} z_1 \arrow[r, "R"] & z_2 \arrow[r, "R"] \arrow[d, "S"'] & z_4 \\ \ldots & z_3 \arrow[ur, "S"] \arrow[r] & z_{17} \quad \ldots \end{tikzcd}$$

Question: computational complexity

Example: $\mathrm{PCSP}(\mathbb{K}_3, \mathbb{K}_7)$

Remarks:
- $\mathrm{CSP}(\mathbb{A}) = \mathrm{PCSP}(\mathbb{A}, \mathbb{A})$
- multisorted version, $\mathrm{PCSP}(\mathbb{A}, \mathbb{B}) \sim \mathrm{PCSP}$(graphs of functions, graphs of functions)

$$R \subseteq D \times D \quad \sim \quad \begin{tikzcd} & R \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ D & & D \end{tikzcd}$$

The story

CSP

- 🙂 sufficient condition for $\mathrm{CSP}(\mathbb{B}) \leq \mathrm{CSP}(\mathbb{A})$ — [Jeavons '98] [Bulatov, Jeavons, Krokhin '05] [Barto, Opršal, Pinsker '16]
- 😃 good enough for NP-hardness — [Bulatov '17, Zhuk '17]

  if $\mathrm{CSP}(\mathbb{A}) \notin \mathrm{P}$ then $\mathrm{CSP}(\mathbb{B}) \leq \mathrm{CSP}(\mathbb{A})$ $\forall \mathbb{B}$

PCSP

- 🙂 the same sufficient condition works — [BKO] = [Bulín, Krokhin, Opršal '19]
- 😃 the reduction is "in fact" trivial — ± —"—
- 😢 not good enough for NP-hardness
- 😐 better conditions for NP-hardness — [Austrin, Guruswami, Håstad '17] [Brakensiek, Guruswami '16] [BKO] + [BBKO]
  - 🙁 does not follow from a general reduction — [Brandts, Wrochna, Živný '20] = [BWŽ]
  - 😢 uses complicated results

this work

- 🙂 better sufficient condition for $\mathrm{CSP}(\mathbb{B}, \mathbb{B}_w) \leq \mathrm{CSP}(\mathbb{A}, \mathbb{A}_w)$
- 🙂 the reduction is "in fact" obvious
- 🙂 implies 😐 and the proof is simple

$\mathrm{PCSP}(\mathbb{B}, \mathbb{B}_w) \leq \mathrm{PCSP}(\mathbb{A}, \mathbb{A}_w)$ whenever $\exists$ minion homomorphism $\xi : \mathrm{Pol}(\mathbb{A}, \mathbb{A}_w) \to \mathrm{Pol}(\mathbb{B}, \mathbb{B}_w)$ [BKO]

- X-ary <u>polymorphism</u> of $(\mathbb{A}, \mathbb{A}_w)$ = homomorphism $\mathbb{A}^X \to \mathbb{A}_w$
- $\mathcal{M} := \mathrm{Pol}(\mathbb{A}, \mathbb{A}_w)$ = all polymorphisms
  - it is a <u>minion</u> ... closed under taking <u>minors</u>

$$\begin{tikzcd} X \arrow[d, "\pi"'] & \mathcal{M}X = \{f : \mathbb{A}^X \to \mathbb{A}_w\} \arrow[d, "\mathcal{M}\pi"] & f \arrow[d, mapsto] \\ Y & \mathcal{M}Y = \{f : \mathbb{A}^Y \to \mathbb{A}_w\} & f^\pi \end{tikzcd}$$

Ex. $f : \mathbb{A}^7 \to \mathbb{A}_w$ $\quad \pi : [7] \to [3]$

$(1,2,3,4,5,6,7) \mapsto (1,1,3,2,3,1,3)$

$$f^\pi(d_1, d_2, d_3) := f(d_1, d_1, d_3, d_2, d_3, d_1, d_3)$$

- $\xi : \mathcal{M} \to \mathcal{N}$ is a <u>minion homomorphism</u> if
  - preserves arities $\xi = \{\xi_X : \mathcal{M}X \to \mathcal{N}X;\ X \text{ a set}\}$
  - preserves minors

$$\begin{tikzcd} X \arrow[d] & \mathcal{M}X \arrow[r, "\xi_X"] \arrow[d, "\mathcal{M}\pi"] & \mathcal{N}X \arrow[d, "\mathcal{N}\pi"] \\ Y & \mathcal{M}Y \arrow[r, "\xi_Y"] & \mathcal{N}Y \end{tikzcd}$$

