Symmetric Promise Constraint Satisfaction Problems

Beyond the Boolean Case

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Note that $\mathrm{CSP}(\mathbf{A}) = \mathrm{PCSP}(\mathbf{A}, \mathbf{A})$.

Examples of CSPs and PCSPs

Many computational problems, such as 3-coloring and 3SAT, can be expressed in the language of CSPs: 3-coloring corresponds to the CSP over the clique on three vertices – $\mathrm{CSP}(\mathbf{K}_3)$ – and 3SAT corresponds to the CSP over a binary domain with all ternary relations.

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Since PCSP is a generalization of CSP, these problems can also be expressed in the language of PCSPs. Moreover, PCSP is capable of expressing a vast number of additional problems, such as the problem of finding an l-coloring of a k-colorable graph when $k \leq l$ – $\mathrm{PCSP}(\mathbf{K}_k, \mathbf{K}_l)$.

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CSPs are known to have a hardness dichotomy – all CSPs are either NP-complete or in P (Bulatov, Zhuk '17). No such dichotomy has yet been shown for PCSPs. The strongest classification result obtained so far in this direction is the dichotomy theorem over Boolean *symmetric* templates, i.e., templates whose relations are all invariant under permutations of coordinates (Brakensiek, Guruswami '18, Ficak et al. '19).

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1in3 is a binary domain $\{0, 1\}$ with a single symmetric ternary relation $\{(0, 0, 1), (0, 1, 0), (1, 0, 0)\}$. $\mathrm{CSP}(\mathbf{1in3})$ corresponds to the positive 1-in-3-SAT.

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These problems have a hypergraph coloring interpretation: given a 3-uniform hypergraph that is **1in3**-colorable (that is, each vertex can be assigned a color from $\{0,1\}$ so that there is exactly one 1 appearing in each hyperedge), find a **B**-coloring (that is, a coloring by B such that the three colors appearing in each hyperedge are from R).

Three Element Symmetric Structures

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So, for example, **1in3** becomes $\to$ and **NAE**, the relation corresponding to Not-All-Equal 3SAT, becomes $\leftrightarrows$.

Diagrams of Three Element Symmetric Structures

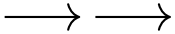
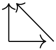
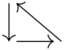
Diagram						
Structure **B**	**1in3**	**NAE**	$\mathbf{D_1}$	$\mathbf{D_2}$	$\mathbf{T_1}$	$\mathbf{T_2}$

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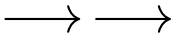
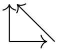
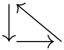
Diagram						
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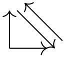
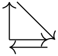
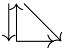
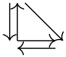
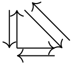
Diagram					
Structure **B**	$\mathbf{Q}_1$	$\mathbf{Q}_2$	$\mathbf{Q}_3$	**C**	**S**

The Hierarchy of Three Element Symmetric Structures

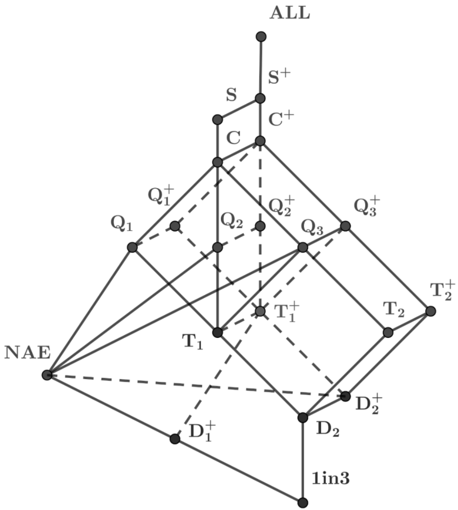


Figure: The templates $\mathbf{B}$ ordered by the relation $\mathbf{B} \leq \mathbf{B}'$ if $\mathbf{B} \rightarrow \mathbf{B}'$.

