19th Jarník lecture

CSPs and Symmetries

Libor Barto, Charles University

erc
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CoCoSym: Symmetry in Computational Complexity

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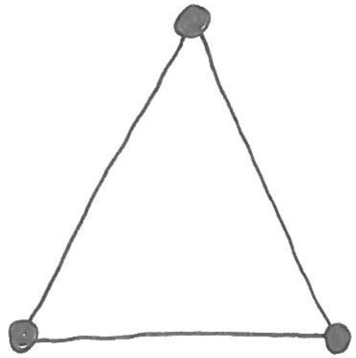
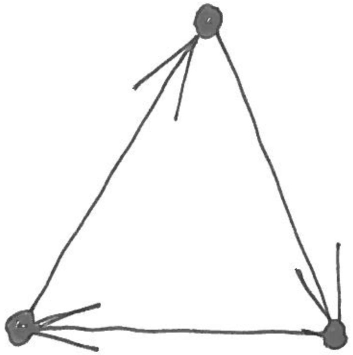
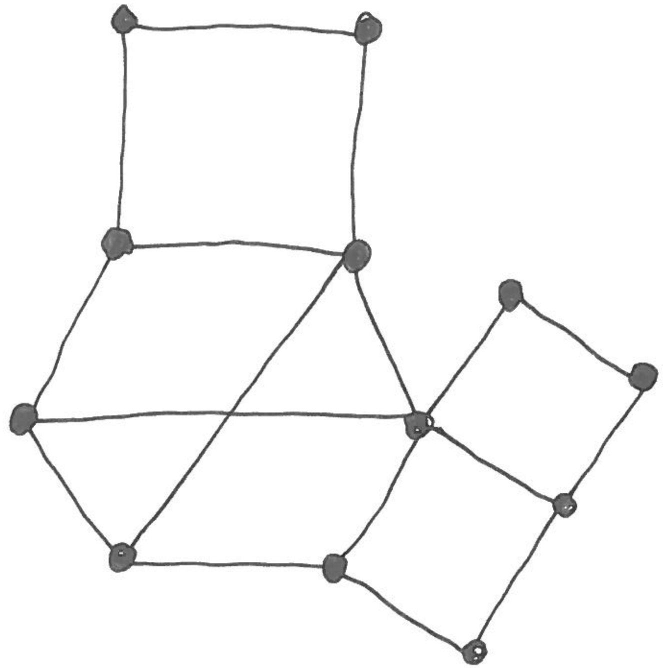
⚠

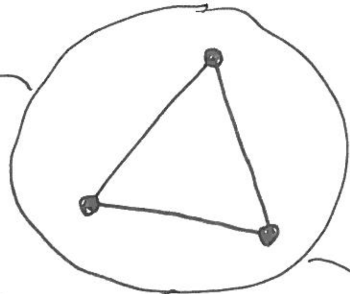
Are these shapes symmetric?