- $\mathcal{I}$ .. trivial minion ... all members come from a single nonconstant $\mathbb{A} \to \mathbb{A}_w$

$\exists$ homo $\mathcal{M} \to \mathcal{I}$

$\Downarrow$

$\mathrm{PCSP}(\mathbb{A}, \mathbb{A}_w)$ NP-hard

$\exists$ homo $\mathcal{M} \to \mathcal{I} \iff \exists \mu = \{\mu_X : \mathcal{M}X \to X\}$ such that

$$\forall \cdot \begin{tikzcd} X \arrow[d, "\pi"] \\ Y \end{tikzcd} \qquad \begin{tikzcd} \mathcal{M}X \arrow[r, "\mu_X"] \arrow[d, "\mathcal{M}\pi"'] & X \arrow[d, "\pi"] \\ \mathcal{M}Y \arrow[r, "\mu_Y"] & Y \end{tikzcd} \text{ commutes, ie.} \qquad \begin{tikzcd} f \arrow[r, mapsto] \arrow[d, mapsto] & \mu(f) \arrow[d, mapsto] \\ & \pi(\mu(f)) \\ f^\pi \arrow[r, mapsto] & \mu(f^\pi) \arrow[u, phantom, "\shortparallel"] \end{tikzcd}$$

$\boxed{\mathrm{PCSP}(\mathbb{A}, \mathbb{A}_w) \sim \mathrm{PCSP}(\text{all relations, some relaxations})\,!}$

Namely $\mathrm{PCSP}(\mathcal{M})$ where $\mathcal{M} = \mathrm{Pol}(\mathbb{A}, \mathbb{A}_w)$

- for graphs of functions $D \xrightarrow{\pi} E$ $\qquad D_w \xrightarrow{\pi_w} E_w := \mathcal{M}D \xrightarrow{\mathcal{M}\pi} \mathcal{M}E$

- for other relations use $R \subseteq D \times D \Leftrightarrow$ $\begin{tikzcd} & R \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ D & & D \end{tikzcd}$

If $\exists \xi : \mathrm{Pol}(\mathbb{A}, \mathbb{A}_w) \to \mathrm{Pol}(\mathbb{B}, \mathbb{B}_w)$ then $\mathrm{PCSP}(\mathbb{B}, \mathbb{B}_w) \leq \mathrm{PCSP}(\mathcal{M})$ $(\sim \mathrm{PCSP}(\mathbb{A}, \mathbb{A}_w))$
by a trivial reduction!

± [BKO]

<u>Note</u> if $\mathcal{M} \to \mathcal{J}$ then ....

we are way too nice to the enemy

PCSP($\mathcal{M}$) is NP-hard if

BKO $\exists \mu = \{\mu_X : \mathcal{M}X \to X\}$ such that $\forall$ $X \xrightarrow{\pi} Y$

$\boxed{\text{"homo"}}$

$$\begin{tikzcd} \mathcal{M}X \arrow[r, "\mu_X"] \arrow[d, "\mathcal{M}\pi"'] & X \arrow[d, "\pi"] \\ \mathcal{M}Y \arrow[r, "\mu_Y"'] & Y \end{tikzcd}$$

commutes, i.e.

$$\begin{tikzcd} f \arrow[r, mapsto] \arrow[d, mapsto] & \mu(f) \arrow[d, mapsto] \\ f^\pi \arrow[r, mapsto] & \mu(f^\pi) = \pi(\mu(f)) \end{tikzcd}$$

Ex • 4-coloring a 3-colorable graph
[Brakensiek, Guruswami '16]

BBKO $\exists \mu = \{\mu_X : \mathcal{M}X \to \binom{X}{\leq 7}\}$ such that ——"—— weakly commutes

$\boxed{\text{"7-homo"}}$

$$\begin{tikzcd} f \arrow[r, mapsto] \arrow[d, mapsto] & \cdot \arrow[d, mapsto] \\ \cdot \arrow[r, mapsto] & \cdot \end{tikzcd}$$

non$\emptyset$ intersection

Ex • good enough for all PCSP(symmetric Boolean, symmetric Boolean)
[Brakensiek, Guruswami '18]
[Ficak, Kozik, Olšák, Stankiewicz '19]

• PCSP($\mathbb{C}_{13597}$, $\mathbb{K}_3$) [Krokhin, Opršal '19]

BWŽ $\exists \mu = \{\mu_X : \mathcal{M}X \to \binom{X}{\leq 7}\}$ such that $\forall$ comm.