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Theorem

Let $(\mathbf{1in3}, \mathbf{B})$ *be a PCSP template, where* $\mathbf{B}$ *has domain-size three.*

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- *If* $\mathbf{B} \to \mathbf{T}_1$ *or* $\mathbf{B} \to \mathbf{D}_1^+$ *or* $\mathbf{B} \to \mathbf{D}_2^+$*, then* $\mathrm{PCSP}(\mathbf{1in3}, \mathbf{B})$ *is NP-hard.*

Three Element Symmetric Structures – Hierarchy of Results

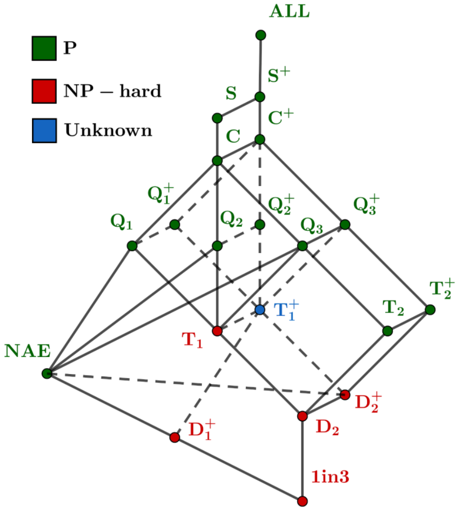

Figure: The templates $\mathbf{B}$ ordered by the relation $\mathbf{B} < \mathbf{B}'$ if $\mathbf{B} \to \mathbf{B}'$.

Preliminaries – Polymorphisms

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Definition (Polymorphism)

Let $(\mathbf{A}, \mathbf{B})$ be a PCSP template. A mapping $f : A^n \to B$ is a *polymorphism of arity* n if, for each pair of corresponding relations R_i and R_i' in the signatures of $\mathbf{A}$ and $\mathbf{B}$, respectively, and any $(r_{1,1}, r_{2,1}, \dots, r_{n,1})$, $\dots$, $(r_{1,\mathrm{ar}_i}, r_{2,\mathrm{ar}_i}, \dots, r_{n,\mathrm{ar}_i})$ with $(r_{j,1}, r_{j,2}, \dots, r_{j,\mathrm{ar}_i}) \in R_i$ for all $j \in [n]$, we have $(f(r_{1,1}, r_{2,1}, \dots, r_{n,1}), \dots, f(r_{1,\mathrm{ar}_i}, r_{2,\mathrm{ar}_i}, \dots, r_{n,\mathrm{ar}_i})) \in R_i'$.

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$$f^\alpha(a_1, \dots, a_m) = f(a_{\alpha(1)}, \dots, a_{\alpha(n)})$$

for every $a_1, \dots, a_m \in A$. A function $g : A^m \to B$ is a *minor of f* if $g = f^\alpha$ for some α.

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The significance of polymorphisms and minors stems from the fact that the computational complexity of $\mathrm{PCSP}(\mathbf{A}, \mathbf{B})$ depends only on the set of all polymorphisms of the template $(\mathbf{A}, \mathbf{B})$. This set is a *minion*, i.e., it is closed under taking minors.

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Definition (Chain of Minors)

A *chain of minors* is a sequence of the form $(f_0, \alpha_{0,1}, f_1, \alpha_{1,2}, \ldots, \alpha_{l-1,l}, f_l)$ where $f_0, \ldots, f_l : A^{n_i} \to B$, $\alpha_{i-1,i} : [n_{i-1}] \to [n_i]$, and $f_{i-1}^{\alpha_{i-1,i}} = f_i$ for every $i \in [l]$. We write $\alpha_{i,j} : [n_i] \to [n_j]$ for the composition of $\alpha_{i,i+1}$, $\alpha_{i+1,i+2}$, $\ldots$, $\alpha_{j-1,j}$. Note that $f_i^{\alpha_{i,j}} = f_j$.

The NP-Hardness Criterion

Theorem (Brandts, Wrochna, Živný '20)

Let $(\mathbf{A}, \mathbf{B})$ *be a PCSP template. Suppose there are constants* $k, l \in \mathbb{N}$ *and an assignment of a set of at most* k *coordinates* $\mathrm{sel}(f) \subseteq [\mathrm{ar}(f)]$ *to every polymorphism* f *of* $(\mathbf{A}, \mathbf{B})$ *such that for every chain of minors* $(f_0, \alpha_{0,1}, \dots, f_l)$ *with each* f_i *a polymorphism of* $(\mathbf{A}, \mathbf{B})$*, there are* $0 \leq i < j \leq l$ *such that* $\alpha_{i,j}(\mathrm{sel}(f_i)) \cap \mathrm{sel}(f_j) \neq \emptyset$ *(or, equivalently,* $\mathrm{sel}(f_i) \cap \alpha_{i,j}^{-1}(\mathrm{sel}(f_j)) \neq \emptyset$*). Then* $\mathrm{PCSP}(\mathbf{A}, \mathbf{B})$ *is NP-hard.*