① 

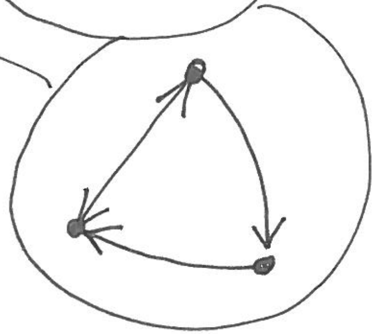
② 

③ 

graphs, digraphs, homomorphisms



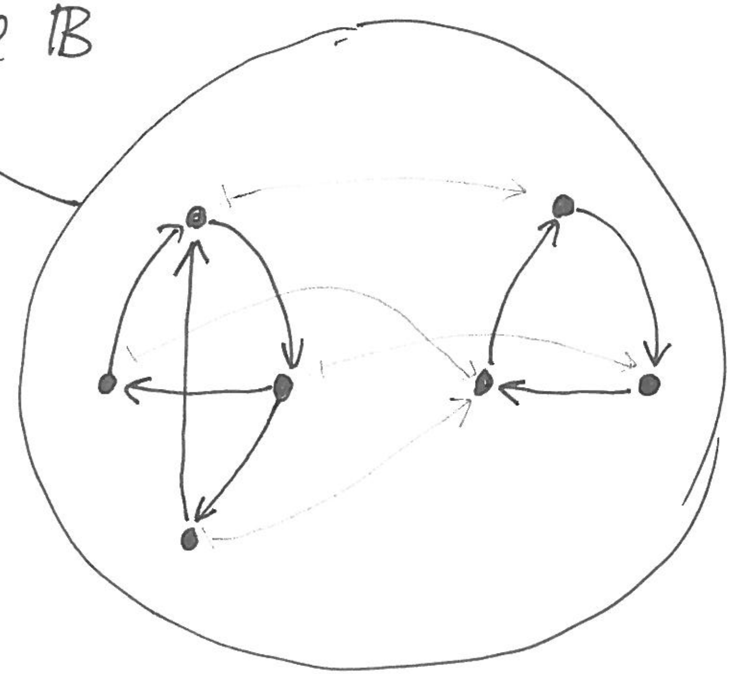
graph $\mathbb{A}$: vertices, edges



digraph $\mathbb{A}$: vertices, directed edges

homomorphism $\mathbb{A} \to \mathbb{B}$:

mapping vertices of $\mathbb{A} \to$ vertices of $\mathbb{B}$
that preserves edges



endomorphism of $\mathbb{A}$: $\mathbb{A} \to \mathbb{A}$

automorphism of $\mathbb{A}$: invertible $\mathbb{A} \to \mathbb{A}$

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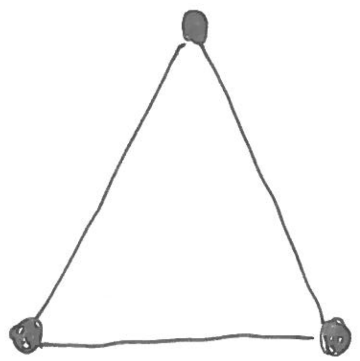
Are these shapes symmetric?

1



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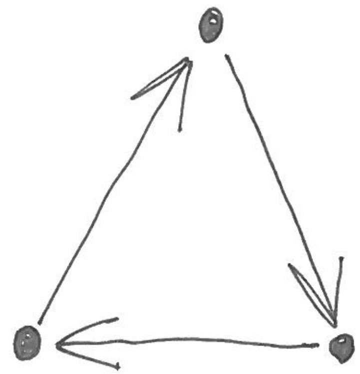

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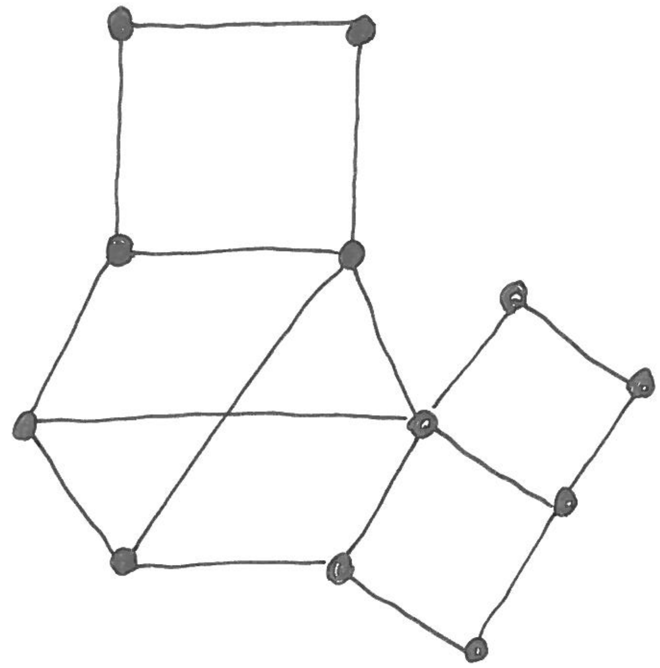