$\boxed{\text{"(7,5)-homo"}}$

$$\begin{tikzcd} X_1 \arrow[d, "\pi_{1,2}"'] \arrow[dd, bend left, "\pi_{1,3}"] \\ X_2 \arrow[d, "\pi_{2,3}"'] \\ \vdots \arrow[d] \\ X_5 \end{tikzcd}$$

$\exists i<j$

$$\begin{tikzcd} \mathcal{M}X_i \arrow[r, "\mu_{X_i}"] \arrow[d, "\mathcal{M}\pi_{i,j}"'] & \binom{X_i}{7} \arrow[d, "\pi_{i,j}"] \\ \mathcal{M}X_j \arrow[r, "\mu_{X_j}"'] & \binom{X_j}{7} \end{tikzcd}$$

weakly commutes

Ex • $10^{10^{10}}$-coloring a 2-colorable 3-uniform hypergraph
[Dinur, Regev, Smyth '05] (+ [BBKO])
CHEATING! ACTUALLY NOT! [Wrochna ?]

• certain symmetric non-Boolean PCSPs
[Brandts, Wrochna, Živný '20], [B, Battistelli, Berg ?]

$$f^\pi(d_1, d_2, d_3) = f(d_1, d_1, d_3, d_2, d_3, d_1, d_3)$$

$\boxed{\text{Main result}}$

If $\exists (7,5)$-homomorphism $\xi : \mathcal{M} = \mathrm{Pol}(\mathbb{A}, \mathbb{A}_w) \to \mathrm{Pol}(\mathbb{B}, \mathbb{B}_w)$

then $\mathrm{PCSP}(\mathbb{B}, \mathbb{B}_w) \leq \mathrm{PCSP}(\mathcal{M})$ $(\sim \mathrm{PCSP}(\mathbb{A}, \mathbb{A}_w))$

by an obvious reduction

$$\xi = \left\{ \xi_X : \mathcal{M}X \longrightarrow \binom{\mathcal{M}Y}{\leq 7} \right\}$$

$\forall$ commuting

$$\begin{array}{c} X_1 \\ \downarrow \\ X_2 \\ \downarrow \\ \vdots \\ X_5 \end{array}$$

$\exists i,j$

$$\begin{array}{ccc} \mathcal{M}X_i & \longrightarrow & \binom{\mathcal{M}X_i}{\leq 7} \\ \downarrow & \text{weakly commutes} & \downarrow \\ \mathcal{M}X_j & \longrightarrow & \binom{\mathcal{M}X_j}{\leq 7} \end{array}$$

original instance

$$z_1 \xrightarrow{R} z_2 \xleftarrow{S} z_3 \quad \cdots \xrightarrow{R} z_8 \cdots$$

new instance

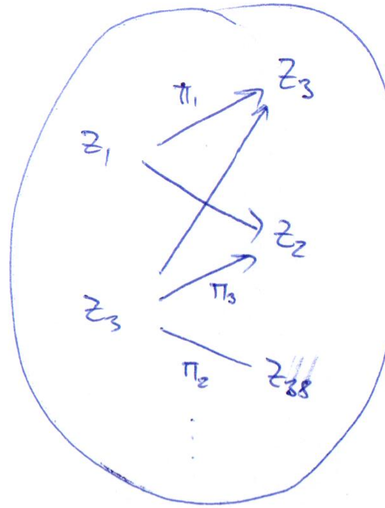
layer 1	layer 2	layer 3	…	layer $10^{10^{10}}$
z_1	$z_{\{12\}}$	$z_{\{123\}}$		$z_{\{123\ldots\}}$
z_2	$z_{\{13\}}$	$z_{\{124\}}$		
z_3	$\vdots$			

domains of variables: partial solutions

constraints: the obvious ones

<u>Remark</u>: only few layers actually needed

Why BBKO gives hardness



Label Cover   CSP(graphs of functions)

Gap Label Cover (1/49)   GIVEN: satisfiable Label Cover instance
FIND: assignment that satisfies 1/49 - fraction of the constraints

NP-hard! < PCP Theorem [Arora, Lund, Motwani, Sudan, Szegedy '98] [Dinur '07]
Parallel Repetition Theorem [Raz '98] [Verbitsky '94]

<u>Remark</u>: PVCSP not PCSP

Gap Label Cover (7)   GIVEN: satisfiable Label Cover instance
FIND: 7-assignment that <u>weakly</u> satisfies all the constraints

variables → ≤7-element subsets of domain elements

hopefully clear

Gap Label Cover (1/49) ≤ Gap Label Cover (7) ≤ PCSP (M)

trivial reductions

whenever BBKO: $\exists \mu = \{\mu_x ; MX \to \binom{X}{7}\}$ such that ...