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Our general approach to showing NP-hardness relies on observing key properties of the polymorphisms for a given template, and using these properties to define "types" of polymorphisms. We then analyze a chain of minors based on these types, and apply the criterion. This is similar to the "smug sets" approach in BWZ '20.

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For our example proof, we consider $\mathrm{PCSP}(\mathbf{1in3}, \mathbf{D}_2^+)$, where $\mathbf{D}_2^+ = (\{0, 1, 2\}, R)$ and R consists of all the permutations of the tuples $(0, 0, 1)$, $(1, 1, 2)$, and $(0, 1, 2)$.

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For our example proof, we consider $\mathrm{PCSP}(\mathbf{1in3}, \mathbf{D}_2^+)$, where $\mathbf{D}_2^+ = (\{0,1,2\}, R)$ and R consists of all the permutations of the tuples $(0,0,1)$, $(1,1,2)$, and $(0,1,2)$.

Let $f : \{0,1\}^n \to \{0,1,2\}$ be a polymorphism of $(\mathbf{1in3}, \mathbf{D}_2^+)$. We adopt two shorthand conventions for $X \subseteq [n]$: $f(X)$ denotes f evaluated with 1 in the ith position for all $i \in X$ and 0 in all other coordinates, and we say X is a *j-set* if $f(X) = j$.

Lemma

Let X and Y be disjoint subsets of $[n]$.

(a) *If $f(\emptyset) = 0$, $f(X) = 0$, and $f(Y) \in \{0,2\}$, then $f(X \cup Y) \in \{0,2\}$.*

(b) *If $f(\emptyset) = 0$, $f(X) = 1$, and $f(Y) \in \{0,1\}$, then $f(X \cup Y) = 1$.*

(c) *If $f(\emptyset) = 1$, $f(X) = f(Y) = 1$, then $f(X \cup Y) \in \{0,1\}$.*

(d) *If $f(\emptyset) = 1$, $f(X) = f(Y) = 0$, then $f(X \cup Y) = 2$.*

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Lemma

If $f(\emptyset) = 1$*, then there exists a 0-set or a 2-set of size at most 2.*

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Conjecture

$\mathrm{PCSP}(\mathbf{1in3}, \mathbf{T}_1^+)$ *and the generalization to larger domains, is NP-complete.*

If this conjecture holds, there is a unique source of hardness for our templates.

Larger Domains

For a 4-element B, the conjecture would resolve all the cases with the exception of the interval between $\check{\mathbf{C}}$ and $\check{\mathbf{C}}^{+}$, where $\check{\mathbf{C}}$ is given by the relation containing the tuples $(0,0,1)$, $(1,1,2)$, $(2,2,3)$, $(3,3,0)$ and their permutations, and $\check{\mathbf{C}}^{+}$ is given by the same relation with all the "rainbow" tuples (i,j,k) such that $|\{i,j,k\}| = 3$.

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Theorem

$\mathrm{PCSP}(\mathbf{1in3}, \check{\mathbf{C}})$ *is NP-hard. The template* $(\mathbf{1in3}, \check{\mathbf{C}}^+)$ *does not have a block symmetric polymorphism with two blocks of sizes 23 and 24 (and therefore fails to satisfy the known sufficient condition for tractability in PCSPs from, e.g. Brakensiek, Guruswami '20).*

Conjectures

Let $\mathbf{LO}_k$ denote the k-element domain structure whose relations are permutations relations (b, b, c) where $b < c$ in the linear order $0 < 1 < 2 < \ldots < k - 1$.

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$\mathrm{PCSP}(\mathbf{1in3}, \check{\mathbf{C}}^+)$ *is NP-hard.*

Negative resolution of this conjecture would also be valuable – it would require a polynomial-time algorithm that has not yet been used for PCSPs.

Thank you for your time!