Outline

- CSP
- CSPs & Symmetries
- Analysis of symmetries

CSPs

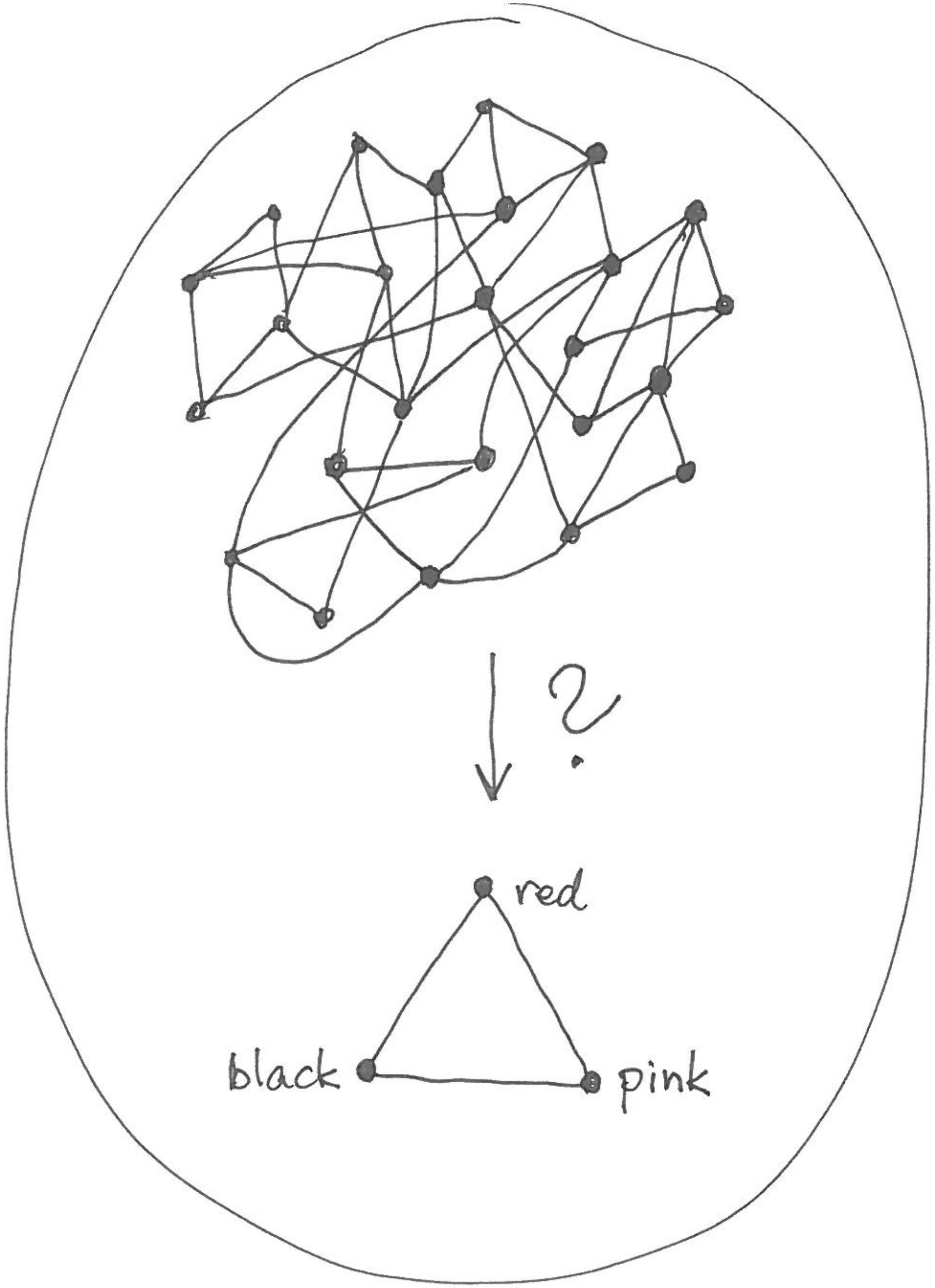
Constraint Satisfaction
Problems

3-coloring problem

INPUT: graph $\mathbb{X}$

OUTPUT: $\mathbb{X} \longrightarrow \triangle$

(if it exists)