Q: Is it nessessary to use ?

Baby PCP Theorem

$\forall \mathbb{A}$ $CSP(\mathbb{A}) \leq$ GapLabelCover(7) by an obvious reduction

say $D = \{1,2,3\}$
$\mathbb{A}$ = binary relations

old instance

$z_1 \xrightarrow{R} z_2 \xleftarrow{S} z_3 \dots$
$z_2 \to z_{51}$
...

new instance

layer a — layer b

$z_{\{12\dots a\}} \xleftarrow{\pi} z_{\{12\dots b\}}$
⋮

domains: partial solutions
constraints: obvious

Need to show: 7-assignment for new instance $\Rightarrow$ solution of the old instance

~~7-assignment~~

7-assignment $\left\{\right.$ $\forall U$ set of variables of size a we have $S_U \subseteq D^U$, $|S_U| = 7$ of partial solutions
$\forall V$ —"— b —"— $S_V \subseteq D^V$, $|S_V| = 7$ —"—
weakly consistent $\forall U \subseteq V$ $S_V \restriction U \cap S_U \neq \emptyset$

V: U: $z_1\ z_2 \dots$; $z_{137} \dots$

$S_U \left\{ \begin{array}{l} 1\,2\,2\,1\,1\,3\,1\,2 \\ 1\,3\,2\,1\,1\,2\,1\,3 \\ 3\,1\,2\,1\,1\,1\,2\,1 \\ \dots \end{array} \right.$

$S_V \left\{ \begin{array}{l} 1\,1\,2\,1\,1\,1\,1\,2\,3\,3\,1\,1\,1\,2\,1\,1\,1\,1\,1\,3 \\ 1\,2\,2\,1\,1\,3\,1\,2\,1\,3\,2\,1 \dots \\ \dots \end{array} \right.$

f: variables $\to D$
such that
$\forall z, z'$ $\exists V \supseteq \{z, z'\}$
$f \restriction \{z,z'\} \in S_V \restriction \{z, z'\}$

solution

$\forall U$ size a $\quad S_U \subseteq D^U$ size $\neq q$
$\forall V$ size b $\quad S_V \subseteq D^V$ size $\neq r$ $\Rightarrow$ solution
weakly consistent

← $\forall q, r$ $\forall$suf. big a $\forall$suf. big b

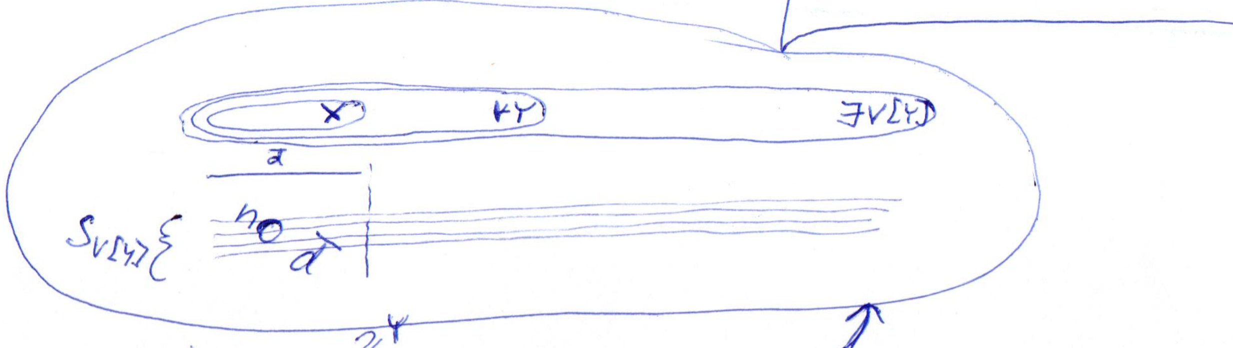
Strategy

$(q,r) \leftarrow (q-1,r)$ I, II

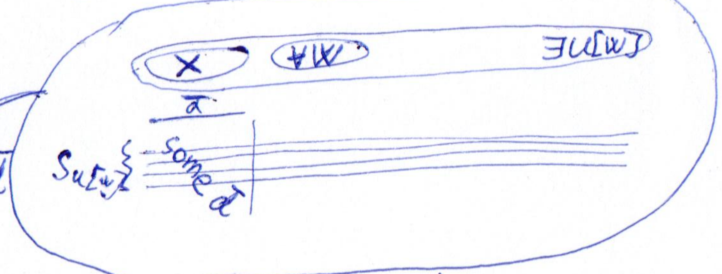
$(1,r) \leftarrow (1,r-1)$

Proof for (q,r) assuming $(q-1,r)$

- say a', b' work for $(q-1,r)$
- a, b, c sufficiently big $\quad a' \ll a$, $\quad b' \ll c \ll b$