Question: how fast can it be solved?

A course in computational complexity

<u>computational problem</u>:
- specified class of inputs
- specified correct outputs

<u>examples</u>: the 3-coloring problem, 2-coloring, ...

it is <u>in P</u> : can be solved by an algorithm running in time $O(n^{\text{const}})$ where n is the size of the input

<u>in NP</u>: correct answers can be verified in P

<u>NP-complete</u>: hardest in NP

$\boxed{P = NP?}$

Examples

- 5-coloring
- 3-SAT : Find a satisfying assignment to

  e.g. $(x \vee \neg y \vee \neg u) \wedge (\neg x \vee z \vee w) \wedge (z \vee \check{r} \vee b) \wedge \ldots$

- LIN-$\mathbb{Z}_2$: Find a solution to

  e.g. $\begin{cases} x + y = 1 \\ y + u + w = 0 \\ u + x = 1 \\ \vdots \end{cases}$ in $\mathbb{Z}_2$

- LP : Find a solution to

  e.g. $\begin{cases} 2x + 3y \geq 1 \\ x - 3u + v \leq 5 \\ \vdots \end{cases}$ in $\mathbb{Q}$

CSP

$\mathbb{A}$: fixed digraph (or other structure)

CSP($\mathbb{A}$)
INPUT: finite digraph $\mathbb{X}$
OUTPUT: Find $\mathbb{X} \to \mathbb{A}$

- the fixed-template CSP
- many variants

- each $\mathbb{A} \rightsquigarrow$ computational problem
- how broad is this class ?
  - general $\mathbb{A}$: all computational problems
  - finite $\mathbb{A}$: 3-coloring (), 3-SAT, LIN-$\mathbb{Z}_2$

always in NP

CSPs

&

Symmetries

Polymorphisms

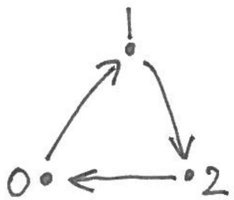
$$\mathbb{A} = (V, \; E \subseteq V^2)$$

vertices   edges

$f : V^n \longrightarrow V$ is a **polymorphism of $\mathbb{A}$** if

$$\begin{array}{cccccc} f(v_1, & v_2, & \dots, & v_n) & = & w \\ \downarrow & \downarrow & \cdots & \downarrow & \Rightarrow & \downarrow \\ f(v_1', & v_2', & \dots, & v_n') & = & w' \end{array}$$

Examples

- $\mathbb{A} =$ 

  $$f(x,y) = \begin{cases} x & \text{if } x \rightarrow y \\ y & \text{if } x \leftarrow y \end{cases}$$

- $\mathbb{A} = (\mathbb{R}, \; E \subseteq \mathbb{R}^2 \text{ convex})$ $f(x,y) = 0.3x + 0.7y$

CSP and symmetry

Theorem [Jeavons '98]

$\mathrm{Pol}(\mathbb{A})$ contains $\mathrm{Pol}(\mathbb{B})$ $\Rightarrow$ $\mathrm{CSP}(\mathbb{A}) \leq \mathrm{CSP}(\mathbb{B})$

all polymorphisms of $\mathbb{A}$ — no harder than

"the more symmetric the easier"

"complexity depends only on symmetries"

- Improvements: [Bulatov, Jeavons, Krokhin '05]
  [Barto, Opršal, Pinsker '18] The Wonderland of Reflections

- Goal: symmetries beyond CSPs

Endomorphisms vs. polymorphisms

	endo/auto morphisms	polymorphisms
what is it	$\mathbb{A} \longrightarrow \mathbb{A}$ symmetry of $\mathbb{A}$	$\mathbb{A}^n \longrightarrow \mathbb{A}$ multivariate symmetry of $\mathbb{A}$
trivial	$v \longmapsto v$ identity	$(v_1, v_2, \dots, v_n) \longmapsto v_i$ $\boxed{\text{dictators}}$
all	endomorphism monoid permutation group	clone
studied in	semigroup theory group theory	universal algebra

Functional equations

Theorem [Bulín, Krokhin, Opršal '19]

CSP($\mathbb{A}$) is equivalent to:

INPUT: trivial system of functional equations* $f(\text{vars}) = g(\text{vars})$

e.g. $m(x, y, z, x) = f(y, z, x)$

$f(x, x, y) = g(y, x)$

$m(x, y, x, y) = g(x, y)$

$\vdots$

OUTPUT: solution in Pol($\mathbb{A}$)

trivial = solvable by dictators

* of some fixed large enough bound on arity

CSP and Symmetry II

Theorem: $CSP(\mathbb{A}) \sim$ solving trivial systems of special functional equations in $Pol(\mathbb{A})$

"the more special equations $Pol(\mathbb{A})$ satisfies, the easier $CSP(\mathbb{A})$ is"

- only trivial equations $\Rightarrow$ NP-complete
- strong enough equations $\Rightarrow$ in P

"complexity depends only on equations satisfied by symmetries"

clone $\xrightarrow{\text{abstraction}}$ equations among its members

analogues to permutation group $\rightsquigarrow$ group

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CSP history

ancient history

- 2-element structures [Schaefer '78]
- graphs [Hell, Nešetřil '90]
- dichotomy conjecture [Feder, Vardi '98]
  P / NP-complete ?

medieval history

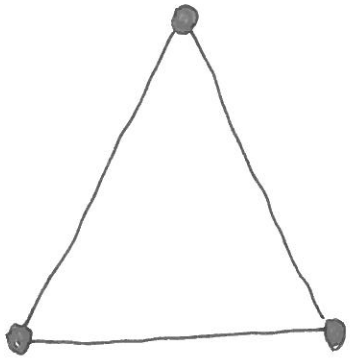
- symmetries [Bulatov, Jeavons, Krokhin, ...]
- describing all homomorphisms [Idziak, Marković, McKenzie, Valeriote, Willard '07]
- consistency [Barto, Kozik '14]
- dichotomy theorem [Bulatov '17, Zhuk '17]

modern history

$\boxed{\text{some nontrivial equations} \Rightarrow \text{in P!}}$

Are these shapes symmetric?