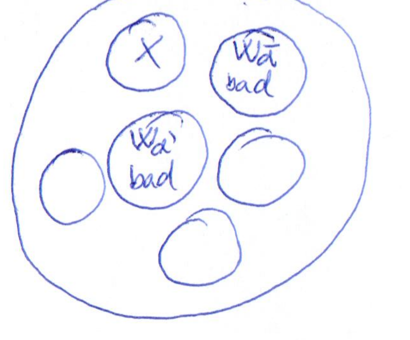


- <u>Assume</u> $(\exists X \text{ size } a')(\forall \bar{a} \in D^X)(\forall Y \supseteq X \text{ size } \leq c)(\exists V[Y] \supseteq Y \text{ size } b)\ S_{V[Y]} \restriction X \not\ni \bar{a}$
- <u>Fix</u> such X
- <u>Find</u> $\bar{a} \in D^X$: $(\forall W \text{ size } a')(\exists U[W] \supseteq X \cup W \text{ size } a)\ S_{U[W]} \restriction X \ni \bar{a}$

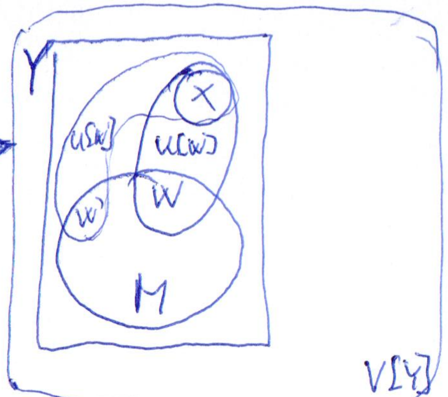


- For W size a' <u>define</u> $S'_W := (S_{U[W]} - \text{tuples with } \restriction X = \bar{a}) \restriction W$

↓proof



- For M size b' <u>take</u> $Y := \bigcup_{\substack{W \subseteq Y \\ \text{size } a'}} U[W]$



<u>define</u> $S'_M := S_{V[Y]} \restriction M$

<u>Remark</u>: seems different from Dinur's proof of PCP theorem

An application

Old Thm: $\exists$ homo $\Rightarrow$ reduction

$10^{10^{10}}$-coloring 2-colorable 3-uniform hypergraph $\leq$ $\mathrm{PCSP}(\mathbb{K}_3, \mathbb{K}_5)$ [BKO]

[Dinur, Regev, Smyth '05]

New thm: $\exists$ (7,5)-homo $\Rightarrow$ reduction

$$\mathrm{PCSP}(\mathcal{M}) \sim \mathrm{PCSP}(\mathcal{M} \cup \{\text{all 7-ary functions}\})$$

intuition: complexity does not depend on low-arity ~~functions~~ polymorphisms

more generally: $\mathrm{PCSP}(\mathcal{M}) \sim \mathrm{PCSP}(\mathcal{M} \cup \mathcal{N})$

if $\exists$ (7,5)-homo $\mathcal{N} \to \mathcal{I}$

Summary

- $\forall$ PCSP $\sim$ PCSP(all relations, some relaxations)
- $(7,5)$-homo $\Rightarrow$ obvious reduction works
- obvious reduction from any CSP proves NP-hardness of GapLabelCover(7)

Questions

- superconstant 7 in $(mX \to \binom{X}{7})$ Recall: was needed for hypergraph coloring
  - is such a version true? (say with $\log|X|$)
  - in particular, $\exists$ analogue of the Parallel Repetition Theorem for ?

- Baby d-to-1 conjecture ...

- capture other NP-hardness results & reductions
  - PCSP($\mathbb{K}_{alot}, \mathbb{K}_{2^{alot^{1/3}}}$) [Huang '13] • d-to-1 $\Rightarrow$ PCSP($\mathbb{K}_k, \mathbb{K}_c$) [Guruswami, Sandeep '20]
  - PCSP($\mathbb{K}_{\binom{4}{2}}, \mathbb{K}_{\binom{38}{19}}$) $\leq$ PCSP($\mathbb{K}_4, \mathbb{K}_{38}$) [Wrochna, Živný '20]
    - does not seem captured, <u>but</u> obvious reduction works!
    - $\exists$ different notion of homomorphism $\rightsquigarrow$ unify!