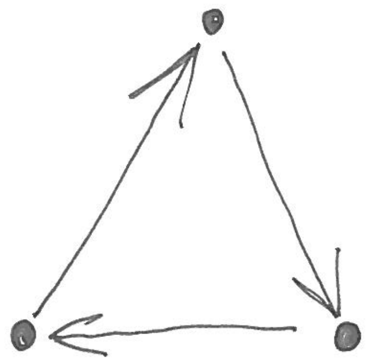
1



NO

only trivial equations

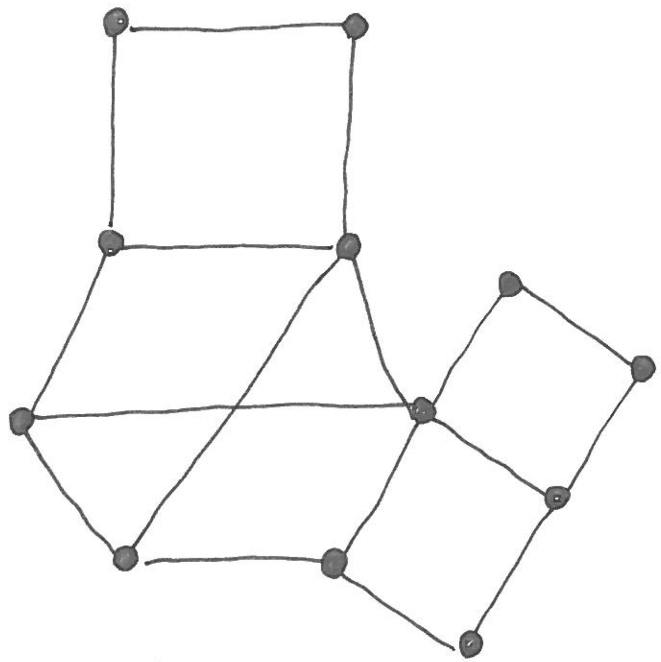
2



YES

$f(x,y) = f(y,x)$

3



YES

$m(x,x,y) = m(x,x,x)$

$m(x,y,z) = m(y,x,z) =$
$= m(z,y,x) = \dots$

Analysis of Symmetries

Cyclic polymorphism

Theorem [Barto, Kozik '12]

some nontrivial system of functional equations satisfied in $Pol(\mathbb{A})$

$\Rightarrow$ this "system" is: $f(x_1, x_2, \dots, x_p) = f(x_2, \dots, x_p, x_1)$ ($\forall$prime $p > |A|$)

Tool: absorbing subset $B \subseteq A$

$$f(B, B, \dots, B, A) \subseteq B$$
$$f(B, B, \dots B, A, B) \subseteq B$$
$$\vdots$$
$$f(A, B, B, \dots, B) \subseteq B$$

compare

ideal $I \subseteq R$ in ring

$$I \cdot R \subseteq \cancel{R}\, I$$
$$R \cdot I \subseteq I$$

3-SAT is hard to approximate

Theorem [Håstad 01]

INPUT: e.g. $(x \vee \neg y \vee z) \wedge \underbrace{(\neg x \vee u \vee \neg v)}_{}^{\text{clause}} \wedge (\neg w \vee \neg z \vee \neg r) \wedge \dots$
which is satisfiable

OUTPUT: assignment satisfying $7/8+\varepsilon$ fraction of clauses

is NP-complete

Tool: Fourièr analysis of $\{0,1\}^n \longrightarrow \{0,1\}$

express them in the basis

$$x_1, x_2, \dots, x_n$$
$$x_1+x_2,\ x_1+x_3, \dots, x_{n-1}+x_n$$
$$x_1+x_2+x_3, \dots$$
$$\vdots$$
$$x_1+x_2+\dots+x_n$$

Promises not helpful for 3-coloring

Theorem [Krokhin, Opršal '19]

INPUT: graph G such that $G \to$ 

OUTPUT: find $G \to$ 

is NP-complete

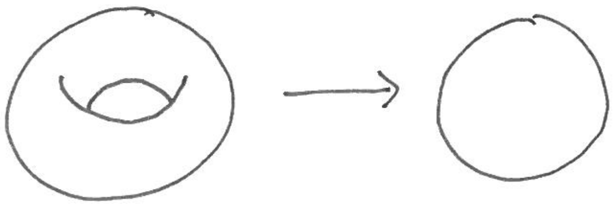
Tool: algebraic topology

instead of

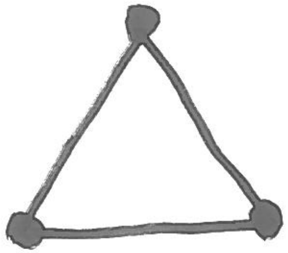
topological combinatorics
[Lovász '78]

consider

e.g. for $n=2$



Conclusion

 is **not** symmetric

complexity determined by symmetry

analysis of symmetries is fun